Self-study: Spivak's Calculus

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Conventions

The set $\mathbb{N}$ is either used to notate
Either $\{0, 1, 2, \cdots\}$
Or $\{1, 2, 3, \cdots\}$

To avoid confusion between the two conventions, we will use
$\mathbb{N}_0 = \{0, 1, 2, \cdots\}$
$\mathbb{N}_{\geqslant 1} = \{1, 2, 3, \cdots\}$
Which hopefully minimises ambiguity on this

Lemons

Lemon 1: Convolution (Actually called The Cauchy Product too) (Math Discord)

1.

Let F and G be functions, such that;

$$F(x) = \sum_{i=0}^{n} a_i x_i$$
$$G(x) = \sum_{i=0}^{n} b_i x^i$$

Now,

$$F(x)G(x) = \sum_{i=0}^{n+m} c_i x_i$$

Where

$$c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

i.e.;

$$F(x)G(x) = \sum_{i=0}^{n+m} \sum_{k=0}^{i} a_k b_{i-k} x_i$$

$\left(\text{Or equivalently, F(x)G(x) = } \sum_{i=0}^{n+m} \sum_{k=0}^{i} a_i - k b_k x^i\right)$

2.

Also, given $\sum_{i=0}^{n} a_i$ and $\sum_{i=0}^{n} b_i$;

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{i=0}^{n} b_i\right) = \sum_{i=0}^{n} c_i$$

Where

$$c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

i.e.

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{i=0}^{n} b_i\right) = \sum_{i=0}^{n} \sum_{k=0}^{i} a_k b_{i-k}$$

★ Works not only for $n \in \mathbb{N}_0$ but also for $\lim_{n\to\infty}$, i.e. infinite series.

Lemon 2: Definition of sums using an index set (Math Discord)

Let I be an index set with $i \in I$ and $f(i) \geq 0$ for all i,

$$\sum_{i \in I} f(i) := \sup \left\{ \sum_{i \in J} f(i) : J \subseteq I \land J \text{ finite} \right\}$$

In the cases where $f(i)$ can be smaller than 0,

You would have to specify a way of exhausting the index set using finite subsets, and the sum may exist for some such choice, but it would be very dependent on this choice in general.
($|f|$ here means the outputs of the function f, or more precisely, elements of its range)

If $|f|$ is summable, which requires that it is nonzero on a countable set,
(Countable means either finite or countably infinite here, including the empty set)
then the definition is to take an exhausting collection of finite sets and take the limit of the associated finite sums.

Facts:
1. This limit exists!
2. This limit is independent of the choice of exhausting collection
(summability of $|f|$ used here to prove both 1 and 2).

If $|f|$ is not summable, then you do not have existence/independence, but you can sometimes still talk about limits over a particular choice exhausting collection of finite subsets.
(This is analogous to summing things like $1 - 1/2 + 1/3 - 1/4 + \cdots$).

Also, something else noteworthy is that for a sum to be convergent, there can at most be countably many nonzero terms!

See:

1. Riemann Rearrangement Theorem
2. https://math.stackexchange.com/questions/1413874/can-we-add-an-uncountable-number-of-positive-elements-and-can-this-sum-be-finit
3. https://math.stackexchange.com/questions/20661/the-sum-of-an-uncountable-number-of-positive-numbers

Another reason to learn real analysis! :D

Lemon 3: The Rational Root Theorem

It's a special case of Gauss' Lemma and C2 Qns 18 is a subcase of the Rational Root Theorem (when $a_n = 1$)

Let $P(x)$ be a polynomial;

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

All rational roots of the polynomial $P(x)$, in the form $x = \frac{p}{q}$ where $\gcd(p, q) = 1$, satisfy

1. $p|a_0$
2. $q|a_n$

Lemon 4: Vieta's Formulas

We define $P(x)$ to be a polynomial;

$$P(x) = \sum_{i=0}^{n} a_i x^i$$

Let $k \in \mathbb{N}_{\geqslant 1}$ where $1 \leqslant k \leqslant n$.
As well as $j \in \mathbb{N}_{\geqslant 1}$, with: $1 \leqslant j \leqslant k$ and $1 \leqslant i_j \leqslant i_{j+1} \leqslant n$, for all j.
Such that it has n (possibly repeated) roots: $r_1, r_2, \cdots, r_n$.

Then;

$$\sum \prod_{j=1}^{k} r_{i_j} = (-1)^k \frac{a_{n-k}}{a_n}$$

Normally this is written as

$$\sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n} \left(\prod_{j=1}^{k} r_{i_j} \right) = (-1)^k \frac{a_{n-k}}{a_n}$$

But I find that subscript under the sum way to confusing for me, since whatever you write on or below the sum, is normally information related to what terms to sum and how to sum them up. However, here it is basically info for the product. So, I prefer to write it as words before writing the formula instead.

The notation I prefer which looks better imo is:

Let $k \in \mathbb{N}_{\geqslant 1}$ where $1 \leqslant k \leqslant n$.
As well as $\mathcal{N} = \{m | (m \in \mathbb{N}) \wedge (m \leqslant n)\}\}$

$$\sum_{\substack{I \subseteq \mathcal{N} \\ |I| = k}} \prod_{i \in I} r_i = (-1)^k \frac{a_{n-k}}{a_n}$$

Lemon 5: Faulhaber's Formula Inspired (to Google) by C2 Qns 7

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{i=0}^{p} (-1)^i \binom{p+1}{i} B_i n^{p+1-i}$$

where B_i notates the ith Bernoulli numbers, which have an explicit formula:

$$B_i = \sum_{j=0}^{i} \sum_{v=0}^{j} (-1)^v \binom{j}{v} \frac{v^i}{j+1}$$

So, I think we can also combine these to form an expliciter formula (and complicated) formula of:

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{i=0}^{p} \left[(-1)^i \binom{p+1}{i} \left(\sum_{j=0}^{i} \sum_{v=0}^{j} (-1)^v \binom{j}{v} \frac{v^i}{j+1} \right) n^{p+1-i} \right]$$

See:

1. https://proofwiki.org/wiki/Faulhaber's_Formula
2. https://www.youtube.com/watch?v=IAMfTu5bzg4
3. https://www.desmos.com/calculator/vrktco6rbh
4. https://www.desmos.com/calculator/bfnslzc5jf

Proof of Lemons

Lemon 1
Let F and G be functions, such that;

$$F(x) = \sum_{i=0}^{n} a_i x_i$$
$$G(x) = \sum_{i=0}^{n} b_i x^i$$

Now, their product is:

$$\begin{aligned}F(x)G(x) &= \sum_{i=0}^{n} a_i x_i \times \sum_{i=0}^{n} b_i x^i \\ &= (a_0x^0 + a_1x^1 + \cdots + a_nx^n)(b_0x^0b_1x^1 + \cdots + b_mx_m) \\ &= (a_0b_0)x^0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_2)x^2 \\ &\quad + \cdots + (a_0b_i + a_1b_{i-1} + \cdots + a_kb_{i-k} + \cdots + a_{i-1}b_1 + a_ib_0)x^i \\ &\quad + \cdots + (a_nb_m)x^{n+m}\end{aligned}$$

So, we notice that the coefficient for a general x^i is: $c_i = \sum_{k=0}^{i} a_k b_{i-k}$
And the product in general has $(n + m + 1)$ terms, (though coefficients can possibly be zero) ranging from $x^0, x, x^2, \cdots$ to x^{n+m}

Thus, the general formula for $F(x)G(x)$ is

$$\sum_{i=0}^{n+m} c_i x^i = \sum_{i=0}^{n+m} \sum_{k=0}^{i} a_k b_{i-k} x^i$$

In the case of the equivalent form of $F(x)G(x) = \sum_{i=0}^{n+m} \sum_{k=0}^{i} a_{i-k} b_k x^i$,
its just indexing in the opposite direction, i.e., a_k starts from i and decreases to 0 while b_k starts from 0 and increases to i, with each increase in k. Or simply swap F(x) and G(x).

Learning Points

1. There's a difference between simple / weak induction and complete / strong induction. (Check out C2 Qns 3 for instance)

2. Trying to list out

   α) What you know already (Info given in qns and prior knowledge that you think might be useful)

   (Even how it links to previous parts of the qns. Previous parts of the qns might give you a direct, simple, and elegant solution instead of having to solve / proof everything from scratch. See C2 Qns 3 lol.)

   β) What the qns is asking, simplifying the question into a simpler form first.

   (I guess basically like play around with the what the qns is asking, to see if you can find a form of the qns that gives you inspiration on how to proceed)

   So like, try to find the most important points needed to solve the qns, then try to think of some ideas that might help to tackle each of those points to work towards the solution.

1 Basic Properties of Numbers

7.

> **7.** Prove that if $0 < a < b$, then
> $$a < \sqrt{ab} < \frac{a+b}{2} < b.$$
> Notice that the inequality $\sqrt{ab} \leq (a+b)/2$ holds for all $a, b \geq 0$. A generalization of this fact occurs in Problem 2-22.

$$\begin{aligned} a &< b \\ a^2 &< ab \\ a &< \sqrt{ab} \end{aligned}$$

$$\begin{aligned} a &< b \\ a + b &< 2b \\ \frac{a+b}{2} &< b \end{aligned}$$

$$\begin{aligned} a &< b \\ 0 &< b - a \\ 0 &< a^2 - 2ab + b^2 \\ 4ab &< a^2 + 2ab + b^2 \\ \sqrt{ab} &< \frac{a+b}{2} \end{aligned}$$

So, $a < \sqrt{ab} < \frac{a+b}{2} < b$.

19.

19. The fact that $a^2 \geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1 y_1 + x_2 y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\,\sqrt{y_1{}^2 + y_2{}^2}.$$

(A more general form occurs in Problem 2-21.) The three proofs of the Schwarz inequality outlined below have only one thing in common—their reliance on the fact that $a^2 \geq 0$ for all a.

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number $\lambda \geq 0$, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no

(a) Assume $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some $\lambda \geqslant 0$, notice that

$$x_1y_1 + x_2y_2 = \lambda(y_1^2 + y_2^2)$$

$$\begin{aligned}\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} &= \sqrt{\lambda^2(y_1^2 + y_2^2)}\sqrt{y_1^2 + y_2^2} \\ &= \sqrt{\lambda^2(y_1^2 + y_2^2)^2} \\ &= \lambda(y_1^2 + y_2^2)\end{aligned}$$

Therefore, since $x_1y_1 + x_2y_2 = \lambda(y_1^2 + y_2^2)$ equality holds for the Schwarz Inequality in this case

Now when $y_1 = y_2 = 0$,

$$\begin{aligned}x_1y_1 + x_2y_2 &= 0 \\ &= \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}\end{aligned}$$

Thence, eqality trivially holds in this case as well.

Again, let $\lambda \geq 0$, suppose that y_1 and y_2 are not both 0, and that there is no λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then,

$$0 \leqslant (\lambda y_1` - x_1)^2 \text{ and } 0 \leqslant (\lambda y_2 - x_2)^2$$
$$\implies 0 \leqslant (\lambda y_1` - x_1)^2 + (\lambda y_2 - x_2)^2$$

The only case where $(\lambda y_` - x_1)^2 + (\lambda y_2 - x_2)^2 = 0$ is if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. However, we supposed that there exists no λ where this occurs. Therefore, we can safely eliminate the equality, leaving us with:

$$\begin{aligned}0 &< (\lambda y_1` - x_1)^2 + (\lambda y_2 - x_2)^2 \\ &= \lambda^2 y_1^2 - 2\lambda x_1y_1 + x_1^2 + \lambda^2 y_2^2 - 2\lambda x_2y_2 + x_2^2 \\ &= \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2)\end{aligned}$$

As desired.

Since our quadratic equation here is greater than 0 for all x, we can apply the discriminant from 18.:

$$\begin{aligned}\left[\frac{2(x_1y_1 + x_2y_2)}{(y_1^2 + y_2^2)}\right]^2 - 4\left[\frac{(x_1^2 + x_2^2)}{(y_1^2 + y_2^2)}\right] &< 0 \\ \left[\frac{2(x_1y_1 + x_2y_2)}{(y_1^2 + y_2^2)}\right] &< \sqrt{4\left[\frac{(x_1^2 + x_2^2)}{(y_1^2 + y_2^2)}\right]} \\ 2(x_1y_1 + x_2y_2) &< 2\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2} \\ x_1y_1 + x_2y_2 &< \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}\end{aligned}$$

Thence, we have exhausted all possible options by showing that equality holds iff $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ or $y_2 = y_2 = 0$, otherwise, the inequality is true but both sides are then not equal. So, we have proven the Schwarz Inequality.

