$\cancel{T = \{n \in \omega \mid \forall k (k \in n \Rightarrow k \in \omega)\}}$

$\cancel{\emptyset \in T}$ ~~vacuously.~~

~~If~~ $\cancel{n \in T}$, ~~then~~ $\cancel{n^+ \in T}$: $\cancel{n^+ = n \cup \{n\}}$

$\cancel{\forall k (k \in n^+ \Leftrightarrow k \in n \vee k = n)}$

For all $n \in \omega$ and $k \in n$, $k \in \omega$.

$T = \{n \in \omega \mid \forall k (k \in n \Rightarrow k \in \omega)\}$

$k \in n \in \omega \Rightarrow k \in \omega$

$\emptyset \in T$ vacuously.

If $n \in T$, then $n^+ \in T$: $n^+ = n \cup \{n\}$

$k \in n^+ \Leftrightarrow k \in n \vee k = n$

$k \in \omega \vee k \in \omega$

$\bigcup(a^+) = a$

$\bigcup(a \cup \{a\}) = a$

$\left(\bigcup a\right) \cup a = a$

$x \in \left(\bigcup a\right) \cup a \Leftrightarrow x \in a$

$\exists y (x \in y \in a) \vee x \in a \Leftrightarrow x \in a$

$c \in b \in \bigcup A \Rightarrow c \in \bigcup A$

$\exists a (c \in b \in a \in A)$

$\exists a (c \in a \in A)$

$c \in \bigcup A$

$c \in b \in \bigcap A \Rightarrow c \in \bigcup A$

$\forall a (c \in b \in a \in A)$

$\forall a (c \in a \in A)$

$c \in \bigcap A$

$c \in b \in \mathcal{P} a \Rightarrow c \in \mathcal{P} a$

Show: $c \in b \in a \Rightarrow c \in a$

$b \in a \Rightarrow b \subseteq a$

$c \in b \in \bigcup a \Rightarrow c \in \bigcup a$

$\exists u (c \in b \in u \in a) \Rightarrow \exists v (c \in v \in a)$

$c \in b \in a$

$c \in \bigcup a$

$d \in c \in b \in a \Rightarrow d \in c \in \bigcup a$

$\Rightarrow d \in c \in a$

$\Rightarrow d \in \bigcup a$

$c \in b \in \mathcal{P} a \Rightarrow c \in b \subseteq a$

$\Rightarrow c \in a$

$\Rightarrow c \subseteq a$

$\Rightarrow c \in \mathcal{P} a$

$d \in c \in b \in \mathcal{P} a \Rightarrow d \in c \in \mathcal{P} a$

$d \in c \in b \subseteq a \Rightarrow d \in c \subseteq a$

$d \in c \in a \Rightarrow d \in a$

$[\forall x (x \notin y \vee y \notin a) \wedge x \notin a] \vee x \in a$

$(a \vee b) \Rightarrow c \quad \neg(a \vee b) \vee c \Leftrightarrow [(\neg a) \wedge (\neg b)] \vee c$

$a \subseteq \mathcal{P} a \Leftrightarrow [(\neg a) \vee c] \wedge [(\neg b) \vee c]$

$c \in b \in a \Rightarrow c \in a$

$c \in b \in a^+ \Rightarrow c \in b \in a \vee c \in b = a$

$\Rightarrow c \in a \subseteq a^+$

Show: $\mathcal{P} a \subseteq \mathcal{P} \mathcal{P} a \Leftrightarrow (b \Rightarrow c)$

$x \in \mathcal{P} a \Rightarrow x \in \mathcal{P} \mathcal{P} a$

$x \subseteq a \Rightarrow x \subseteq \mathcal{P} a$

$\mathcal{P}\{\emptyset, \{\{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\{\{\emptyset\}\}\}\}$

$\mathcal{P}\{\{a\}\} = \{\emptyset, \{\{a\}\}\}$

$c \in b \in a \Rightarrow c \in a$

$c \in b \in \mathcal{P} a \Rightarrow c \in \mathcal{P} a \Rightarrow c \subseteq a$

$c \in b \in \mathcal{P} a$

$c \in b \in \mathcal{P} a \Leftrightarrow c \in b \subseteq a \in \mathcal{P} a$

$\cancel{\Rightarrow c \in a \subseteq \mathcal{P} a}$

$\Rightarrow c \in \mathcal{P} a$

$x \in a \vee x \in c$

$\Rightarrow x \in a$

$x \in a \Rightarrow x \in a \vee$

$\forall x \forall y [(x \notin y \vee y \notin a) \vee x \in a]$

$\forall x \forall y [(x \in y \wedge y \in a) \Rightarrow x \in a]$

$\forall x [\forall y (x \notin y \vee y \notin a) \vee x \in a]$

$$h_1(n) = h_2(n)$$

$$h_1(n^+) = F(h_1(n)) \qquad h_2(n^+) = F(h_2(n))$$

$$h_1(n^+) = h_2(n^+)$$

$$\begin{aligned} h(0) = F(h(-1)) &= h(-1)+1 \\ &= h(-2)+2 \\ &= h(-n)+n = 0 \\ &\qquad h(-n) = -n \end{aligned}$$

$$\text{OR}: h(0) = F(h(-1)) = h(-1) = 0$$

Hello there!

$$\begin{tikzcd} \omega \arrow[r, "s"] \arrow[d, "h"'] & \omega \arrow[d, "h"] \\ A \arrow[r, "F"] & A \end{tikzcd}$$

$$h(n+1) = F(h(n))$$

$$\langle n^+, \alpha \rangle \quad \langle h(n), \alpha \rangle$$

$$\operatorname{dom}(F \circ h) = \omega$$

h surjective

$$h \Rightarrow \quad \langle 0, a \rangle$$

$$\langle 1, F(a) \rangle$$

$$\langle 2, F(F(a)) \rangle \qquad \langle n, \overbrace{F(F(\cdots F(a)))}^{n \text{ times}} \rangle$$

$$T = \{ n \in \omega \mid \overbrace{F(F(\cdots F(a)))}^{n \text{ times}} \vee n = 0 \}$$

$$T = \{ n \in \omega \mid \exists t_1, t_2, \ldots, t_k, t_n (nFt_1 \wedge t_1Ft_2 \wedge \ldots \wedge t_kFt_n) \}$$

$$\sigma(0) = 0$$
$$\sigma(n) =$$

$$\ell : \omega \longrightarrow S$$
$$\sigma : \omega \longrightarrow S$$
$$\sigma(0) = 0$$
$$\sigma(n) = F(\sigma(n-1))$$
$$= \sigma(n-1) + \ell(n)$$