(b)

$2xy \leqslant x^2 + y^2$ is derived as follows:

$$\begin{aligned}
0 &\leqslant (x-y)^2 \\
0 &\leqslant x^2 - 2xy + y^2 \\
2xy &\leqslant x^2 + y^2
\end{aligned}$$

Let $x = \frac{x_i}{\sqrt{x_1^2+x_2^2}}$ and $y = \frac{y_i}{\sqrt{y_1^2+y_2^2}}$ where $i = 1, 2$;

$$\begin{aligned}
2xy &\leqslant x^2 + y^2 \\
2\left(\frac{x_i}{\sqrt{x_1^2+x_2^2}}\right)\left(\frac{y_i}{\sqrt{y_1^2+y_2^2}}\right) &\leqslant \left(\frac{x_i}{\sqrt{x_1^2+x_2^2}}\right)^2 + \left(\frac{y_i}{\sqrt{y_1^2+y_2^2}}\right)^2 \\
\frac{2x_iy_i}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leqslant \frac{x_i^2(y_1^2+y_2^2) + y_i^2(x_1^2+x_2^2)}{(x_1^2+x_2^2)(y_1^2+y_2^2)}
\end{aligned}$$

Now, summing the cases of $i = 1$ and $i = 2$, we get;

$$\begin{aligned}
\frac{2x_1y_1}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} + \frac{2x_2y_2}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leqslant \frac{x_1^2(y_1^2+y_2^2) + y_1^2(x_1^2+x_2^2)}{(x_1^2+x_2^2)(y_1^2+y_2^2)} + \frac{x_2^2(y_1^2+y_2^2) + y_2^2(x_1^2+x_2^2)}{(x_1^2+x_2^2)(y_1^2+y_2^2)} \\
\frac{2x_1y_1 + 2x_2y_2}{\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}} &\leqslant \frac{2(x_1^2+x_2^2)(y_1^2+y_2^2)}{(x_1^2+x_2^2)(y_1^2+y_2^2)} \\
x_1y_1 + x_2y_2 &\leqslant \sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}
\end{aligned}$$

We indeed get that the Schwarz Inequality holds true!

(c)

We first see that:

$$\begin{aligned}
(x_1^2+x_2^2)(y_1^2+y_2^2) &= x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2 \\
&= [(x_1y_1)^2 + 2x_1y_1 + (x_2y_2)^2] + [(x_1y_2)^2 - 2(x_1y_2)(x_2y_1) + (x_2y_1)^2] \\
&= (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2
\end{aligned}$$

2 Numbers of Various Sorts

3.

3. If $0 \le k \le n$, the "binomial coefficient" $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{if } k \neq 0, n$$

$\binom{n}{0} = \binom{n}{n} = 1$ (a special case of the first formula if we define $0! = 1$),

and for $k < 0$ or $k > n$ we just define the binomial coefficient to be 0.

(a) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

(The proof does not require an induction argument.)

This relation gives rise to the following configuration, known as "Pascal's triangle"—a number not on one of the sides is the sum of the two numbers above it; the binomial coefficient $\binom{n}{k}$ is the $(k+1)$st number in the $(n+1)$st row.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \cdots & & & & &
\end{array}$$

(b) Notice that all the numbers in Pascal's triangle are natural numbers. Use part (a) to prove by induction that $\binom{n}{k}$ is always a natural number. (Your entire proof by induction will, in a sense, be summed up in a glance by Pascal's triangle.)

(c) Give another proof that $\binom{n}{k}$ is a natural number by showing that $\binom{n}{k}$ is the number of sets of exactly k integers each chosen from $1, \ldots, n$.

(d) Prove the "binomial theorem": If a and b are any numbers and n is a natural number, then

$$
\begin{aligned}
(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \\
&= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.
\end{aligned}
$$

(e) Prove that

(i) $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \cdots + \binom{n}{n} = 2^n.$

(ii) $\displaystyle\sum_{j=0}^{n} (-1)^j \binom{n}{j} = \binom{n}{0} - \binom{n}{1} + \cdots \pm \binom{n}{n} = 0.$

(iii) $\displaystyle\sum_{l \text{ odd}} \binom{n}{l} = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$

(iv) $\displaystyle\sum_{l \text{ even}} \binom{n}{l} = \binom{n}{0} + \binom{n}{2} + \cdots = 2^{n-1}.$

(a)

$$
\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} \\
&= \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} + \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{kn(n-1)\cdots(n-k+2)}{k!} + \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{(k+n-k+1)(n)(n-1)\cdots(n-k+2)}{k!} \\
&= \frac{(n+1)(n)(n-1)\cdots(n-k+2)}{k!} \\
&= \frac{(n+1)!}{(n+1-k)!k!} \\
&= \binom{n+1}{k}
\end{aligned}
$$

Q.E.D. ■

(b)

Notice that $\binom{0}{0} = 1$ is a natural number. In fact this holds for $\binom{n}{0}$ and $\binom{n}{n}$, for all n, because:

$$\binom{n}{0} = \frac{n!}{n!0!} = 1 = \frac{n!}{0!n!} = \binom{n}{n}$$

Suppose that for all k and m, such that $0 \leqslant k \leqslant m < n$, $\binom{m}{k}$ is a natural number.

Consider $\binom{n+1}{k}$; In the case where $k = 0$ or $k = n + 1$, $\binom{n+1}{k} = 1$ trivially. Now look at the other cases where $k < m < n + 1$.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

By our hypothesis, $\binom{n}{k-1}$ and $\binom{n}{k}$ are both natural numbers. So, $\binom{n+1}{k}$, which is a sum of those two natural numbers above, must be a natural number itself too.

Q.E.D. ■

Note to self: This induction hypothesis (IH) is allowed legally (rigorously) because

It is not: you are NOT taking 0 as n of the IH, i.e. this is not saying that since $\binom{0}{0}$ is a natural number, anything below it is and trying to proof that $\binom{n}{k}$ is thus a natural number.

What this induction hypothesis is saying is that since we know $\binom{0}{0}$ is a natural number, if we take $0+1=1$ as the 'new' n, then the our m can only be 0. But note that we cannot apply the formula here since $\binom{1}{0} = \binom{0}{-1} + \binom{0}{0}$ and $\binom{1}{1} = \binom{0}{0} + \binom{0}{1}$. But it doesn't matter since we know that they are both 1. $(\binom{n}{n} = 1 = \binom{n}{0})$

Let us take $n = 2$ instead. Now, we need to note the case where $m = 1$ and $k = 0$ or $k = 1$. Noting this, $\binom{2}{1} = \binom{1}{0} + \binom{1}{1}$. As we know that $\binom{1}{r}$ is just 1, $\binom{2}{1}$ which is a sum of 2 natural numbers, specifically $1+1 = 2$ in this case, therefore it must also be a natural number. By induction, you can just continue this chain.

(c)

For a set N with $|N| = n$, $\binom{n}{k}$ is the number of subsets of N with cardinality k that can be chosen, which is of course a natural number.
(i.e.: The number of subsets that exist with k elements chosen from N)
Equivalently, let the set of all subsets of N with cardinality k be S. Then, $\binom{n}{k} = |S|$, again this must be a natural number.

This is easily verifiable for $n = 0$ and $k = 0$;

The set with cardinality 0 is $\varnothing$, so there is only 1 subset from $\varnothing$ with 0 elements, this being $\varnothing$ itself. $|\{\varnothing\}| = 1$. This agrees with our factorial definition of $\binom{n}{k}$.

Similarly, $\binom{n}{0}$ and $\binom{n}{n}$ are the number of subsets that can be chosen with 0 elements and n elements, respectively, from a set with cardinality n. There exists only one such subset for each of these cases; $\varnothing$ and the set with cardinality n itself respectively.

Now, suppose that for all natural numbers k and m such that $0 \leqslant k \leqslant m \leq n$, $\binom{m}{k}$ is a natural number.
Consider the case of $\binom{n+1}{k}$
In the cases where $\binom{n+1}{0}$ or $\binom{n+1}{n+1}$, they equal 1 by the same argument as above.
In the other cases where $1 \leqslant k \leqslant n$, we simply apply Pascal's Rule;

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Since $\binom{n}{k-1}$ $\binom{n}{k}$ are both natural numbers by our hypothesis, their sum, $\binom{n+1}{k}$ must be natural numbers as well.
Therefore, by induction, for all natural n and k, $\binom{n}{k}$ is a natural number and is the number of subsets with k elements of a set N, where $|N| = n$, that can be chosen.

(d)

The Binomial Theorem trivially works for $n = 0$, as

$$(a+b)^0 = 1$$

$$\sum_{j=0}^{n} \binom{0}{j} a^{0-0} b^0 = a^0 b^0 = 1$$

Suppose The Binomial Theorem works for some n, i.e. $(a+b)^n = \sum\limits_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$, then we will see that it works for $n+1$ too;

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b) \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \\
&= a \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j + b \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \\
&= \sum_{j=0}^{n} \binom{n}{j} a^{n-j+1} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} \\
&= \sum_{j=0}^{n} \binom{n}{j} a^{n-j+1} b^j + \sum_{j=1}^{n+1} \binom{n}{j-1} a^{n-j+1} b^j
\end{aligned}$$

To proceed, let us define $\binom{c}{c+k} = 0$ and $\binom{c}{-k} = 0$ for $c \in \mathbb{N}_0$ and $k \in \mathbb{N}$. This definition makes sense in two ways:
1. $\binom{c}{-k}$ and $\binom{c}{c+k}$ is similar to the number of ways there is to choose $-k < 0$ and $c+k > c$ number of items, respectively, from a set with $c > 0$ items. But there is no such way to choose a negative number of items or a number of items greater than which the set contains. Thus, it makes sense to define it as 0.
2. For any m, we know $\binom{m+1}{0} = 0$. And defining $\binom{c}{-k}$ and $\binom{c}{c+k}$ this way means that Pascal's Formula works even more generally, $\binom{m+1}{0} = \binom{m}{-1} + \binom{m}{0} = 0 + 1 = 1$ and $\binom{m+1}{m+1} = \binom{m}{m} + \binom{m}{m+1} = 1 + 0 = 1$ which is what we want to see.
Therefore, we can continue our proof;

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} a^{n-j+1} b^j + \sum_{j=1}^{n+1} \binom{n}{j-1} a^{n-j+1} b^j \\
&= \sum_{j=0}^{n+1} \binom{n}{j} a^{n-j+1} b^j + \sum_{j=0}^{n+1} \binom{n}{j-1} a^{n-j+1} b^j \\
&= \sum_{j=0}^{n+1} \left[\binom{n}{j} + \binom{n}{j-1} \right] a^{n-j+1} b^j \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} a^{n-j+1} b^j
\end{aligned}$$

So, by induction, we have proven that for any $n \in \mathbb{N}_0$, $a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j+1} b^j$$

Q.E.D. ■

(e)(i)

Trivially, $\sum\limits_{j=0}^{n} \binom{n}{j} = 2^n$ works for $n = 0$;

$$\begin{aligned}
&\sum_{j=0}^{0} \binom{0}{j} \\
=&\binom{0}{0} \\
=&1 \\
=&2^0
\end{aligned}$$

Suppose $\sum\limits_{j=0}^{n} \binom{n}{j} = 2^n$ for some n, then it is also true for $n+1$;

$$\begin{aligned}
&\sum_{j=0}^{n+1} \binom{n+1}{j} \\
=&\sum_{j=0}^{n+1} \left[\binom{n}{j-1} + \binom{n}{j}\right] \\
=&\sum_{j=0}^{n+1} \binom{n}{j-1} + \sum_{j=0}^{n+1} \binom{n}{j}
\end{aligned}$$

$$=\sum_{j=0}^{n} \binom{n}{j} + \sum_{j=0}^{n} \binom{n}{j}$$

(Since $\binom{n}{n+k} = \binom{n}{-k} = 0$ for $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, as justified in (d))

$$\begin{aligned}
&\sum_{j=0}^{n} \binom{n}{j} + \sum_{j=0}^{n} \binom{n}{j} \\
=&2^n + 2^n \\
=&2(2^n) \\
=&2^{n+1}
\end{aligned}$$

So, by induction, $\sum\limits_{j=0}^{n} \binom{n}{j} = 2^n$ for all (natural) values of n.