Let the set $S = \{ i \in \omega \mid h(i) \subseteq C^* \}$.

$$x \in h(0) \Rightarrow x \in A \Rightarrow \forall X (A \subseteq X \Rightarrow x \in X)$$
$$\Rightarrow \forall X ((A \subseteq X \subseteq B \land f[\![X]\!] \subseteq X) \Rightarrow x \in X)$$
$$\Rightarrow x \in C^*$$

$T = \{ i \in \omega \mid h(i) \subseteq A \}$

$0 \in \omega$ by definition

If $i \in \omega$, then $i^+ \in \omega$:

Thus, $0 \in S$.

If $i \in S$, then $i^+ \in S$:

$$y \in h(i^+) \Leftrightarrow y \in h(i) \cup f[\![h(i)]\!]$$
$$\Leftrightarrow y \in h(i) \lor \underbrace{y \in f[\![h(i)]\!]}$$
$$\cancel{\Leftrightarrow y \in h(i) \lor \exists x (x \in h(i) \land \langle x, y \rangle \in f)}$$
$$\Rightarrow y \in C^*$$

$f : B \to B$

$h(i) \subseteq C^*$

$\forall X (A \subseteq h(i) \subseteq X \subseteq B \land f[\![X]\!] \subseteq X)$

$\Rightarrow \forall X (A \subseteq h(i) \subseteq X \subseteq B \land f[\![h(i)]\!] \subseteq X)$

$\Rightarrow f[\![h(i)]\!] \subseteq C^*$

$h(i) \subseteq C^*$

$x \in h(i) \Rightarrow x \in C^*$

$\Rightarrow \forall X [(A \subseteq X \subseteq B \land f[\![X]\!] \subseteq X) \Rightarrow x \in X]$

$\forall X [(A \subseteq X \subseteq B \land f[\![X]\!] \subseteq X) \Rightarrow h(i) \subseteq X]$

$\Rightarrow h(i) \subseteq X \land f[\![h(i)]\!] \subseteq X)$

$$\Rightarrow y \in C^*$$

Therefore, the subset S of ω is inductive by definition. By the induction principle for ω, $S = \omega$.

Consequently, $y \in C_* \Leftrightarrow \exists i (i \in \omega \land y \in h(i))$
$\Rightarrow y \in C^*$

So, $C_* \subseteq C^*$. Wherefore, since $C^* \subseteq C_*$ and $C_* \subseteq C^*$, $C^* = C_*$ easily follows.

Clearly, h is a function from ω into $\mathcal{P}B$. We see that $A \subseteq C_* \subseteq B$:

$A \subseteq X$: $\quad x \in A \iff x \in h(0) \implies (0 \in \omega \wedge x \in h(0) = A)$

$\implies x \in \bigcup_{i\in\omega} h(i)$

$X \subseteq B$: $\quad x \in \bigcup_{i \in \omega} h(i) \iff \exists i(i \in \omega \wedge x \in h(i) \in \mathcal{P}B)$

$\implies x \in B$

$$y \in f[\![C_*]\!] = f[\![\underbrace{\bigcup_{i\in\omega} h(i)}_{h(0) \cup h(0) \cup f[\![h(0)]\!] \cup h(1) \cup h(1) \cup f[\![h(1)]\!] \cup \cdots}]\!]$$

$\iff \exists x(x \in \bigcup_{i\in\omega} h(i) \wedge \langle x, y\rangle \in f)$

$\iff \exists i \exists x(i \in \omega \wedge x \in h(i) \wedge \langle x, y\rangle \in f)$

$\implies \exists i(i \in \omega \wedge y \in h(i^+))$ by the construction of h

$\implies y \in C_*$

Hence, $f[\![C_*]\!] \subseteq C_*$. Now, it follows that

$x \in C^* \iff x \in \bigcap\{X \mid A \subseteq X \subseteq B \wedge f[\![X]\!] \subseteq X\}$

$\iff \forall X((A \subseteq X \subseteq B \wedge f[\![X]\!] \subseteq X) \Rightarrow x \in X)$

$\implies x \in C_*$ as $A \subseteq C_* \subseteq B$ and $f[\![C_*]\!]$

which means, we successfully shown that $C^* \subseteq C_*$.

We need only show $C_* \subseteq C^*$ after this.

17. Let the set $S = \{p \in \omega \mid \forall n(n \in \omega \Rightarrow m^{n+p} = m^n \cdot m^p)\}$. We know

$$
\begin{aligned}
m^{n+0} &= m^n && (A1)\\
&= m^n + 0 && (A1)\\
&= 0 + m^n && \text{by Theorem 4K(2)}\\
&= m^n \cdot 0 + m^n && (M1)\\
&= m^n \cdot 0^+ && (M2)\\
&= m^n \cdot 1 &&\\
&= m^n \cdot m^0 && (E1)
\end{aligned}
$$

So, $0 \in S$. Suppose $p \in S$:

$$
\begin{aligned}
m^{n+p^+} &= m^{(n+p)^+} && (A2)\\
&= m^{(p+n)^+} && \text{by Theorem 4K(2)}\\
&= m^{p+n^+} && (A2)\\
&= m^{n^+ + p} && \text{by Theorem 4K(2)}\\
&= m^{(n^+)} + m^p && \text{as } p \in S\\
&= (m^n \cdot m) \cdot m^p && (E2)\\
&= m^n \cdot (m \cdot m^p) && \text{by Theorem 4K(4)}\\
&= m^n \cdot (m^p \cdot m) && \text{by Theorem 4K(5)}\\
&= m^n \cdot m^{(p^+)} && (E2)
\end{aligned}
$$

Therefore, $p^+ \in S$. Now, S is an inductive subset of ω. Consequently, by the Induction Principle for ω, $S = \omega$. Wherefore, for all $m, n, p \in \omega$; $m^{n+p} = m^n \cdot m^p$.