(e)(ii)

It is easily seen that $\sum_{j=0}^{n}(-1)^j\binom{n}{j}=0$ is true for $n=1$;

$$\sum_{j=0}^{1}(-1)^j\binom{1}{j}=\binom{1}{0}-\binom{1}{1}=1-1=0$$

Suppose $\sum_{j=0}^{m}(-1)^j\binom{n}{j}=0$ holds for all natural numbers m, where
$0\leqslant k\leqslant m<n$,
Now consider $\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}$;

$$\begin{aligned}
&\sum_{j=0}^{n+1}(-1)^j\binom{n}{j}\\
=&\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}\\
=&\sum_{j=0}^{n+1}(-1)^j\left[\binom{n}{j-1}+\binom{n}{j}\right]\\
=&\sum_{j=0}^{n+1}(-1)^j\binom{n}{j-1}+\sum_{j=0}^{n+1}(-1)^j\binom{n}{j}\\
=&\sum_{j=1}^{n+1}(-1)^j\binom{n}{j-1}+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\\
=&\sum_{j=0}^{n}(-1)^j\binom{n}{j}+\sum_{j=0}^{n}(-1)^j\binom{n}{j}
\end{aligned}$$

Since we know that $\sum_{j=0}^{m}(-1)^j\binom{m}{j}$ for $0\leqslant m\leqslant n$:

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^j\binom{n}{j}+\sum_{j=0}^{n}(-1)^j\binom{n}{j}\\
=&0+0\\
=&0
\end{aligned}$$

So, by induction, $\sum_{j=0}^{n}(-1)^j\binom{n}{j}=0$ for all $n\in\mathbb{N}_{\geqslant 1}$

(e)(iii)

We see that $\sum\limits_{\ell \text{ odd}}^{n} \binom{n}{\ell} = 2^{n-1}$ and $\sum\limits_{\ell \text{ even}}^{n} \binom{n}{\ell} = 2^{n-1}$, rather simply in the case of $n = 1$,

$$\sum_{\ell \text{ odd}}^{1} \binom{1}{\ell} = \binom{1}{1} = 2^0 = 2^{1-1}$$

$$\sum_{\ell \text{ even}}^{1} \binom{1}{\ell} = \binom{1}{0} = 2^0 = 2^{1-1}$$

Suppose that $\sum\limits_{\ell \text{ odd}}^{m} \binom{m}{\ell} = 2^{m-1}$ and $\sum\limits_{\ell \text{ even}}^{m} \binom{m}{\ell} = 2^{m-1}$ and are true for all natural numbers m, where $0 \leqslant \ell \leqslant m < n$

Then, it holds true that $\sum\limits_{\ell \text{ odd}}^{n+1} \binom{n+1}{\ell} = 2^{(n+1)-1}$:

$$\begin{aligned}
&\sum_{\ell \text{ odd}}^{n+1} \binom{n+1}{\ell} \\
&= \sum_{\ell \text{ odd}}^{n+1} \left[\binom{n}{\ell-1} + \binom{n}{\ell}\right]
\end{aligned}$$

By our hypothesis, we know that $\sum\limits_{\ell \text{ odd}}^{n} \binom{n}{\ell}$ is the same as 2^{n-1}, and $\sum\limits_{\ell \text{ even}}^{n} \binom{n}{\ell}$ the same as 2^{n-1}.

$$\begin{aligned}
&\sum_{\ell \text{ odd}}^{n+1} \left[\binom{n}{\ell-1} + \binom{n}{\ell}\right] \\
&= \sum_{\ell \text{ odd}}^{n+1} \binom{n}{\ell-1} + 2^{n-1} \\
&= \sum_{\ell \text{ even}}^{n} \binom{n}{\ell} + 2^{n-1} \\
&=2^{n-1} + 2^{n-1} \\
&=2\left(2^{n-1}\right) \\
&=2^n
\end{aligned}$$

Therefore, by induction, $\sum\limits_{\ell \text{ odd}}^{n} \binom{n}{\ell} = 2^{n-1}$ is true for all $n \in \mathbb{N}_{\geqslant 1}$

(e)(iv) Just take (e)(ii)-(e)(iii) lol.

Answer

3.(a) Don't really have to say anything for this, just slap that factorial definition down
(b) Should be ok, about the same
(c)

(c) There are $n(n-1)\cdots(n-k+1)$ k-tuples of distinct integers each chosen from $1,\ldots,n$, since the first can be picked in n ways, the next in $n-1$ ways, etc. Now each set of exactly k integers can be arranged in $k!$ k-tuples, so there are $n(n-1)\cdots(n-k+1)/k! = \binom{n}{k}$ such sets.

(d) Should be good too :)
(e)(i) "lmao just apply the binomial theorem" "$(1+1)^n = \sum_{j=0}^{n}\binom{n}{j}1^{n-j}1^j = \sum_{j=0}^{n}\binom{n}{j}$
(e)(ii) Apply TBT: $(1-1)^n$ e(iii) "lol just take (i)-(ii)" Basically,

$$\begin{aligned}
\sum_{j=0}^{n}\binom{n}{j} - \sum_{j=0}^{n}(-1)^j\binom{n}{j} &= 2^n \\
\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n} - \left[\binom{n}{0}-\binom{n}{1}+\cdots\pm\binom{n}{n}\right] &= 2^n \\
2\sum_{j\ \mathrm{odd}}^{n}\binom{n}{j} &= 2^n \\
\sum_{j\ \mathrm{odd}}^{n}\binom{n}{j} &= 2^{n-1}
\end{aligned}$$

(e)(iv) Yeah just take (i)-(iii) or (i)+(ii)

4. (a) Prove that

$$\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} = \binom{n+m}{l}.$$

Hint: Apply the binomial theorem to $(1+x)^n(1+x)^m$.

(b) Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(a)
First, observe that:

$$(1+x)^{1+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} x^i$$

Now, let's look at an equivalent formulation using the fact that $(1+x)^{n+m} = (1+x)^n(1+x)^m$:

$$\begin{aligned}(1+x)^n(1+x)^m &= \sum_{i=0}^{n} \binom{n}{i} x^i \times \sum_{i=0}^{m} \binom{m}{i} x^i \\ &= \sum_{i=0}^{n+m} \varphi_i x^i\end{aligned}$$

By Lemon 1, convolution, φ_i is:

$$\varphi_i = \sum_{k=0}^{i} \binom{n}{i}\binom{m}{i-k}$$

Therefore, by substituting this back into our second sum above, we get

$$\sum_{i=0}^{n} \varphi_i x^i = \sum_{i=0}^{n+m} \sum_{k=0}^{i} \binom{n}{i}\binom{m}{i-k} x^i$$

So,

$$\sum_{i=0}^{n+m} \binom{n+m}{i} x^i = \sum_{i=0}^{n+m} \sum_{k=0}^{i} \binom{n}{i}\binom{m}{i-k} x^i$$

Since we know that for any polynomials to be equal, they must have identical coefficients for each x^i. Thus, by comparing coefficients;

$$\binom{n+m}{i} = \sum_{k=0}^{i} \binom{n}{i}\binom{m}{i-k}$$

$$\binom{n+m}{\ell} = \sum_{k=0}^{\ell} \binom{n}{i}\binom{m}{\ell-k}$$

You can choose not to use Lemon 1 (directly) by expanding $(1+x)^n$ and $(a+x)^m$ then factorising the powers, using a similar argument as Lemon 1

Q.E.D. ■

(b)

Notice that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

So, we can apply our identity from (a) now;

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

Q.E.D. ■

Answer

(a) The answer basically said "its obvious lmao" But the proof I wrote should be alright.
(b) Yeah its good :D

5.

5. (a) Prove by induction on n that

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$ (if $r = 1$, evaluating the sum certainly presents no problem).

(b) Derive this result by setting $S = 1 + r + \cdots + r^n$, multiplying this equation by r, and solving the two equations for S.

(a)

We easily see that the cases where $n = 0$ holds true;

$$\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{1 - r^{0+1}}{1 - r}$$

Now assume there exists some n such that $\sum\limits_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ holds true. Then, we shall see that it also holds for $n+1$:

$$\begin{aligned}
\sum_{i=0}^{n+1} r^i &= r^{n+1} + \sum_{i=0}^{n} r^i \\
&= r^{n+1} + \frac{1 - r^{n+1}}{1 - r} \text{ (Since it holds for some n)} \\
&= \frac{(1 - r)r^{n+1} + 1 - r^{n+1}}{1 - r} \\
&= \frac{r^{n+1} - r^{n+2} + 1 - r^{n+1}}{1 - r} \\
&= \frac{1 - r^{(n+1)+1}}{1 - r}
\end{aligned}$$

So, by induction, $\sum\limits_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$ holds true for all $n \in \mathbb{N}_0$ and all $r \neq 1$.

Q.E.D. ■

(b)

Let $S = \sum\limits_{i=0}^{n} r^i$ and $Sr = \sum\limits_{i=1}^{n+1} r^i$. Now, we can do a simple substitution:

$$\begin{aligned} Sr = \sum_{i=1}^{n+1} r^i &= \sum_{i=0}^{n} r^i - 1 + r^{n+1} \\ S(r-1) &= -1 + r^{n+1} \\ S(1-r) &= 1 - r^{n+1} \\ S &= \frac{1 - r^{n+1}}{1 - r} \end{aligned}$$

Answer

Its so trivial that there's no answer inside lol

6.

6. The formula for $1^2 + \cdots + n^2$ may be derived as follows. We begin with the formula

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \ldots, n$ and adding, we obtain

$$\begin{aligned}
2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\
3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
&\ \ \vdots \\
(n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \\
\hline
(n+1)^3 - 1 &= 3[1^2 + \cdots + n^2] + 3[1 + \cdots + n] + n.
\end{aligned}$$

Thus we can find $\sum_{k=1}^{n} k^2$ if we already know $\sum_{k=1}^{n} k$ (which could have been found in a similar way). Use this method to find

(i) $1^3 + \cdots + n^3$.
(ii) $1^4 + \cdots + n^4$.
(iii) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)}$.
(iv) $\dfrac{3}{1^2 \cdot 2^2} + \dfrac{5}{2^2 \cdot 3^2} + \cdots + \dfrac{2n+1}{n^2(n+1)^2}$.

(i)

We know that by The Binomial Theorem,

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$$

Notice that when we use $k+1$ instead of k,

$$(k+2)^4 - (k+1)^4 = 4(k+1)^3 + 6(k+1)^2 + 4(k+1) + 1$$

This means that when we sum them up, the $(k+1)^4$ term will be removed:

$$\begin{aligned}
(k+1)^4 - k^4 + (k+2)^4 - (k+1)^4 &= 4k^3 + 6k^2 + 4k + 1 \\
&\quad + (k+1)^3 + 6(k+1)^2 + 4(k+1) + 1 \\
(k+2)^4 - k^4 &= 4[k^3 + (k+1)^3] + 6[k^2 + (k+1)^2]^2 + 4[k + (k+1)] + 2
\end{aligned}$$

So, summing up k from 1 to n;

$$\begin{aligned}
\sum_{k=1}^{n} (k+1)^4 - k^4 &= \sum_{k=1}^{n} 4k^3 + 6k^2 + 4k + 1 \\
(n+1)^4 - 1 &= 4\left[\sum_{k=1}^{n} k^3\right] + 6\left[\sum_{k=1}^{n} k^2\right] + 4\left[\sum_{k=1}^{n} k\right] + n
\end{aligned}$$

Note that -1 comes from the first term (where $k = 1$) since there are no other terms with +1 (on the left hand side), while $(n+1)^4$ comes from the last term where there is no no case where $k = n+1$ in which $-(n+1)^4$ would 'cancel' it away, as n is the upper bound of the summation. (If you haven't noticed, the n on the right hand side is the result of $\sum_{k=0}^{n} 1 = n$.)