If $n \in S_5$, we show that $n^+ \in S_5$ as well. Let the set $T_5' = \{m \in \omega \mid m \cdot n^+ = n^+ \cdot m\}$. Notice that

$$
\begin{aligned}
0 \cdot n^+ &= 0 \cdot n + 0 && \text{(M2)} \\
&= n \cdot 0 + 0 && \text{as } n \in S_5 \\
&= 0 + 0 && \text{(M1)} \\
&= 0 && \text{(A1)} \\
&= n^+ \cdot 0 && \text{(M1)}
\end{aligned}
$$

As a result, $0 \in T_5'$. Whenever $m \in T_5'$,

$$
\begin{aligned}
m^+ \cdot n^+ &= m^+ \cdot n + m^+ && \text{(M2)} \\
&= n \cdot m^+ + m^+ && \text{because } n \in S_5 \\
&= (n \cdot m + n) + m^+ && \text{(M2)} \\
&= (m \cdot n + n) + m^+ && \text{because } n \in S_5 \\
&= m \cdot n + (n + m^+) && (1) \\
&= m \cdot n + (m + n)^+ && (2) \\
&= (m \cdot n + m) + n^+ && (1) \\
&= m \cdot n^+ + n^+ && \text{(M2)} \\
&= n^+ \cdot m + n^+ && \text{since } m \in T_5' \\
&= n^+ \cdot m^+ && \text{(M2)}
\end{aligned}
$$

Consequently, $m^+ \in T_5'$ and T_5' is an inductive subset of ω. By the Induction Principle for ω, $T_5' = \omega$. i.e. For all $m \in \omega$, $m \cdot n^+ = n^+ \cdot m$. Wherefore, $n^+ \in S_5$ and S_5 is an inductive subset of ω. Once more, by the Induction Principle for ω, $S_5 = \omega$; For all $m, n \in \omega$, $m \cdot n = n \cdot m$.

Exercises

13. Assume $m \neq 0$ and $n \neq 0$. We shall show that $m \cdot n \neq 0$ in such cases. Let $S = \{k \in \omega \mid m \cdot k^+ \neq 0\}$. Since we know

$$
\begin{aligned}
m \cdot 0^+ &= m \cdot 0 + m && \text{(M2)} \\
&= 0 + m && \text{(M1)} \\
&= m + 0 && \text{by Theorem 4K (2)} \\
&= m, && \text{(A1)} \\
&\neq 0 && \text{by assumption}
\end{aligned}
$$

thus $0 \in S$. If $k \in S$, then $k^+ \in S$: First let $T = \{p \in \omega \mid m + p \neq 0\}$. $m + 0 = m \neq 0$ by assumption, so $0 \in T$. When $p \in T$,

$$
\begin{aligned}
m + p^+ &= (m+p)^+ \\
&\neq 0
\end{aligned}
$$

because by Theorem 4D, 0 is not the successor of any natural number. Hence, $p^+ \in T$ and T is an inductive subset of ω. By the Induction Principle for ω, $T = \omega$; meaning the sum of any nonzero natural number with another natural number is always nonzero. As such,

$$
\begin{aligned}
m \cdot k^+ &= m \cdot k + m \\
&\neq 0 && \text{since } m \neq 0 \text{ by assumption}
\end{aligned}
$$

Thence, $k^+ \in S$ and S is an inductive subset of ω. Again, by the Induction Principle for ω, $S = \omega$. Combined with the fact that for all nonzero $n \in \omega$, there exists some $k \in \omega$ such that $n = k^+$, by Theorem 4C; this means that for all nonzero $m, n \in \omega$, $m \cdot n \neq 0$. Taking the contrapositive of this conditional statement, we conclude that for all $m, n \in \omega$; $m \cdot n = 0$ implies $m = 0$ or $n = 0$.

$$
\begin{aligned}
\neg(m = 0 \vee n = 0) &\Rightarrow \neg(m \cdot n = 0) \\
m \cdot n = 0 &\Rightarrow (m = 0 \vee n = 0)
\end{aligned}
$$

The converse is simple; Consider $n = 0$, then $m \cdot 0 = 0$ by (M1). If $m = 0$,

$$
\begin{aligned}
0 \cdot n &= n \cdot 0 && \text{by Theorem 4K(5)} \\
&= 0 && \text{(M1)}
\end{aligned}
$$

Therefore, for all $m, n \in \omega$: If $m = 0$ or $n = 0$, then $m \cdot n = 0$.
Wherefore, we can conclude that $m = 0$ or $n = 0$ iff $m \cdot n = 0$.

Self-Proof of Theorem 4K:

(1) Let $S_1 = \{p \in \omega \mid \forall m \forall n[(m \in \omega \wedge n \in \omega) \Rightarrow m+(n+p) = (m+n)+p]\}$. When $p=0$, $m+(n+0) = m+n = (m+n)+0$ by (A1), meaning $0 \in S_1$.
If $p \in S_1$, $p^+ \in S_1$:

$$\begin{aligned} m+(n+p^+) &= m+(n+p)^+ && (A2) \\ &= [m+(n+p)]^+ && (A2) \\ &= [(m+n)+p]^+ && \text{since } p \in S_1 \\ &= (m+n)+p^+ && (A2) \end{aligned}$$

Thus, S_1 is an inductive subset of ω and by the Induction Principle for ω, $S_1 = \omega$. Which means that for all $m, n, p \in \omega$, $m+(n+p) = (m+n)+p$.

(2) Let $S_2 = \{n \in \omega \mid \forall m(m \in \omega \Rightarrow m+n = n+m)\}$ and the set $T_2 = \{m \in \omega \mid m+0 = 0+m\}$. Immediately, $0+0 = 0+0$ and $0 \in T_2$.
If $m \in T_2$,

$$\begin{aligned} 0+m^+ &= (0+m)^+ && (A2) \\ &= (m+0)^+ && \text{since } m \in T_2 \\ &= m^+ && (A1) \\ &= m^+ + 0 && (A1) \end{aligned}$$

Consequently, $m^+ \in T_2$. By the Induction Principle for ω, $m+0 = 0+m$ for all $m \in \omega$; so $0 \in S_2$. Now assume $n \in S_2$; letting the set $T'_2 = \{m \in \omega \mid m+n^+ = n^+ + m\}$:

$$\begin{aligned} 0+n^+ &= (0+n)^+ && (A2) \\ &= (n+0)^+ && \text{as } n \in S_2 \\ &= n^+ && (A1) \\ &= n^+ + 0 && (A1) \end{aligned}$$

Accordingly, $0 \in T'_2$. Whenever $m \in T'_2$,

$$\begin{aligned} m^+ + n^+ &= (m^+ + n)^+ && (A2) \\ &= (n+m^+)^+ && \text{because } n \in S_2 \\ &= [(n+m)^+]^+ && (A2) \\ &= [(m+n)^+]^+ && \text{since } n \in S_2 \\ &= (m+n^+)^+ && \\ &= (n^+ + m)^+ && \text{as } m \in T'_2 \\ &= n^+ + m^+ && (A2) \end{aligned}$$

As a result, $m^+ \in T_2$. By the Induction Principle for ω, $T_2 = \omega$, and for all $m \in \omega$, $m + n^+ = n^+ + m$. Hence, $n^+ \in S_2$ follows and by the Induction Principle for ω, S_2

we have successfully proven that for all $n, m \in \omega$; $m + n = n + m$.

(3) Let $S_3 = \{p \in \omega \mid \forall m \forall n [m, n \in \omega \implies m\cdot(n+p) = (m\cdot n) + (m\cdot p)]\}$.

when $p = 0$,

$$\begin{aligned} m\cdot(n+0) &= m\cdot n && (A1)\\ &= m\cdot n + 0 && (A1)\\ &= m\cdot n + m\cdot 0 && (M1) \end{aligned}$$

So, $0 \in S_3$.

If $p \in S_3$;

$$\begin{aligned} m\cdot(n+p^+) &= m\cdot(n+p)^+ && (A2)\\ &= [m\cdot(n+p)] + m && (M2)\\ &= (m\cdot n + m\cdot p) + m && \text{Since } p \in S_3\\ &= (m\cdot n) + (m\cdot p + m) && (1)\\ &= m\cdot n + m\cdot p^+ && (M2) \end{aligned}$$

Hence, $p^+ \in S_3$. S_3 is an inductive subset of ω. By the Induction Principle for ω, $S_3 = \omega$. Thence, for all $m, n, p \in \omega$, $m\cdot(n+p) = m\cdot n + m\cdot p$.

(4) Let $S_4 = \{p \in \omega \mid \forall m \forall n [m, n \in \omega \implies m\cdot(n\cdot p) = (m\cdot n)\cdot p\}$. Since $m\cdot(n\cdot 0) = m\cdot 0 = 0 = (m\cdot n)\cdot 0$ by (M1), $0 \in S_4$. Whenever $p \in S_4$, $p^+ \in S_4$

$$\begin{aligned} m\cdot(n\cdot p^+) &= m\cdot[(n\cdot p) + n] && (M2)\\ &= m\cdot(n\cdot p) + m\cdot n && (3)\\ &= (m\cdot n)\cdot p + m\cdot n && \text{as } p \in S_4\\ &= (m\cdot n)\cdot p^+ && (M2) \end{aligned}$$

Therefore, we see that $p^+ \in S_4$ indeed. S_4 is now an inductive subset of ω. By the Induction Principle for ω, $S_4 = \omega$.
In other words: For all $m, n, p \in \omega$; $m\cdot(n\cdot p) = (m\cdot n)\cdot p$.

(5) Let the set $S_5 = \{n \in \omega \mid \forall m (m \in \omega \implies m\cdot n = n\cdot m)\}$ and the set $T_5 = \{m \in \omega \mid m\cdot 0 = 0\cdot m\}$. We see that $0\cdot 0 = 0\cdot 0$, so $0 \in$
When $m \in T_5$,

$$\begin{aligned} 0\cdot m^+ &= (0\cdot m) + 0 && (M2)\\ &= (m\cdot 0) + 0 && \text{Since } m \in T_5\\ &= 0 + 0 && (M1)\\ &= 0 && (A1)\\ &= m^+\cdot 0 && (M1) \end{aligned}$$

Thus, $m^+ \in T_5$; i.e. T_5 is an inductive subset of ω. By the Induction Principle for ω, $T_5 = \omega$. In other words, for all $m \in \omega$, $m\cdot 0 = 0\cdot m$. Acc
$0 \in S_5$.

With this, we can [illegible] the notation [illegible] awkward:

First, let the function $\hat{G}_i = \{\langle\langle m,n\rangle, [G_i(m)](n)\rangle \mid m,n\in\omega\}$. We now show that it must always be a function as we claimed.

Let $\langle m_1, n_1\rangle = \langle m_2, n_2\rangle$. We have previously shown that G_i is a function mapping from ω to ${}^{\omega}\omega$. Thus, $G_i(m_1) = G_i(m_2) \in {}^{\omega}\omega$ is a function; consequently, $[G_i(m_1)](n_1) = [G_i(m_2)](n_2)$. Which means that $\hat{G}_i(m_1,n_1) = \hat{G}_i(m_2,n_2)$. Indeed, we can conclude $\hat{G}_i$ is a function.

Secondly, let $G = \{\langle\langle i,m,n\rangle, \hat{G}_i(m,n)\rangle \mid i,m,n\in\omega\}$.

We first show that, for all $i\in\omega$, the function $\hat{G}_i$ is unique:

Let $T'' = \{i\in\omega \mid \forall G_i \forall G_i' [(G_i:\omega\to{}^{\omega}\omega \wedge G_i':\omega\to{}^{\omega}\omega \wedge G_i \text{ and } G_i' \text{ satisfy conditions } (1)-(4)) \Longrightarrow G_i = G_i']\}$.
In other words, T'' is the set of all natural i so that the function G_i is unique. $G_0(m) = A_m = G_0'(m)$ for all m in their common domain of $\operatorname{dom} G_0 = \operatorname{dom} G_0' = \omega$. Thus, $G_0 = G_0'$, and $0\in T''$.