Therefore, using the fact that $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum\limits_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$;

$$\begin{aligned}
(n+1)^4 - 1 &= 4\left[\sum_{k=1}^{n} k^3\right] + 6\left[\sum_{k=1}^{n} k^2\right] + 4\left[\sum_{k=1}^{n} k\right] + n \\
(n+1)^4 - 1 &= 4\left[\sum_{k=1}^{n} k^3\right] + 6\left[\frac{n(n+1)(2n+1)}{6}\right] + 4\left[\frac{n(n+1)}{2}\right] + n \\
\sum_{k=1}^{n} k^3 &= \frac{(n+1)^4 - n - 1 - n(n+1)(2n+1) - 2n(n+1)}{4} \\
&= \frac{(n^4 + 4n^3 + 6n^2 + 4n + 1) - n - 1 - (2n^3 + 3n^2 + n) - (2n^2 + 2n)}{4} \\
&= \frac{n^4 + 2n^3 + n^2}{4} \\
&= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

(ii) Just repeat the same procedure :p

(iii)

We see that:

$$\frac{1}{(k+1)^2} = \frac{1}{k} - \frac{1}{k+1}$$

Notice that, again, for $k+1$;

$$\frac{1}{(k+2)^2} = \frac{1}{k+1} - \frac{1}{k+2}$$

Similar to that of in (i) and (ii), when we add these two together, we get:

$$\begin{aligned}
\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} &= \frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+1} - \frac{1}{k+2} \\
&= \frac{1}{k} - \frac{1}{k+2}
\end{aligned}$$

So, for $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$;

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)} &= 1 - \frac{1}{n+1} \\
&= \frac{n}{n+1}
\end{aligned}$$

(iv)

We first see that:

$$\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2}$$

Again, for $k+1$ instead of k,

$$\frac{2(k+1)+1}{(k+1)^2(k+2)^2} = \frac{1}{(k+1)^2} - \frac{1}{(k+2)^2}$$

Similarly to previous parts, when we sum them up,

$$\begin{aligned}\frac{2k+1}{k^2(k+1)^2} + \frac{2(k+1)+1}{(k+1)^2(k+2)^2} &= \frac{1}{k^2} - \frac{1}{(k+1)^2} + \frac{1}{(k+1)^2} - \frac{1}{(k+2)^2} \\ &= \frac{1}{k^2} - \frac{1}{(k+2)^2}\end{aligned}$$

So, now for $\sum\limits_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2}$;

$$\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Answer

6.(i) Should be right (ii) Yeah just repeat lmao
(iii) I guess shld be ok (iv) Yah seems correct

7.

***7.** Use the method of Problem 6 to show that $\sum_{i=1}^{n} k^p$ can always be written in the form

$$\frac{n^{p+1}}{p+1} + An^p + Bn^{p-1} + Cn^{p-2} + \cdots .$$

(The first 10 such expressions are

$$\begin{aligned}
\sum_{k=1}^{n} k &= \tfrac{1}{2}n^2 + \tfrac{1}{2}n \\
\sum_{k=1}^{n} k^2 &= \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n \\
\sum_{k=1}^{n} k^3 &= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 \\
\sum_{k=1}^{n} k^4 &= \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n \\
\sum_{k=1}^{n} k^5 &= \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2 \\
\sum_{k=1}^{n} k^6 &= \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n \\
\sum_{k=1}^{n} k^7 &= \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2 \\
\sum_{k=1}^{n} k^8 &= \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n \\
\sum_{k=1}^{n} k^9 &= \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{3}{20}n^2 \\
\sum_{k=1}^{n} k^{10} &= \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - 1n^7 + 1n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n.
\end{aligned}$$

Notice that the coefficients in the second column are always $\frac{1}{2}$, and that after the third column the powers of n with nonzero coefficients decrease by 2 until n^2 or n is reached. The coefficients in all but the first two columns seem to be rather haphazard, but there actually is some sort of pattern; finding it may be regarded as a super-perspicacity test. See Problem 27-17 for the complete story.)

By The Binomial Theorem,

$$(k+1)^{p+1} - k^{p+1} = \sum_{i=0}^{p} \binom{p+1}{i} k^i$$

We see that when $k+1$ is used instead of k,

$$(k+2)^{p+1} - (k+1)^{p+1} = \sum_{i=0}^{p} \binom{p+1}{i} (k+1)^i$$

Notice that

$$\begin{aligned}
&\cancel{2^{p+1}} - 1^{p+1} + \cancel{3^{p+1}} - \cancel{2^{p+1}} + \cancel{\cdots} + \cancel{(n-1+1)^{p+1}} - \cancel{(n-1)^{p+1}} + (n+1)^{p+1} - \cancel{n^{p+1}} \\
&= \binom{p+1}{1}[1^p + 2^p + \cdots + (n-1)^p + n^p] + \binom{p+1}{2}[1^{p-1} + 2^{p-1} + \cdots + (n-1)^{p-1} + n^{p-1}] + \cdots + \\
&\binom{p+1}{p+1}[1 + 2 + \cdots + (n-1) + n] + n \\
\Rightarrow (n+1)^{p+1} - 1 &= \sum_{i=0}^{p} \left[\binom{p+1}{i} \left(\sum_{k=1}^{n} k^i \right) \right]
\end{aligned}$$

Therefore, using this fact;

$$\begin{aligned}
(n+1)^{p+1} - 1 &= \sum_{i=0}^{p} \left[\binom{p+1}{i} \left(\sum_{k=1}^{n} k^i \right) \right] \\
&= \binom{p+1}{p} \sum_{k=1}^{n} k^p + \sum_{i=0}^{p-1} \left[\binom{p+1}{i} \left(\sum_{k=1}^{n} k^i \right) \right] \\
&= (p+1) \sum_{k=1}^{n} k^p + \sum_{i=0}^{p-1} \left[\binom{p+1}{i} \left(\sum_{k=1}^{n} k^i \right) \right] \\
\Rightarrow \sum_{k=1}^{n} k^p + \frac{1}{p+1} &= \frac{(n+1)^{p+1} - \sum_{i=0}^{p-1} \left[\binom{p+1}{i} \left(\sum_{k=1}^{n} k^i \right) \right]}{p+1} \\
&= \sum_{i=0}^{p+1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} n^{p+1-i} \right] - \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} \left(\sum_{k=1}^{n} k^i \right) \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{p+1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} n^{p+1-i} \right] + \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} n^i \right] \\
&\quad - \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} \left(\sum_{k=1}^{n-1} k^i \right) \right] \\
&= \frac{n^{p+1}}{p+1} + (p+1)n^p + \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{p+1-i} n^i \right] \\
&\quad + \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} n^i \right] - \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} \left(\sum_{k=1}^{n-1} k^i \right) \right] \\
&= \frac{n^{p+1}}{p+1} + (p+1)n^p + \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \left[\binom{p+1}{p+1-i} + \binom{p+1}{p+1-i} \right] n^i \right] \\
&\quad - \sum_{i=0}^{p-1} \left[\left(\frac{1}{p+1} \right) \binom{p+1}{i} \left(\sum_{k=1}^{n-1} k^i \right) \right]
\end{aligned}$$

So,

$$\Rightarrow \sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + (p+1)n^p + \sum_{i=0}^{p-1}\left[\left(\frac{1}{p+1}\right)\left[\binom{p+1}{p+1-i} + \binom{p+1}{p+1-i}\right]n^i\right]$$
$$-\sum_{i=0}^{p-1}\left[\left(\frac{1}{p+1}\right)\binom{p+1}{i}\left(\sum_{k=1}^{n-1} k^i\right)\right] - \frac{1}{p+1}$$

Answer

~~Should be correct even though~~ Nope wrong; Other than the fact that there might have been a careless mistake turning $-$ into $+$, $\sum_{i=0}^{p-1}\left[\left(\frac{1}{p+1}\right)\binom{p+1}{i}\left(\sum_{k=1}^{n-1} k^i\right)\right]$ still involves n and is not fully simplified as necessary. Which is also why the coefficient of n^p is wrong for instance. (Even if I derived a correct formula, I forgot to prove the formula) Answer used induction instead to prove the statement.

7. The proof is by complete induction on p. The statement is true for $p = 1$, since

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + n.$$

Suppose that the statement is true for all natural numbers $\leq p$. The binomial theorem yields the equations

$$(k+1)^{p+1} - k^{p+1} = (p+1)k^p + \text{terms involving lower powers of } k.$$

Adding for $k = 1, \ldots, n$, we obtain

$$\frac{(n+1)^{p+1}}{p+1} = \sum_{k=1}^{n} k^p + \text{terms involving } \sum_{k=1}^{n} k^r \text{ for } r < p.$$

Chapter 2

By assumption, we can write each $\sum_{k=1}^{n} k^r$ as an expression involving powers n^s with $s \leq p$. It follows that

$$\sum_{k=1}^{n} k^p = \frac{(n+1)^{p+1}}{p+1} + \text{terms involving powers of } n \text{ less thatn } p+1.$$

13.

13. (a) Prove that $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$ are irrational. Hint: To treat $\sqrt{3}$, for example, use the fact that every integer is of the form $3n$ or $3n+1$ or $3n+2$. Why doesn't this proof work for $\sqrt{4}$?
(b) Prove that $\sqrt[3]{2}$ and $\sqrt[3]{3}$ are irrational.

(a)

Assume that $\sqrt{3}$ is a rational number, i.e.:

$$\begin{aligned} \sqrt{3} &= \frac{a}{b}, \text{ where } \gcd(a,b) = 1, \text{ and } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1} \\ 3 &= \frac{a^2}{b^2} \\ 3b^2 &= a^2 \end{aligned}$$

This means that 3 is a factor of a^2. Since 3 is a prime number, it cannot be formed by a combination of two other factors of a. Therefore, 3 is a factor of a as well:

$$\begin{aligned} a^2 &= (3k)^2 = 9k^2 \text{ for some } k \in \mathbb{N}_0 \\ \Rightarrow 3b^2 &= 9k^2 \\ b^2 &= 3k^2 \end{aligned}$$

Thus, by the same reasoning as shown above for a, 3 is also a factor of b, meaning $\gcd(a,b) \geqslant 3$. However, this contradicts our assumption that $\sqrt{3}$ is a rational number and can be expressed as $\frac{a}{b}$ such that $\gcd(a,b) = 1$. So, $\sqrt{3}$ must not be rational and is instead irrational.

Note to self, now $\frac{a^2}{b^2} = \frac{9k^2}{3k^2} = \frac{3}{1}$ which supports what we know, that 3 is a prime number. Also, $\frac{a}{b} = \frac{3k}{\sqrt{3}k} = \frac{3\sqrt{3}k}{3k} = \frac{\sqrt{3}}{1}$, again supporting what we know that $\sqrt{3}$ is irrational.

Assume that $\sqrt{5}$ is a rational number, i.e.:

$$\begin{aligned}
\sqrt{5} &= \frac{a}{b}, \text{ where } \gcd(a,b)=1, \text{ and } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1} \\
5 &= \frac{a^2}{b^2} \\
5b^2 &= a^2
\end{aligned}$$

This means that 5 is a factor of a^2. Since 5 is a prime number, it cannot be formed by a combination of other factors of a. Therefore, 5 is a factor of a as well:

$$\begin{aligned}
a^2 &= (5k^2) = 25k^2 \text{ for some } k \in \mathbb{N}_0 \\
\Rightarrow 5b^2 &= 25k^2 \\
b^2 &= 5k^2
\end{aligned}$$

Thus, by the same argument as shown above for a having 5 as its factor, b must also have 5 as a factor in this case, meaning that $\gcd(a,b) \geqslant 5$. However, this contradicts our assumption that that $\sqrt{5}$ is a rational number and as such can be expressed as $\frac{a}{b}$ with $\gcd(a,b)=1$. So, $\sqrt{5}$ must not be rational, i.e. it must be irrational.

Assume that $\sqrt{6}$ is a rational number, i.e.:

$$\begin{aligned}
\sqrt{6} &= \frac{a}{b}, \text{ where } \gcd(a,b)=1, \text{ and } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1} \\
6 &= \frac{a^2}{b^2} \\
6b^2 &= a^2
\end{aligned}$$

The prime factorisation of $6 = (2)(3)$ and 6 is a factor of a. Since a has identical factors with itself and the prime factors of 6 are not repeated (i.e. they are of power 1), a must contain 6 as a factor. (It is clearly impossible that one of the two a's will have a factor of 2 (without having a factor of 3 as well), vice versa)

$$\begin{aligned}
a^2 &= (6k)^2 = 36k^2 \text{ for some } k \in \mathbb{N}_0 \\
\Rightarrow 6b^2 &= 36k^2 \\
b^2 &= 6k^2
\end{aligned}$$

Thus, by the same argument as shown above for the case of 6 being a factor of a, 6 must be a factor of b as well. This means that $\gcd(a,b) \geqslant 6$. However, this contradicts our assumption that $\gcd(a,b)=1$. So, $\sqrt{6}$ must not be rational, i.e. it is irrational.

(b)

Assume that $\sqrt[3]{2}$ is rational, i.e.:

$$\begin{aligned} \sqrt[3]{2} &= \frac{a}{b}, \text{ where } \gcd(a,b)=1, \text{ and } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1} \\ 2 &= \frac{a^3}{b^3} \\ 2b^3 &= a^3 \end{aligned}$$

Since 2 is a prime number, it cannot be formed by a combination of other factors of a (besides 2 itself) such that 2 is a factor of a^3, meaning 2 must be a factor of a as well.

$$\begin{aligned} a^3 &= (2k)^3 = 8k^3, \text{ for some } k \in \mathbb{N}_0 \\ \Rightarrow 2b^3 &= 8k^3 \\ b^3 &= 4k^3 \end{aligned}$$

Thus, 4 must be a factor of b by a similar argument as above for showing 2 is a factor of a, i.e. that:
As there exists no possible combination of factors of b (besides with 4 itself) such that 4 is a factor of b^3
This means that $\gcd(a,b) \geqslant 2$. However, this contradicts our assumption that $\sqrt[3]{2}$ is a rational number, which can be expressed as $\frac{a}{b}$ where $\gcd(a,b)=1$. So, $\sqrt[3]{2}$ must not be rational, i.e. it must be irrational.

Just the same procedure to show $\sqrt[3]{3}$ is irrational.

Answer

Yah seems correct.
14.

14. Prove that

(a) $\sqrt{2}+\sqrt{6}$ is irrational.

(b) $\sqrt{2}+\sqrt{3}$ is irrational.

(a)

Assume that $\sqrt{2}+\sqrt{6}$ is irrational, i.e. that:

$$\begin{aligned}
\sqrt{2}+\sqrt{6} &= \frac{a}{b},\ \text{where } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1} \\
8+2\sqrt{2}\sqrt{6} &= \frac{a^2}{b^2} \\
8+4\sqrt{3} &= \frac{a^2}{b^2} \\
\sqrt{3} &= \frac{a^2}{4b^2} - 2 = \frac{a^2-8b^2}{4b^2}
\end{aligned}$$

We easily see that $a^2, -8b^2, 4b^2$ are all natural numbers, since they are (natural) multiples of natural numbers a, b. Therefore, this would mean that $\sqrt{3}$ is rational as $\frac{a^2-8b^2}{4b^2}$ is a fraction of 2 integers. However, we know that $\sqrt{3}$ is actually irrational.

So, since $\sqrt{2}+\sqrt{6}$ being rational would imply $\sqrt{3}$ is rational, which contradicts the fact that $\sqrt{3}$ is actually irrational, thus $\sqrt{2}+\sqrt{6}$ must be irrational.

(b)

Assume that $\sqrt{2}+\sqrt{3}$ is rational, i.e. that:

$$\begin{aligned}
\sqrt{2}+\sqrt{3} &= \frac{a}{b},\ \text{where } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1} \\
5+2\sqrt{2}\sqrt{3} &= \frac{a^2}{b^2} \\
\sqrt{6} &= \frac{a^2-5b^2}{2b^2}
\end{aligned}$$

We again easily see that $a^2, 5b^2, 2b^2$ are all natural numbers, since they are (natural) multiples of natural numbers a, b. Therefore, this would mean that $\sqrt{6}$ is rational as $\frac{a^2-5b^2}{2b^2}$ is a fraction of 2 integers. However, we know that $\sqrt{6}$ is actually irrational.

So, since $\sqrt{2}+\sqrt{3}$ would imply that $\sqrt{6}$ is rational, which contradicts the fact that $\sqrt{6}$ is actually irrational, thus $\sqrt{2}+\sqrt{3}$ must be irrational.

Answer

Yah seems correct.

15.

15. (a) Prove that if $x = p + \sqrt{q}$ where p and q are rational, and m is a natural number, then $x^m = a + b\sqrt{q}$ for some rational a and b.
(b) Prove also that $(p - \sqrt{q}\,)^m = a - b\sqrt{q}$.

(a)

Let $x = p + \sqrt{q}$ where $p, q \in \mathbb{Q}$ and $m, k \in \mathbb{N}_0$,
Now, for x^m,

$$\begin{aligned} x^m &= (p+\sqrt{q})^m \\ &= \sum_{k=0}^{m} \binom{m}{k} p^{m-k} \sqrt{q}^k \\ &= \sum_{k=0}^{m} \binom{m}{k} p^{m-k} + \sum_{k \text{ even}}^{m} \binom{m}{k} q^{\frac{1}{2}k} + \sum_{k \text{ odd}}^{m} \binom{m}{k} \sqrt{q}^k \\ &= \sum_{k=0}^{m} \binom{m}{k} p^{m-k} + \sum_{k \text{ even}}^{m} \binom{m}{k} q^{\frac{1}{2}k} + \left[\sum_{k \text{ odd}}^{m} \binom{m}{k} \sqrt{q}^{k-1} \right] \sqrt{q} \end{aligned}$$

So, we thus notice 3 facts as $\binom{m}{k}$ is a natural number for all m, k:

1. $m-k$ is an integer, and hence p^{m-k} is rational too. This means that $\binom{m}{k}p^{m-k}$ and $\sum\limits_{k=0}^{m} \binom{m}{k}p^{m-k}$ are rational too.
2. For any even k, $\frac{1}{2}k \in \mathbb{N}_0$, thus, $q^{\frac{1}{2}k}$ and $\sum\limits_{k \text{ even}}^{m} \binom{m}{k}q^{\frac{1}{2}k}$ are rationals.

   (It should be easy to see that $\sum\limits_{k=0}^{m} \binom{m}{k}p^{m-k} + \sum\limits_{k \text{ even}}^{m} \binom{m}{k}q^{\frac{1}{2}k}$ is a sum of 2 rationals and must be rational itself.)
3. Given any odd k, $k-1$ is an even natural number. Therefore, $\sqrt{q}^{k-1} = q^{\frac{k-1}{2}}$ is again q raised to some natural power, meaning $\sqrt{q}^{k-1}$ is rational. Thence, $\sum\limits_{k \text{ odd}}^{m} \binom{m}{k}\sqrt{q}^{k-1}$ is rational too.

Therefore, we have shown that $x^m = a + b\sqrt{q}$, where
$a = \sum\limits_{k=0}^{m} \binom{m}{k}p^{m-k} + \sum\limits_{k \text{ even}}^{m} \binom{m}{k}q^{\frac{1}{2}k}$ and $b = \sum_{k \text{ odd}}^{m} \binom{m}{k}\sqrt{q}^{k-1}$
which are both rational.

Note to self: Btw if you're reading this again don't forget why the odd can't necessarily be changed to even for all k. (Specifically for even m)
E.g.: Let $m = 4$

$$\sum_{k \text{ odd}}^{m} \binom{m}{k} \sqrt{q}^{k-1} = \sum_{k \in \{1,3\}} \binom{4}{k} \sqrt{q}^{k-1} = \binom{4}{0} + \binom{4}{2} q$$
$$\neq \sum_{k \in \{0,2,4\}} \binom{4}{k} \sqrt{q}^{k} = \binom{4}{0} + \binom{4}{2} q + \binom{4}{4} q^2$$

(b) Just repeat (a) lol

Answer

Yeah shld be alright, but another way to do it which is easier is just to use induction

15. (a) The assertion is true for $m = 1$. If it is true for m, then

$$(p + \sqrt{q})^{m+1} = (p + \sqrt{q})(a + b\sqrt{q}) = (ap + bq) + (a + pb)\sqrt{q},$$

and $ap + bq$ and $a + bp$ are rational.

(b) The assertion is true for $m = 1$. If it is true for m, then

$$(p - \sqrt{q})^{m+1} = (p - \sqrt{q})(a - b\sqrt{q}) = (ap + bq) - (a + pb)\sqrt{q},$$

whereas $(p + \sqrt{q})^{m+1} = (ap + bq) + (a + pb)\sqrt{q}$ by part (a).

16.

16. (a) Prove that if m and n are natural numbers and $m^2/n^2 < 2$, then $(m + 2n)^2/(m + n)^2 > 2$; show, moreover, that

$$\frac{(m + 2n)^2}{(m + n)^2} - 2 < 2 - \frac{m^2}{n^2}.$$

(b) Prove the same results with all inequality signs reversed.

(c) Prove that if $m/n < \sqrt{2}$, then there is another rational number m'/n' with $m/n < m'/n' < \sqrt{2}$.

(a) Let $m, n \in \mathbb{N}_{\geqslant 1}$ such that $m^2/n^2 < 2$.
We now observe that since $m, n \geqslant 1$:

$$m + 2n > 3 \qquad m + n > 2$$
$$(m + 2n)^2 > 9 \qquad (m + n)^2 > 4$$

Therefore,

$$\frac{(m + 2n)^2}{(m + n)^2} > 2\frac{1}{4}$$
$$\Rightarrow \frac{(m + 2n)^2}{(m + n)^2} > 2$$

(c)

If $m/n < \sqrt{2}$, then $\sqrt{2} - \frac{m}{n} > 0$. Now, let $p, q \in \mathbb{N}_{\geqslant 1}$ such that $\frac{p}{q} < \sqrt{2} - \frac{m}{n}$. We can now obtain our m' and n' we want;

$$\frac{m}{n} < \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} < \sqrt{2}$$

It is trivial to see that $mq + np, nq \in \mathbb{N}_{\geqslant 1}$ as they are just sums and products of natural numbers. So, our $m' = mq + np$ and our $n' = nq$.

Proof 2:
We first take the average of m/n and $\sqrt{2}$;

$$\frac{1}{2}\left(\frac{m}{n} + \sqrt{2}\right)$$

We obviously know that $\sqrt{2}$ is irrational and so we can replace it with some $\frac{m}{n} < \frac{p}{q} < \sqrt{2}$, where $p, q \in \mathbb{N}_{\geqslant 1}$, so that the m' and n' are natural numbers:

$$\begin{aligned} \frac{m'}{n'} &= \frac{1}{2}\left(\frac{m}{n} + \frac{p}{q}\right) \\ &= \frac{mq + np}{2nq} \end{aligned}$$

Now, we see that:

$$\begin{aligned} \frac{m}{n} &< \frac{p}{q} \\ \frac{m}{n} + \frac{p}{q} &< 2\left(\frac{p}{q}\right) \\ \frac{1}{2}\left(\frac{m}{n} + \frac{p}{q}\right) &< \frac{p}{q} \\ \frac{m'}{n'} &< \frac{p}{q} < \sqrt{2} \end{aligned}$$

$$\begin{aligned} \frac{m}{n} &< \frac{p}{q} \\ 2\left(\frac{m}{n}\right) &< \frac{m}{n} + \frac{p}{q} \\ \frac{m}{n} &< \frac{1}{2}\left(\frac{m}{n} + \frac{p}{q}\right) \\ \frac{m}{n} &< \frac{m'}{n'} \end{aligned}$$

Therefore, this again satisfies our other condition that $\frac{m}{n} < \frac{m'}{n'} = \frac{mq+np}{2nq} < \sqrt{2}$.

Q.E.D. ■

Answer

Should be right

Spivak's Calculus Answer Book:

16. (a) The inequality $(m+2n)^2/(m+n)^2 > 2$ is equivalent to

$$m^2 + 4mn + 4n^2 > 2m^2 + 4mn + 2n^2,$$

or simply $2n^2 > m^2$.

The second inequality is equivalent to

$$n^2[(m+2n)^2 - 2(m+n)^2] < (2n^2 - m^2)(m+n)^2,$$

or

$$n^2(2n^2 - m^2) < (2n^2 - m^2)(n^2 + [2mn + m^2]),$$

or

$$0 < (2n^2 - m^2)(2mn + m^2).$$

(b) Reverse all inequality signs in the solution for part (a).