Assume $i\in T''$. Now let the set $T''' = \{n\in\omega \mid \forall m(m\in\omega \Longrightarrow [G_{i^+}(m)](n) = [G'_{i^+}(m)](n))\}$. By definition, there exists some $c'\in\omega$ for all $m\in\omega$ so $[G_i(m)](c') = [G_i'(m)](c') = m$ since $i\in T''$, and thus $[G_{i^+}(m)](0) = c' = [G'_{i^+}(m)](0)$. Accordingly, $0\in T'''$. If $n\in T'''$, then

…oof of
…iqueness
…s simply
…[illegible]
…of
…istence.

$$
\begin{aligned}
[G_{i^+}(m)](n^+) &= [G_i(m)]([G_{i^+}(m)](n)) \\
&= [G_i'(m)]([G'_{i^+}(m)](n)) \quad \text{since } i\in T'' \text{ and } n\in T''' \\
&= [G'_{i^+}(m)](n^+)
\end{aligned}
$$

Which means that $n^+\in T'''$ and T''' is an inductive subset of ω. By the Induction Principle on ω, $T''' = \omega$. Therefore, for all $n\in\omega$ and $m\in\omega$, $[G_{i^+}(m)](n) = [G'_{i^+}(m)](n)$. We know the functions $G_{i^+}(m)$ and $G'_{i^+}(m)$ both have domain ω, on which they agree on, so $G_{i^+}(m) = G'_{i^+}(m)$. Again, these have the same domain of ω, on which they agree on, thus $G_{i^+} = G_{i^+}'$. As a result, $i^+\in T''$ and T'' is an inductive subset of ω. By the Induction Principle for ω, $T'' = \omega$. i.e. For all $i\in\omega$, the function G_i is unique.

Consequently, whenever $\langle i_1, m_1, n_1 \rangle = \langle i_2, m_2, n_2 \rangle$, $G_{i_1} = G_{i_2}$. These are functions; as such, $G_{i_1}(m_1) = G_{i_2}(m_2)$. Once again, these are functions, in this ca mapping from ω to ω. So, $[G_{i_1}(m_1)](n_1) = [G_{i_2}(m_2)](n_2)$. i.e. $\hat{G}_{i_1}(m_1, n_1) = \hat{G}_{i_2}(m_2, n_2)$. So, G is indeed a function.

Sanity check: Does this all make sense or did we create some random functions?

1. Multiplication:

$$M_m(1) = m \qquad (2)$$

$$M_m(0) = 0 \text{ and } A_m(0) = m \qquad (3)$$

$$\begin{aligned} M_m(n^+) &= M_m(n) + m \\ &= A_m(M_m(n)) \quad \text{Thm 4K(2)} \end{aligned} \qquad (4)$$

Hence, $G_1(m) = M_m$ for all $m \in \omega$.

2. Exponentiation.

$$E_m(1) = m \qquad (2)$$

$$E_m(0) = 1 \text{ and } M_m(1) = m \qquad (3)$$

$$\begin{aligned} E_m(n^+) &= E_m(n) \cdot m \\ &= M_m(E_m(n)) \quad \text{Thm 4K(5)} \end{aligned} \qquad (4)$$

Indeed, $G_2(m) = E_m$ for all $m \in \omega$.

3. Tetration

$$T_m(1) = m \qquad (2)$$

$$T_m(0) = 1 \text{ and } E_m(1) = m \qquad (3)$$

$$T_m(n^+) = E_m(T_m(n)) \qquad (4)$$

We see that $G_3(m) = T_m$ for all $m \in \omega$.

Proof Sketch:

Well, $\lim_{n\to\infty}\left(\underbrace{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}_{n}\right) = 2$ isn't formal notation, but we can formalise it via the Recursion Theorem on ω:

if we wanted to. ~~Either ways, this is besides the point.~~

~~The main thing is that $\underbrace{\sqrt{2}^{\cdot^{\cdot^{\cdot^{\sqrt{2}}}}}}_{n}$ can be rewritten as $2^{\frac{1}{2^n}}$. Thus, with this identity we can easily conclude that~~

$$\cancel{\lim_{n\to\infty}\left(\underbrace{\sqrt{2}^{\cdot^{\cdot^{\sqrt{2}}}}}_{n}\right) = \lim_{n\to\infty}\left(2^{\frac{1}{2n}}\right) = 2^{\lim_{n\to\infty}\left(\frac{1}{2^n}\right)} = 2}$$

There exists a function $h : \omega \to \mathbb{R}^+_0$ so that

$$h(0) = 1$$
$$h(n+1) = \sqrt{2}^{\,h(n)}$$

Eg: $h(1) = \sqrt{2}$
$h(2) = \sqrt{2}^{\sqrt{2}}$

$$G(0) = 1$$
$$G(n+1) = 2^{\frac{1}{2^{G(n)}}}$$

$G(1) = 2^{\frac{1}{2}} = \sqrt{2}$
$G(2) = 2^{\frac{1}{2^{\sqrt{2}}}}$

Generalising 'Arithmetical Functions':

Sketch

Claim: For all $k \in \omega$, there exists a $G_k : \omega \longrightarrow \underbrace{{}^{\omega}\omega}_{\text{Recursion Thm on } \omega}$ so that

$\tilde{G} : \omega \longrightarrow \omega \times (\omega \times \omega)$

$\tilde{G} : k \longmapsto G_k$

$[G_k(m)](n) = g_m(n)$

$g_m(0) =$

$\cancel{G_0 =}\ [G_0(m)](n) = n^+ \qquad G_k \qquad F : B \to B$

$k \in B$

Exists a function $h : \omega \longrightarrow B$ s.th.

$h(0) = k$

$h(n^+) = F(h(n))$

${}^{n^+}m = ({}^{n}m)^m$

${}^{n}m = \overbrace{m^{m^{\cdot^{\cdot^{\cdot^{m}}}}}}^{n \text{ times}} \qquad m^0 = 1$

$[G_0(m)](m) = m$

$\hat{m} = \overbrace{m^{\cdot^{\cdot^{\cdot^{m}}}}}^{n \text{ times}} m = m^{m^{m}} \ n \text{ times}$

$[G_0(m)](n) = A_m(n) \qquad [G_{k^+}(m)](0) = c$ where $[G_k(m)](c) = m$

$[G_{k^+}(m)](n^+) = [G_k(m)]([G_{k^+}(m)](n))$

$\underbrace{[G_{k^+}(m)]}_{h}$

$T_m(0) = 1$

$T_m(n^+) = [T_m(n)]^m$

$\cancel{[G_k(m)](c) = m}$

Exists $c \in \omega$ such that $[G_k(m)](c) = m$

Proof: Let the set $S = \{k \mid G_k \text{ exists}\}$. By definition, we know $0 \in S$ as $[G_0(m)](0) = A_m(0) = m$, meaning condition (i) is satisfi-