(c) Let $m_1 = m + 2n$ and $n_1 = m + n$, and then choose

$$\begin{aligned} m' &= m_1 + 2n_1 = 3m + 4n, \\ n' &= m_1 + \ \ n_1 = 2m + 3n. \end{aligned}$$

Since $R(y)$ is a polynomial of degree 3, i.e.: $R(y) = a_1x + a_2x^2 + a_3x^3$ where $a_i \in \mathbb{R}$, and $B(y) = R''(y)$, the degree of $B(y)$ must be 3-2=1. This means that $B(y) = my + k$, for some $m, k \in \mathbb{R}$.

We know 2 points, D and E, on $B(y)$, which we can use to find m and k:

$$\begin{aligned} m &= \frac{10-1}{\left(2-\frac{1}{2}\right)} = 6 \\ k &= 10 - 6(2) = -2 \end{aligned}$$

Therefore, we see that $B(y) = 6y - 2$. We know point A is a stationary point of $P(y)$, i.e. when $P'(y) = 66(y-2) = 0$. Using this, we can find the coordinates of A

$$\begin{aligned} P'(y) = 66(y-2) &= 0 \\ y &= 2 \\ x &= 33(2-2)^2 = 0 \\ \Rightarrow & A(0,2) \end{aligned}$$

$B(y) = R''(y)$, so by integrating $B(y)$ twice and using the fact that point $A(0,2)$ is a stationary point of $R(y)$, i.e.: $R'(2) = 0$ and $R(2) = 0$:

$$\begin{aligned} R'(y) = \int B(y)\,dy &= \int 6y - 2\,dy \\ &= 3y^2 - 2y + c_1 \end{aligned}$$

$$\begin{aligned} R'(2) &= 0 \\ 8 + c_1 &= 0 \\ c_1 &= -8 \end{aligned}$$

$$\Rightarrow R'(y) = 3y^2 - 2y - 8$$

$$\begin{aligned} R(y) = \int R'(y)\,dy &= \int 3y^2 - 2y - 8\,dy \\ &= y^3 - y^2 - 8y + c_2 \end{aligned}$$

$$\begin{aligned} R(2) &= 0 \\ -12 + c_2 &= 0 \\ c_2 &= 12 \end{aligned}$$

$$\Rightarrow R(y) = y^3 - y^2 - 8y + 12$$

To find the coordinates of C, since C is the point at which $R(y) = B(y)$,

$$\begin{aligned} R(y) &= B(y) \\ y^3 - y^2 - 8y + 12 &= 6y - 2 \\ y^3 - y^2 - 14y + 14 &= 0 \end{aligned}$$

Let $\varphi(y) = y^3 - y^2 - 14y + 14$, now notice that;

$$\varphi(1) = 1^3 - 1^2 - 14 + 14 = 0$$

Thus, by The Factor Theorem, $(y-1)$ is a factor of $\varphi(y)$. We continue solving for the coordinates of C,

$$\begin{aligned} y^3 - y^2 - 14y + 14 &= 0 \\ (y-1)(y^2-14) &= 0 \\ y-1 = 0 \text{ OR } y^2 - 14 &= 0 \\ y = 1 \text{ OR } (y+\sqrt{14})(y-\sqrt{14}) &= 0 \\ \text{OR } y &= \pm\sqrt{14} \end{aligned}$$

By the graph given, $y_c > 0$, so $y_c \neq -\sqrt{14}$, and $y_c < P(x_c)$;

$$\begin{aligned} y_c &= \sqrt{14} \\ x_c &= 6\sqrt{14} - 2 \\ P(x_c) &= 33(6\sqrt{14} - 4)^2 = 11233 \\ &\text{(nearest whole no.)} \end{aligned} \qquad \begin{aligned} y_c &= 1 \\ x_c &= 6 - 2 = 4 \\ 33(2)^2 &= 132 \end{aligned}$$

Obviously, $1 < 132$ while $-\sqrt{14} \not< 11233$ (nearest whole no.). We easily see that $C(4,1)$. As for the coordinate of B, we repeat a similar procedure since it is the intersection between $P(y)$ and $B(y)$;

$$\begin{aligned} P(y) &= B(y) \\ 33(y-2)^2 &= 6y - 2 \\ 33y^2 - 138y + 134 &= 0 \\ y &= \frac{-(-138) \pm \sqrt{(-138)^2 - 4(33)(134)}}{2(33)} \\ y &= 2\frac{1}{11} \pm \frac{\sqrt{339}}{33} \end{aligned}$$

Once again, utilising the graph given we see that there are two intersections between $P(y)$ and $B(y)$, and B has a lower value for its y-coordinate compared to the other intersection point. Thus, $y_B = 2\frac{1}{11} - \frac{\sqrt{339}}{33}$ and $x_B = 6\left(2\frac{1}{11} - \frac{\sqrt{339}}{33}\right) - 2 = 10\frac{6}{11} - \frac{2\sqrt{339}}{11}$, $B\left(10\frac{6}{11} - \frac{2\sqrt{339}}{11}, 2\frac{1}{11} - \frac{\sqrt{339}}{33}\right)$ We can finally compute the shaded area: (Goddamn finally after 10 mil-)
These are the coordinates of the points we need: $A(0,2)$, $B\left(10\frac{6}{11} - \frac{2\sqrt{339}}{11}, 2\frac{1}{11} - \frac{\sqrt{339}}{33}\right)$, $C(4,1)$.

$$
\begin{aligned}
&\int_{2\frac{1}{11}-\frac{\sqrt{339}}{33}}^{2} P(y)\,dy + \frac{1}{2}\left(10\frac{6}{11} - \frac{2\sqrt{339}}{11} + 4\right)\left(2\frac{1}{11} - \frac{\sqrt{339}}{33} - 1\right) - \int_{1}^{2} R(y)\,dy \\
=&\int_{2\frac{1}{11}-\frac{\sqrt{339}}{33}}^{2} 33(y-2)^2\,dy + \left(7\frac{3}{11} - \frac{\sqrt{339}}{11}\right)\left(1\frac{1}{11} - \frac{\sqrt{339}}{33}\right) - \int_{1}^{2} y^3 - y^2 - 8y + 12\,dy \\
=&11\left[(y-2)^3\right]_{2\frac{1}{11}-\frac{\sqrt{339}}{33}}^{2} + 8\frac{105}{121} - \frac{116\sqrt{339}}{363} - \left[\frac{y^4}{4} - \frac{y^3}{3} - 4y^2 + 12y\right]_1^2 \\
=&11\left[(2-2)^3 - \left(2\frac{1}{11} - \frac{\sqrt{339}}{33} - 2\right)^3\right] + 8\frac{105}{121} - \frac{116\sqrt{339}}{363} - 1\frac{5}{12} \\
=&11\left[\left(1\frac{1}{11} - \frac{\sqrt{339}}{33}\right)^3\right] + 8\frac{105}{121} - \frac{116\sqrt{339}}{363} - 1\frac{5}{12} \\
=&12 - 1\frac{23}{121}\sqrt{339} + 11\frac{25}{121} + \frac{113\sqrt{339}}{1089} + 8\frac{105}{121} - \frac{116\sqrt{339}}{363} - 1\frac{5}{12} \\
=&30\frac{955}{1452} - \frac{1531\sqrt{339}}{1089}
\end{aligned}
$$

Well im no computer, not gonna stare till this answer becomes right.

9.

9. Prove that if a set A of natural numbers contains n_0 and contains $k+1$ whenever it contains k, then A contains all natural numbers $\geq n_0$.

We know that set A must contain n_0 as stated in the question. But let's assume that set A does not contain all natural numbers greater than n_0.

Let the set of all natural numbers, $n > n_0$, such that $n \notin A$, be S. Then, there must be some smallest natural number, n_s, in S. The more technical construction of S is: Using the Axiom Schema Of Specification,

$$\forall n(n \in S \Leftrightarrow n \in \mathbb{N}_0 \wedge n \notin A)$$

Thereafter, we know that $n_s - 1 \in A$. (n_0 is always in A, so the smallest possible n in S is $n_0 + 1$, in which case this still trivially holds)

We know that the set A contains $k+1$ whenever it contains k, and $n_s - 1$ is contained in A, thus n_s must be contained in A as well.

However, by our construction of the set S, it contains all natural numbers not in A, i.e. they are disjoint sets. So this creates a contradiction because n_0 cannot be in A and S at the same time.

Thence, the set A must contain all natural numbers greater than or equal to n_0.

Answer

Should be correct

11. Prove the principle of complete induction from the ordinary principle of induction. Hint: If A contains 1 and A contains $n+1$ whenever it contains $1, \ldots, n$, consider the set B of all k such that $1, \ldots, k$ are all in A.

11.

The logical statement of the ordinary principle of induction is that:
Let $P(n)$ be some predicate, $k, n \in \mathbb{N}_0$

$$(P(1) \wedge \forall k[P(k) \Rightarrow P(k+1)]) \Leftrightarrow \forall n P(n)$$

While that of the principle of complete induction is:

$$\left(P(1) \wedge \forall k \left[\left(\bigwedge_{i=1}^{k} P(i) \right) \Rightarrow P(k+1) \right] \right) \Leftrightarrow \forall n P(n)$$

We see that;

$$\begin{aligned}(P(1) \wedge \forall k[P(k) \Rightarrow P(k+1)]) \Leftrightarrow & \left(P(1) \wedge \forall k \left[\bigwedge_{i=1}^{k} [P(i) \Rightarrow P(i+1)] \right] \right) \\ & \left(P(1) \wedge \forall k \left[\bigwedge_{i=1}^{k} P(i) \Rightarrow P(k+1) \right] \right)\end{aligned}$$

Therefore since the ordinary principle of induction, which we know to be true, implies the principal of complete induction, this principle of complete induction must be true as well

Answer

11. Clearly 1 is in B. If k is in B, then $1, \ldots, k$ are all in A, so $k+1$ is in A, so $1, \ldots, k+1$ are in A, so $k+1$ is in B. By (ordinary) induction, $B = \mathbf{N}$, so also $A = \mathbf{N}$.

Let A be a set such that $1 \in A$, and $n+1 \in A$ iff $1, 2, \cdots, n \in A$, while B be a set such that $k \in B$ iff $1, 2, \cdots, k \in A$.

Notice that $1 \in B$ trivially since $1 \in A$.

Now suppose there exists some $k \in B$, meaning $1, 2, \cdots, k \in A$. So, we see that $k+1 \in A$. Then, $1, 2, \cdots, k, k+1 \in A$, thus $k+1 \in B$.

By the ordinary principle of induction, $B = \mathbb{N}_{\geqslant 1}$, therefore, $A = \mathbb{N}_{\geqslant 1}$

Or logically, let $n, k \in \mathbb{N}_{\geqslant 1}$, and A, B be sets such that:

$$1 \in A \wedge (n+1 \in A \Leftrightarrow 1, 2, \cdots, n \in A)$$

$$k \in B \Leftrightarrow 1, 2, \cdots, k \in A$$

We see that $1 \in B$ since $1 \in A \Leftrightarrow 1 \in B$.

Now, suppose that there exists some k such that $k \in B$, then $k+1 \in B$ must be true, because:

$$\begin{aligned} k \in B &\Leftrightarrow 1, 2, \cdots, k \in A \\ &\Leftrightarrow k+1 \in A \\ &\Leftrightarrow 1, 2, \cdots, k, k+1 \in A \\ &\Leftrightarrow k+1 \in B \end{aligned}$$

By the ordinary principle of induction, $B = \mathbb{N}_{\geqslant 1}$, so $A = \mathbb{N}_{\geqslant 1}$;

$$\begin{aligned} B = \mathbb{N}_{\geqslant 1} &\Leftrightarrow (x \in \mathbb{N}_{\geqslant 1} \Rightarrow x \in B) \\ &\Leftrightarrow [x \in \mathbb{N}_{\geqslant 1} \Rightarrow (x \in B \Leftrightarrow 1, 2, \cdots, x \in A)] \\ &\Leftrightarrow (x \in \mathbb{N}_{\geqslant 1} \Rightarrow 1, 2, \cdots, x \in A) \\ &\Leftrightarrow (x \in \mathbb{N}_{\geqslant 1} \Rightarrow x \in A) \\ &\Leftrightarrow A = \mathbb{N}_{\geqslant 1} \end{aligned}$$

17.