If $k \in S$, then $k^+ \in S$:

Notice that for all $m \in \omega$; there indeed exists a function mapping from ω to ω: $G_k(m) \in {}^{\omega}\omega$; and there is some $c \in \omega$ (so that $[G_k(m)](c) = m$) by our assumption that $k \in S$. Hence, all conditions for applying the Recursion Theorem on ω are satisfied; for all $m \in \omega$, there now exists a function $h : \omega \longrightarrow \omega$ where

$$h(0) = c$$
$$h(n^+) = [G_k(m)](h(n))$$

$${}^{1}m = m^m$$

$$h(n) = m$$

$\Longrightarrow c' = m$ OR $[G_k(m)](c') = m$

$c' = c$

$\Longrightarrow h(1) = [G_k(m)](h(0))$

$= [G_k(m)](c)$

$= m$

Thus, this function h satisfies (ii) – (iv). Since for all $m \in \omega$, there exists such a $h_m \in {}^{\omega}\omega$, so we can now construct the function $G_{k^+} = \{\langle m, h_m \rangle \mid m \in \omega\}$.

Prove that for all $i \in \omega$, there exists a unique function $G_i : \omega \longrightarrow {}^{\omega}\omega$ so that

$$[G_0(m)](n) = A_m(n) \quad (1)$$

There exists some $c \in \omega$ such that

$$[G_i(m)](c) = m \quad \text{(for all } m \in \omega) \quad (2)$$

If there exists some $j \in \omega$ with $i = j^+$, then

There exists some $c' \in \omega$ for all $m \in \omega$ so

$$[G_{j^+}(m)](0) = c' \text{ and } [G_j(m)](c') = m \quad (3)$$

$$[G_{j^+}(m)](n^+) = [G_j(m)]\left([G_{j^+}(m)](n)\right) \quad (4)$$

$$\exists c'\left(c' \in \omega \wedge \forall m\left[m \in \omega \Rightarrow \left([G_j(m)](c') = m \wedge [G_{j^+}(m)](0) = c'\right)\right]\right)$$

Proof:

Let the set $S = \{k \mid G_k \text{ exists}\}$. By definition, we know $0 \in S$ because $[G_0(m)](0) = A_m(0) = m$, meaning condition (2) is satisfied. ... Critically G_0 is a function: Let $m_1 = m_2$ and the set $T = \{n \in \omega \mid A_{m_1}(n) = A_{m_2}(n)\}$. Since $A_{m_1}(0) = m_1 = m_2 = A_{m_2}(0)$, $0 \in T$. When $n \in T$,

$$A_{m_1}(n^+) = A_{m_1}(n)^+ = A_{m_2}(n)^+ = A_{m_2}(n^+)$$

which means $n^+ \in T$ and the set T is an inductive subset of ω. By the Induction Principle for ω, $T = \omega$. As $A_{m_1}(n) = A_{m_2}(n)$ for all n in their common domain of ω, $A_{m_1} = A_{m_2}$. Thus, G_0 is indeed a function.

By Theorem 4D, 0 is not the successor of any natural number, so conditions (3) and (4) are not necessary here (the conditional statement above implicating them is immediately true already). Critically G_0 is a function: If $k \in S$, then we will see that $k^+ \in S$ must be true too:

Notice that by our assumption that $k \in S$, for all $m \in \omega$ there indeed exists a function mapping from ω to ω: $G_k(m) \in {}^{\omega}\omega$; and there is also some $c \in \omega$ (so that $[G_k(m)](c) = m$ for all $m \in \omega$). Hence all conditions for applying the Recursion Theorem on ω are satisfied. (For all $m \in \omega$) There now exists a function $h_m : \omega \longrightarrow \omega$ where

$$h_m(0) = c$$
$$h_m(n^+) = [G_k(m)](h(n))$$

We claim that $h_m = G_{k^+}(m)$. Clearly, conditions (3) and (4) are satisfied by definition. Condition (2) is also satisfied as

$$h_m(1) = [G_k(m)](h(0)) = [G_k(m)](c) = m$$

Lastly, $G_{k^+} = \{\langle m, h_m\rangle \mid m \in \omega\}$ is a function: $\Leftarrow$ Let $T' = \{n \in \omega \mid h_{m_1}(n) = h_{m_2}(n)\}$. By definition, $h_{m_1}(0) = c = h_{m_2}(0)$. Whenever $n \in T'$, $n^+ \in T'$ is also true:

$$h_{m_1}(n^+) = [G_k(m_1)](h_{m_1}(n)) = [G_k(m_2)](h_{m_2}(n)) = h_{m_2}(n^+)$$

since G_k is a function as $k \in S$, and $n \in T'$.

As a result, T' is an inductive subset of ω and $T' = \omega$ by the Induction Principle for ω. So, since $h_{m_1}(n) = h_{m_2}(n)$ for all n in their common domain, $h_{m_1} = h_{m_2}$. Accordingly, $G_{k^+}(m_1) = G_{k^+}(m_2)$ and G_{k^+} is a function.

Thence, $h_m = G_{k^+}(m)$ indeed as conditions (2)–(4) are all satisfied. Which means $k^+ \in S$ and S is an inductive subset of ω. By the Induction Principle for ω, $S = \omega$. Therefore, for all $i \in \omega$, such $G_i : \omega \rightarrow {}^{\omega}\omega$ exists.