***17.** It seems likely that $\sqrt{n}$ is irrational whenever the natural number n is not the square of another natural number. Although the method of Problem 13 may actually be used to treat any particular case, it is not clear in advance that it will always work, and a proof for the general case requires some extra information. A natural number p is called a **prime number** if it is impossible to write $p = ab$ for natural numbers a and b unless one of these is p, and the other 1; for convenience we also agree that 1 is *not* a prime number. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19. If $n > 1$ is not a prime, then $n = ab$, with a and b both $< n$; if either a or b is not a prime it can be factored similarly; continuing in this way proves that we can write n as a product of primes. For example, $28 = 4 \cdot 7 = 2 \cdot 2 \cdot 7$.

(a) Turn this argument into a rigorous proof by complete induction. (To be sure, any reasonable mathematician would accept the informal argument, but this is partly because it would be obvious to her how to state it rigorously.)

A fundamental theorem about integers, which we will not prove here, states that this factorization is unique, except for the order of the factors. Thus, for example, 28 can never be written as a product of primes one of which is 3, nor can it be written in a way that involves 2 only once (now you should appreciate why 1 is not allowed as a prime).

(b) Using this fact, prove that $\sqrt{n}$ is irrational unless $n = m^2$ for some natural number m.

(c) Prove more generally that $\sqrt[k]{n}$ is irrational unless $n = m^k$.

(d) No discussion of prime numbers should fail to allude to Euclid's beautiful proof that there are infinitely many of them. Prove that there cannot be only finitely many prime numbers $p_1, p_2, p_3, \ldots, p_n$ by considering $p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$.

(a)

Let's start off by looking at $n = 2$, obviously it has a prime factorisation of just the product of a single number, 2 itself.

Suppose that there exists some natural number $n \geqslant 2$, such that all natural numbers $m \leqslant n$ have prime factorisations. Now consider the case of $n + 1$, and notice that:

1. There is either some m_1 such that m_1 divides $n + 1$, which would mean that $n + 1 = m_1 m_2$. Since we know all $m \leqslant n$ have prime factorisations, this means that $n + 1$ is equal to the product of the prime factorisations of m_1 and m_2, i.e. It is a product of prime numbers. So $n + 1$ must have a prime factorisation as well.
2. Or there exists no m that divides $n + 1$, meaning $n + 1$ is prime, and its prime factorisation is simply itself.

Therefore, $n + 1$ must have a prime factorisation. So, by (complete) induction we have shown that all natural numbers $n \geqslant 2$ have prime factorisations.

(b)

Not using the fact:

Let n be a natural number which cannot be expressed as a square of any natural number, m^2. Assume that $\sqrt{n}$ is rational,

$$\begin{aligned}\sqrt{n} &= \frac{a}{b}, \text{ for some } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1}; \gcd(a,b) = 1\\ n &= \frac{a^2}{b^2}\\ nb^2 &= a^2\end{aligned}$$

Clearly, n is a factor of a^2. Therefore, by our assumption, there exists no natural number m such that $nb^2 = (mb)^2 = a^2$. i.e: No natural number m exists such that $a = mb$, so n must be a factor of a, in order for a to be a natural number such that n to be a factor of a^2 as shown. After we conclude this fact;

$$\begin{aligned}a &= nk \text{ for some } k \in \mathbb{N}_0\\ nb^2 &= (nk)^2\\ nb^2 &= n^2k^2\\ b^2 &= nk^2\end{aligned}$$

By the same argument as above, we see that n must also be a factor of b. Thence, $\gcd(a,b) \geqslant n$. However, this contradicts our assumption that n is rational and thus can be expressed as $\frac{a}{b}$ such that $\gcd(a,b) = 1$. So, for all natural numbers, n, such that it cannot be expressed as a square of any natural number, $\sqrt{n}$ must be irrational.

<u>Using the fact:</u>

Let n be a natural number such that $\sqrt{n}$ be rational, i.e.:

$$\begin{aligned}\sqrt{n} &= \frac{a}{b}, \text{ for some } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1}; \gcd(a,b) = 1\\ n &= \frac{a^2}{b^2}\\ nb^2 &= a^2\end{aligned}$$

Since we know prime factorisation is unique (by the information given), and $nb^2 = a^2$, this means that nb^2 and a^2 must have the same prime factorisation, comprising of exactly the same prime factors of the same powers. Notice that the prime factorisation of a^2 and b^2 must both consist of prime factors all of even powers. Therefore, the prime factorisation of n must also consist of prime factors of power 2, i.e.: there is some natural number $n = m^2$.

(Because since all prime factors of b^2 have even powers, if n had prime factor(s) of odd power(s), nb^2 would have prime factor(s) of odd power(s) which contradicts the fact that all prime factors of a^2 have even powers.)

(Prime factors and powers here refer to that of in their prime factorisations obviously)

Conversely, if there exists some natural number m such that $n = m^2$, $\sqrt{n} = \sqrt{m^2} = m$ which is trivially rational.

Thus, we can conclude that $\sqrt{n}$ is a rational number iff there exists some natural number m such that $n = m^2$, and is irrational otherwise.

(c)

Let n, k be natural numbers such that $\sqrt[k]{n}$ is rational, i.e.:

$$\begin{aligned}\sqrt[k]{n} &= \frac{a}{b}, \text{ for some } a \in \mathbb{N}_0, b \in \mathbb{N}_{\geqslant 1}; \gcd(a,b) = 1 \\ n &= \frac{a^k}{b^k} \\ nb^k &= a^k\end{aligned}$$

We know that prime factorisation is unique and $nb^k = a^k$, this means that nb^k and a^k have the same prime factorisation, comprising of exactly the same prime factors, with the same powers each of which is some factor of k.

Since the prime factors of b^k already have powers which are factors of k, therefore, this must also be true for n (in order for $nb^k = a^k$), which would mean $n = m^k$ for some natural number m.

To see this, consider when n has a prime factor not of factor k, then nb^k would also have some prime factor not being a factor of k. But this would violate what we said above that $nb^k = a^k$ (that a^k, and hence nb^k, must have prime factors of powers which are factors of k)

Conversely, let n, m, k be natural numbers such that $n = m^k$. Rather easily, we see that $\sqrt[k]{n} = \sqrt[k]{m^k} = m$ which is a rational number.

So, we can conclude that $\sqrt[k]{n}$ is rational iff $n = m^k$ for some natural number, m, and is irrational otherwise.

(d) × Good Try still! :)

Assume that there are only n number of primes.

Let the set $I = \{i | i \in \mathbb{N} \land i \leqslant n\}$. Now, taking the product of all primes p_n and adding 1,

$$\left(\prod_{i \in I} p_i\right) + 1$$

Given some arbitrary kth prime number, p_k, where $1 \leqslant k \leqslant n$;

$$\begin{aligned}&\frac{\left(\prod_{i \in I} p_i\right) + 1}{p_k} \\ &= \left(\prod_{i \in I \setminus \{k\}} p_i\right) + \frac{1}{p_k}\end{aligned}$$

$\prod_{i \in I \setminus \{k\}} p_i$ is a natural number, while $\frac{1}{p_k}$ is an noninteger rational number. So, their sum is a noninteger too. Therefore, as our selection of p_k was arbitrary, this means that our $\left(\prod_{i \in I} p_i\right) + 1$ does not have any divisors (other than 1 and itself), meaning it must be prime! However, this contradicts our assumption that there are only a finite n number of primes. Thence, there must be an infinite number of prime numbers!

Answer

(a) Yeah should be good (b) Should be ok (c) Should be good too (d) Wrong rip, note that

$\left(\prod_{i\in I} p_i\right) + 1$ is not necessarily prime. But given a collection of n prime numbers, the $(n+1)$th prime number is less than or equal to $\left(\prod_{i\in I} p_i\right) + 1$.

Good try nonetheless!

17. (a) Suppose that every number $< n$ can be written as a product of primes. If $n > 1$ is not a prime, then $n = ab$ for $a, b < n$. By assumption, a and b are each products of primes, so $n = ab$ is also.

(b) If $\sqrt{n} = a/b$, then $nb^2 = a^2$, so the factorization into primes of nb^2 and of a^2 must be the same. Now every prime appears an even number of times in the factorization of a^2, and of b^2, so the same must be true of the factorization of n. This implies that n is a square.

(c) Repeat the same argument, using the fact that every prime occurs a multiple of k times in a^k and b^k.

(d) If $p_1, \ldots, p_n$ were the only primes, then $(p_1 \cdot p_2 \cdots p_n) + 1$ could not be a prime, since it is larger than all of them (and is not 1), so it must be divisible by a prime. But $p_1, \ldots, p_n$ clearly do not divide it, a contradiction. (Although this is a proof by contradiction, it can be used to obtain some positive information: If $p_1, \ldots, p_n$ are the first n primes, then the $(n+1)^{\text{st}}$ prime is $\leq (p_1 \cdot p_2 \cdots p_n) + 1$. It is not necessarily true, however, that the number $(p_1 \cdot p_2 \cdots p_n) + 1$ is a prime; for example, $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 1 = 30{,}031 = 59 \cdot 509$.)

18. (Rational Root Theorem)

***18.** (a) Prove that if x satisfies

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

for some integers $a_{n-1}, \ldots, a_0$, then x is irrational unless x is an integer. (Why is this a generalization of Problem 17?)

(b) Prove that $\sqrt{6} - \sqrt{2} - \sqrt{3}$ is irrational.

(c) Prove that $\sqrt{2} + \sqrt[3]{2}$ is irrational. Hint: Start by working out the first 6 powers of this number.

(a)

Assume there exists some rational noninteger x such that it satisfies the equation given, i.e. $x = \frac{b}{c}$ for some $b \in \mathbb{Z}\backslash\{0\}$ and $c \in \mathbb{Z}\backslash\{0, 1, -1\}$ such that $b \neq c$ and $\gcd(b, c) = 1$, then:

$$x^n + \sum_{i=0}^{n-1} a_i x^i = 0$$

$$\left(\frac{b}{c}\right)^n + \sum_{i=0}^{n-1} a_i \left(\frac{b}{c}\right)^i = 0$$

$$\frac{b^n}{c^n} + \sum_{i=0}^{n-1} \frac{a_i b^i}{c^i} = 0$$

$$b^n + \sum_{i=0}^{n-1} a_i b^i c^{n-i} = 0$$

$$\begin{aligned} b^n &= -\sum_{i=0}^{n-1} a_i b^i c^{n-i} \\ &= -a_0 c^n - a_1 b c^{n-1} - \cdots - a_{n-1} b^{n-1} c \\ &= c(-a_0 c^{n-1} - a_1 b c^{n-2} - \cdots - a_{n-1} b^{n-1}) \end{aligned}$$

We see that b^n is equal to c times an integer which means $c|b^n$.

($a_0 c^{n-1} + a_1 b c^{n-2} + \cdots + a_{n-1} b^{n-1}$ is an integer as its just a sum of products of integers)

Since we know $\gcd(b, c) = 1$, c is not just b to some natural power, this means that $c|b$ in order for $c|b^n$. However, this leads us to the conclusion that $\gcd(b, c) \geq c > 1$ (as $|c| > 1$), obviously contradicting the assumption that $\gcd(b, c) = 1$.

So, there must not exist any rational noninteger x that satisfies the given equation, i.e.: the x's that satisfy the given equation must be either irrational or an integer.

Equivalently, x is irrational unless x is an integer.

It is a generalisation of 17. as 17. uses an argument of natural numbers only, while we use an argument of all real numbers.

Q.E.D. ■

(b)

Let $x = \sqrt{6} - \sqrt{2} - \sqrt{3}$, we see that;

$$\begin{aligned}
x^2 &= 11 + 2\sqrt{6} - 6\sqrt{2} - 4\sqrt{3} \\
x^2 - 11 &= 2(\sqrt{6} - 3\sqrt{2} - 2\sqrt{3}) \\
(x^2 - 11)^2 = x^4 - 22x^2 + 121 &= 144 - 48\sqrt{2} - 48\sqrt{3} + 48\sqrt{6} \\
x^4 - 22x^2 - 23 &= 48(\sqrt{6} - \sqrt{2} - \sqrt{3}) \\
x^4 - 22x^2 - 48x - 23 = 0
\end{aligned}$$

Since a monic polynomial with the root of $x = \sqrt{6} - \sqrt{2} - \sqrt{3}$ and integer coefficients exists, as well as that $\sqrt{6} - \sqrt{2} - \sqrt{3}$ is obviously not an integer, by 18.(a), we know that $\sqrt{6} - \sqrt{2} - \sqrt{3}$ must certainly be irrational.