(Insert proof of uniqueness here)

Now for the converse. Let $S' = \{k \in \omega \mid 2k \in C_*\}$. By definition, $0 \in \omega$ as $2 \cdot 0 = 0 \in \{0,2\} = h(0) \subseteq C_*$. whenever $k \in \omega$, there exists some $n \in \omega$ so that $2 \cdot k \in h(n)$. As such, there also exists the function $G \in \prod_{i \in \omega} h(n)$ (by applying a subset axiom to $\omega \times h(n)$) with

$$G = \{\langle x, y\rangle \mid (x = 1 \implies y = 2 \cdot k) \wedge (x \in \omega \setminus \{1\} \implies y = 2)\}$$

because $2 \in h(n)$ for all n. We shall do a quick proof of this small fact. Let $S'' = \{n \in \omega \mid 2 \in h(n)\}$. Then, $2 \in \{0,2\} = h(0)$ by definition, meaning $0 \in S''$. If $n \in S''$, $n^+ \in S''$ since $h(n) \subseteq h(n^+) = h(n) \cup f[\![\prod_{i \in \omega} h(n)]\!]$. Thus, S'' is an inductive subset of ω. By the <u>Induction Principle for ω</u>, $S'' = \omega$. Returning to the previous part,

$$
\begin{aligned}
f(G) \in f[\![\textstyle\prod_{i \in \omega} h(n)]\!] &\implies f(G) \in h(n^+) \\
&\implies G(1) + G(2) \in C_* \\
&\implies 2 \cdot k + 2 \in C_* \\
&\implies 2 \cdot k^+ \in C_*
\end{aligned}
$$

Consequently, $k^+ \in S'$; S' is an inductive subset of ω. Using the <u>Induction Principle for ω</u>, $S' = \omega$. Which means that S contains all even (natural) numbers. Combined with our previous fact proven, that C_* is the set containing only even numbers, we conclude that C_* is the set of all even numbers.

kept: We know ran F contains only all even naturals as well (by def.). Hence, $F \in \prod_{i \in \omega} C_*$. However, $f(F) = 1001 \notin C_*$, because 1001 is not even. Accordingly, $f[\![\prod_{i \in \omega} C_*]\!] \not\subseteq C_*$. This contradicts our previous claim, derived from our assumption, that $f[\![\prod_{i \in \omega} C_*]\!] \subseteq C_*$. Wherefore, it must be that $C^* \neq C_*$.

$$f(x) = \begin{cases} 1001 \text{ if } x = F \\ \\ x(1) + x(2) \text{ otherwise} \end{cases}$$

where $F = \{\langle n, 2n\rangle \mid n \in \omega\} \in \prod_{i\in\omega} \omega$.

Assume $C^{*} = C_{*}$,

$\vdots$

Keep

Accordingly, by our assumption, $f[\![\prod_{i\in\omega} C_*]\!] \subseteq C_*$ also.

Let $S = \{n \in \omega \mid \forall e\,(e \in h(n) \Rightarrow \exists m\,(m \in \omega \wedge e = 2\cdot m))\}$. In English, S is the set of natural numbers n such that $h(n)$ contains only even (natural) numbers.

$0 \in S$ immediately by definition, because $h(0) = \{0, 2\}$ indeed contains only the even numbers 0 and 2. If $n \in S$, then $n^+ \in S$ too: $e \in h(n^+)$ implies that $e \in h(n)$ or $e \in f[\![\prod_{i\in\omega} h(n)]\!]$. In the first case, $e \in h(n)$ easily satisfies the desired property as $n \in S$. As for the second case:

$$\begin{aligned}
e \in f[\![\textstyle\prod_{i\in\omega} h(n)]\!] &\Rightarrow \exists g\,(g \in \textstyle\prod_{i\in\omega} h(n) \wedge f(g) = e) \\
&\Rightarrow \exists g\,(g : \omega \to \textstyle\bigcup_{i\in\omega} h(n) \wedge f(g) = e) \\
&\Rightarrow \exists g\,(g : \omega \to h(n) \wedge g(1) + g(2) = e) \\
&\Rightarrow \exists g \exists m_1 \exists m_2 \left(\begin{aligned} &g : \omega \to h(n) \wedge m_1 \in \omega \wedge m_2 \in \omega \wedge g(1) = 2\cdot m_1 \\ &\wedge g(2) = 2\cdot m_2 \wedge 2\cdot m_1 + 2\cdot m_2 = e\end{aligned}\right) \quad \text{since } n \in S,\ g(1) \text{ and } g(2) \text{ are even numbers} \\
&\Rightarrow \exists m_1 \exists m_2\,(2\cdot(m_1 + m_2) = e) \\
&\Rightarrow \exists m\,(m \in \omega \wedge e = 2\cdot m)
\end{aligned}$$

Therefore, $n^+ \in S$ as the desired property holds true in both cases. By definition, S is inductive. Using the <u>Induction Principle on ω</u>, $S = \omega$. Which also means that C_* is the set containing only even numbers.

Let $I = \omega$, $A = \{0,2\}$, $B = \omega$, and

$$f(x) = \begin{cases} 1001 & \text{if } x = F \\ x(1) + x(2) & \text{otherwise} \end{cases}$$

where $F = \{\langle n, 2n\rangle \mid n \in \omega\} \in \prod_{i\in\omega} \omega$.

Also let the set $T = \{n \in \omega \mid 2\cdot n \in C^*\}$. $0, 1 \in T$ since for all X; $\{0,2\} \subseteq X$ implies $2\cdot 0 = 0 \in X$ and $2\cdot 1 = 2 \in X$. If $n \in T$, we shall see that $n^+ \in T$ as well. For all X such that $\{0,1\} \subseteq X \subseteq \omega$, there exists a $G \in \prod_{i\in\omega} X$ with

$$G = \{\langle x, y\rangle \mid (x = 1 \Rightarrow y = 2\cdot n) \wedge (x \in \omega\setminus\{1\} \Rightarrow y = 2)\}$$

because $1, n \in T$. Consequently, for all X,

$$\left(\{0,2\} \subseteq X \subseteq \omega \wedge f[\![\textstyle\prod_{i\in\omega} X]\!] \subseteq X\right) \Longrightarrow \forall g\left[g \in \prod_{i\in\omega} X \Rightarrow f(g) \in X\right]$$

$$\Longrightarrow f(G) = 2\cdot n + 2 = 2\cdot n^+ \in X$$

Which means that $2\cdot n^+ \in C^*$. Hence, $n^+ \in T$ and T is an inductive subset of ω.

Just retain the rest and place in the necessary amendments from the corrected Answer 1.

Self-Proof of Lemma 4L

(a)

$$m^+ \in n^+ \Longleftrightarrow m^+ \in n \cup \{n\}$$
$$\Longleftrightarrow (m^+ \in n \vee m^+ = n)$$
$$\Longleftrightarrow$$

$$S = \{n \in \omega \mid \forall m [m \in \omega \Longrightarrow (m \in n \Longleftrightarrow m^+ \in n^+)]\}$$

$$m^+ \in 0^+ \Longrightarrow m^+ \in 1$$
$$\Longrightarrow m^+ = 0$$
$$\Longrightarrow m \in m \cup \{m\} = 0$$

Thus, $0 \in S$.