(c) Teach me your ways, master. (I couldn't solve this myself rip) Math Discord, Drake: (I phrased this myself but yeah the main crucial parts come frm the disc)

Let $x = \sqrt{2} + \sqrt[3]{2}$. Now we see that:

$$\begin{aligned}
x - \sqrt{2} &= \sqrt[3]{2} \\
(x - \sqrt{2})^3 &= 2 \\
x^3 - 3\sqrt{2}x^2 + 6x + 2\sqrt{2} = 2 \\
x^3 + 6x - 2 &= (3x^2 - 2)\sqrt{2} \\
(x^3 + 6x - 2)^2 &= 2(3x^2 + 2)^2 \\
x^6 + 12x^4 - 4x^3 + 36x^2 - 24x + 4 &= 2(9x^4 + 12x^2 + 4) \\
x^6 + 12x^4 - 4x^3 + 36x^2 - 24x + 4 &= 18x^4 + 24x^2 + 8 \\
x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 = 0
\end{aligned}$$

Therefore, we have found a monic polynomial with integer coefficients which has $\sqrt{2} + \sqrt[3]{2}$ as its root, meaning that by 18.(a), $\sqrt{2} + \sqrt[3]{2}$ must either be an integer or irrational number. Of course, $\sqrt{2} + \sqrt[3]{2}$ is not an integer, so it must be an irrational number.

Answer

Should be correct! :D

19.

19. Prove Bernoulli's inequality: If $h > -1$, then

$$(1+h)^n \geq 1 + nh$$

for any natural number n. Why is this trivial if $h > 0$?

In the cases where $h < 1$, We see easily the inequality holds true for $n = 0$:

$$(1+h)^0 = 1 = 1 + 0h$$
$$(1+h)^0 \geqslant 1 + 0h$$

Suppose that the inequality is true for some $n \in \mathbb{N}_0$, then it holds too for $n+1$;

$$\begin{aligned}
(1+h)^n &\geq 1 + nh \\
(1+h)^{n+1} &\geq (1+nh)(1+h) \\
(1+h)^{n+1} &\geq 1 + nh + h + nh^2 \geqslant 1 + nh + h \text{ as } nh^2 \geq 0 \\
(1+h)^{n+1} &\geq 1 + nh + h + nh^2 = 1 = (n+1)h
\end{aligned}$$

This is trivial in the cases of $h > 0$ as by The Binomial Theorem,

$$(1+h)^n = 1 + nh + \sum_{i=2}^{n} \binom{n}{i} h^i$$

Every term in the sum is greater than or equal to 0,
so $(1+h)^n = 1 + nh + \sum_{i=2}^{n} \binom{n}{i} h^i \geq 1 + nh$.

Therefore, by induction, the inequality $(1+h)^n \geqslant 1 + nh$ is true for all $n \in \mathbb{N}_0$.

Answer

Think it shld be correct, seems ok

20.

20. The Fibonacci sequence $a_1, a_2, a_3, \ldots$ is defined as follows:

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 1, \\ a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3. \end{aligned}$$

This sequence, which begins 1, 1, 2, 3, 5, 8, ... , was discovered by Fibonacci (circa 1175–1250), in connection with a problem about rabbits. Fibonacci assumed that an initial pair of rabbits gave birth to one new pair of rabbits per month, and that after two months each new pair behaved similarly. The number a_n of pairs born in the nth month is $a_{n-1} + a_{n-2}$, because a pair of rabbits is born for each pair born the previous month, and moreover each pair born two months ago now gives birth to another pair. The number of interesting results about this sequence is truly amazing—there is even a Fibonacci Association which publishes a journal, *The Fibonacci Quarterly*. Prove that

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

One way of deriving this astonishing formula is presented in Problem 24-16.

In the case of $n = 1$,$n = 2$,$n = 3$, the formula holds true as:

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{1+\sqrt{5}-\left(1-\sqrt{5}\right)}{2\sqrt{5}} = \frac{2\sqrt{5}}{2} = 1 = a_1$$

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{1+2\sqrt{5}+5-\left(1-2\sqrt{5}+5\right)}{2\sqrt{5}} = \frac{2\sqrt{5}}{2} = 1 = a_2$$

$$\begin{aligned} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3}{\sqrt{5}} &= \frac{\left[\frac{1+3\sqrt{5}+15+5\sqrt{5}-\left(1-3\sqrt{5}+15-5\sqrt{5}\right)}{8}\right]}{\sqrt{5}} \\ &= \frac{3\sqrt{5}+3\sqrt{5}+5\sqrt{5}+5\sqrt{5}}{8\sqrt{5}} \\ &= \frac{16\sqrt{5}}{8\sqrt{5}} \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_3 &= a_2 + a_1 \\ &= 1 + 1 \\ &= 2 \\ &= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3}{\sqrt{5}} \end{aligned}$$

Suppose the statement holds for all k such that $k \leqslant n$ where $n \geqslant 3$ and $k, n \in \mathbb{N}_0 \backslash \{0, 1, 2\}$, then we shall see that it also holds for $n + 1$;

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(1 + \frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(1 + \frac{2}{1-\sqrt{5}}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(1 - \frac{2-2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(1 - \frac{2+2\sqrt{5}}{4}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{2+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{2-2\sqrt{5}}{4}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}\left(\frac{1+1\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}
\end{aligned}$$

Therefore, by complete induction, for all $n \in \mathbb{N}_{\geqslant 1}$, $a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}$

Q.E.D. ■

Answer

Shld be correct

Mr Cheng summation qns:

We see that the series' denominators follows a pattern of 6+6+(6+2)+(6+4)+(6+6)+(6+8)+⋯.
Thence, we can derive our formula for any term, s_n, of the given sequence of real numbers:

Notice that s_n has denominators are made up of n number of sixes and $\sum_{i=1}^{n-2} i$ number of twos. i.e.:
$s_n = \frac{1}{6n+2\sum_{i=1}^{n-2} i}$.

Lemma 1

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof:
We see that it trivially holds for $n = 1$,

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

Now, assume it holds for some n, then it also holds for $n+1$;

$$\begin{aligned}\sum_{i=1}^{n+1} i &= n+1+\sum_{i=1}^{n} i \\ &= \frac{2n+2}{2} + \frac{n(n+1)}{2} \\ &= \frac{2n+2+n^2+n}{2} \\ &= \frac{n^2+3n+2}{2} \\ &= \frac{(n+1)(n+1+1)}{2}\end{aligned}$$

Therefore, by induction, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ holds for all natural n.

Applying Lemma 1 onto our formula for s_n,

$$\begin{aligned}s_n &= \left(6n+2\sum_{i=1}^{n-2} i\right)^{-1} \\ &= \left(6n+2\left[\frac{(n-2)(n-2+1)}{2}\right]\right)^{-1} \\ &= (6n+n^2+3n+2)^{-1} \\ &= (n^2+3n+2)^{-1} \\ &= \frac{1}{n^2+3n+2}\end{aligned}$$

Now, all that is left is to find and proof an explicit formula for $\sum_{i=1}^{m} s_n$:

Observe that we can decompose our $\frac{1}{n^2+3n+2}$ into two partial fractions:

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

If we add s_n with s_{n+1}, some terms cancel out:

$$\begin{aligned}
\frac{1}{(n+1)(n+2)} + \frac{1}{(n+1+1)(n+1+2)} &= \frac{1}{n+2} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} \\
&= \frac{1}{n+1} - \frac{1}{n+3}
\end{aligned}$$

This can be easily extended for $\sum\limits_{i=1}^{m} s_n$ as we see;

$$\begin{aligned}
\sum_{i=1}^{m} s_n &= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{m} - \frac{1}{m+1} + \frac{1}{m+1} - \frac{1}{m+2} \\
&= \frac{1}{2} - \frac{1}{m+2}
\end{aligned}$$

Proof:
This is trivially true in the case of $m = 1$,

$$\sum_{i=1}^{m} \frac{1}{i^2+3i+2} = \frac{1}{1+3+2} = \frac{1}{6} = \frac{1}{2} - \frac{1}{1+2}$$

Now, suppose that this is true for some natural m, then it also is true for $m+1$;

$$\begin{aligned}
\sum_{i=1}^{m+1} \frac{1}{i^2+3i+2} &= \frac{1}{(m+1)^2+3(m+1)+2} + \sum_{i=1}^{m} \frac{1}{i^2+3i+2} \\
&= \frac{1}{m^2+5m+6} + \frac{1}{2} - \frac{1}{m+2} \\
&= \frac{1}{(m+2)(m+3)} + \frac{1}{2} - \frac{m+3}{(m+2)(m+3)} \\
&= \frac{1}{2} - \frac{m+2}{(m+2)(m+3)} \\
&= \frac{1}{2} - \frac{1}{m+1+2}
\end{aligned}$$

Therefore, by induction, $\sum\limits_{i=1}^{m} s_n = \frac{1}{2} - \frac{1}{m+2}$ is true for all natural m.
We can FINALLY apply our last step!

$$\begin{aligned}
\sum_{i=1}^{\infty} s_i &= \lim_{m\to\infty} \sum_{i=1}^{m} s_i \\
&= \lim_{i\to\infty} \frac{1}{2} - \frac{1}{m+2} \\
&= \frac{1}{2} - \lim_{m\to\infty} \frac{1}{m+2} \\
&= \frac{1}{2} - 0 \\
&= \frac{1}{2}
\end{aligned}$$

We can also prove $\lim_{m\to\infty} \frac{1}{m+2} = 0$ using the epsilon-delta definition of the limit.

We need to find a choice for M such that;

$$\forall\varepsilon\exists M\forall m \left(\varepsilon > 0 \land m > M \Rightarrow \left|\frac{1}{m+2} - 0\right| < \varepsilon\right)$$

Note that:

$$\left|\frac{1}{m+2}-0\right|=\frac{1}{m+2}<\varepsilon$$

Our choice of M is rather obvious here, we choose $M=\frac{1}{\varepsilon}$

$$\left|\frac{1}{m+2}\right|<|m+2|<\frac{1}{\varepsilon}$$
$$\left|\frac{1}{m+2}-0\right|<\frac{1}{\varepsilon}$$

So, we have proven that

$$\forall\varepsilon\exists M\forall m\left(\varepsilon>0\wedge m>\frac{1}{\varepsilon}\Rightarrow\left|\frac{1}{m+2}-0\right|<\varepsilon\right)$$

So, we have shown and proven that the sum of this infinite series is $\frac{1}{2}$.

Find max($\mathbf{x^2y}$) if x+y=2022.

Rearranging the above condition, we get $y=-x+2022$.

Substituting it into our $\max(x^2y)$, we get that $\max(x^2y)=\max(-x^3+2022x^2)$.

Now let $\vartheta(x)=-x^3+2022x^2$,

$$\frac{d\vartheta(x)}{dx}=-3x^2+4044x$$

To find the maximum value of $\vartheta(x)$, we take $\frac{d\vartheta(x)}{dx}=0$:

$$\begin{aligned}\frac{d\vartheta(x)}{dx}&=0\\-3x^2+4044x&=0\\x(-3x+4044)&=0\\x=0\text{ OR }x&=1348\end{aligned}$$

Rather trivially, we see that $x=0$ must not be the maximum point:
$\vartheta(0)=0<\vartheta(1348)=1247280896$

$$\begin{aligned}\frac{d^2\vartheta(x)}{dx^2}&=-6x+4044\\\left.\frac{d^2\vartheta(x)}{dx^2}\right|_{x=1348}&=-12132<0\end{aligned}$$

We can now be certain that $\vartheta(1348)$ must be the maximum value of $\vartheta(x)=-x^3+2022x^2=x^2y$.

So, $\max(x^2y)=-(1348)^3+2022(1348)^2=1224728096$.

$\mathbb{C}$

When you are in inline math mode (i.e. $ $) or your expression is very big in display mathmode and you use the big symbols, with \sum_{$i = 0$}^{n} you'll get:

$\sum_{i=0}^{n} f(i)$

But if you add in \limits, i.e. \sum\limits_{$i = 0$}^{n} , you get:

$$\sum\limits_{i=0}^{n} f(i)$$