$m \in n^+$

$n \in \omega$

$$m^+ \in n^{++} \Longrightarrow m^+ \in n^+ \cup \{n^+\}$$
$$\Longrightarrow (m^+ \in n^+ \vee m^+ = n^+)$$
$$\Longrightarrow (m \in m^+ \in n^+ \vee m \in m^+ = n^+)$$
$$\Longrightarrow m \in n^+$$

$$m \in n^+ \Longrightarrow m \in n \cup \{n\}$$
$$\Longrightarrow (m \in n \vee m = n)$$
$$\Longrightarrow (m^+ \in n^+ \vee m \in n \cup \{n\})$$ since $n \in \omega$
$$\Longrightarrow m^+ \in n^+ \cup \{n^+\}$$
$$\Longrightarrow m^+ \in n^{++}$$

So, $n^+ \in \omega$

713767

[illegible] $m \in 0$ implies $m^+ \in 0^+$ is vacuously true (for all natural m) since there exists no such $m \in 0$.
Conversely,

$$\begin{aligned} m^+ \in 0^+ &\Rightarrow m^+ \in 1 \\ &\Rightarrow m^+ = 0 \\ &\Rightarrow m \in m \cup \{m\} = 0 \\ &\Rightarrow m \in 0 \end{aligned}$$

So, $m \in 0$ iff $m^+ \in 0^+$; meaning $0 \in S$. Whenever $n \in S$,

$$\begin{aligned} m \in n^+ &\Rightarrow m \in n \cup \{n\} \\ &\Rightarrow (m \in n \vee m = n) \\ &\Rightarrow (m^+ \in n^+ \vee m^+ = n^+) && \text{since } n \in S \\ &\Rightarrow m^+ \in n^+ \cup \{n^+\} \\ &\Rightarrow m^+ \in n^{++} \end{aligned}$$

In addition,

$$\begin{aligned} m^+ \in n^{++} &\Rightarrow m^+ \in n^+ \cup \{n^+\} \\ &\Rightarrow (m^+ \in n^+ \vee m^+ = n^+) \\ &\Rightarrow (m \in m^+ \in n^+ \vee m \in m^+ = n^+) \\ &\Rightarrow m \in n^+ && \text{by Theorem 4F} \end{aligned}$$

Thus, $n^+ \in S$. Hence, S is an inductive subset of ω. By the Induction Principle for ω, $S = \omega$. Wherefore, for all $m, n \in \omega$: $m \in n$ iff $m^+ \in n^+$.

(b) Let the set $T = \{n \in \omega \mid n \notin n\}$. Since $\varnothing \notin \varnothing$ by definition, $0 \notin 0$ and therefore $0 \in T$. If $n \in T$; Assume $n^+ \in n^+$, then

$$\begin{aligned} n^+ \in n^+ &\Rightarrow n^+ \in n \cup \{n\} \\ &\Rightarrow (n^+ \in n \vee n^+ = n) \\ &\Rightarrow (n \in n^+ \in n \vee n \in n^+ = n) \\ &\Rightarrow n \in n && \text{by Theorem 4F} \end{aligned}$$

However, $n \notin n$ since $n \in T$. Wherefore, by contradiction, $n^+ \notin n^+$. i.e. $n^+ \in T$, meaning T is an inductive subset of ω. By the Induction Principle for ω,
Wherefore, for all $n \in \omega$: $n \notin n$.

Remarks: for part (b) we can actually just use part (a) to say $n \notin n \Leftrightarrow n^+ \notin n^+$ for the inductive step.

For all natural numbers m and n, either $m=n$ or $m\neq n$. In the first case, by Lemma 4L(b) we know $m\notin n$ and $n\notin m$. Thus, trichotomy holds. In the second case, let the set $S=\{n\in\omega \mid \forall m\,(m\in\omega\Rightarrow[n\neq m\Rightarrow(n\in m \vee m\in n)])\}$ and $T=\{m\in\omega \mid m=0 \vee 0\in m\}$. Immediately, $0\in T$. If $m\in T$,

$$m=0 \qquad\qquad 0\in m$$
$$m^+=1 \qquad\qquad 0\in m\cup\{m\}$$
$$0\in m^+ \qquad\qquad 0\in m^+$$

Hence, $m^+\in T$ and T is an inductive subset of ω. By the Induction Principle for ω: $T=\omega$. Thus, for all natural $m\neq 0$, $0\in m$; so, $0\in S$. Whenever $n\in S$,

$$\begin{aligned}
n^+\neq m &\Rightarrow (n=m \vee n\neq m) && \\
&\Rightarrow (m\in n^+ \vee n\in m \vee m\in n) && \text{since } n\in S \\
&\Rightarrow (m\in n^+ \vee n^+\in m^+ \vee m^+\in n^+) && \text{by Lemma 4L(a)} \\
&\Rightarrow (m\in n^+ \vee n^+=m \vee n^+\in m \vee m\in m^+\in n^+) && \\
&\Rightarrow (m\in n^+ \vee n^+\in m \vee m\in m^+\in n^+) && \text{as } n^+\neq m \\
&\Rightarrow (m\in n^+ \vee n^+\in m) && \text{by Theorem 4F}
\end{aligned}$$

Wherefore, $n^+\in T$ and T is an inductive subset of ω. So, $T=\omega$. In other words, For all natural numbers m and n, either $m=n$ or $m \neq n$. In the first case, by Lemma 4L(b) we know $m \notin n$ and $n \notin m$. Thus, trichotomy holds. In the second case, let the set $S = \{n \in \omega \mid \forall m\, [m \in \omega \Rightarrow (n \neq m \Rightarrow [(n \in m \wedge m \notin n) \vee (n \notin m \wedge m \in n)])]\}$ and the set $T = \{m \in \omega \mid (m = 0 \wedge 0 \notin m) \vee (m \neq 0 \wedge 0 \in m)\}$

For all natural numbers m and n, either $m=n$ or $m\neq n$. In the first case, by Lemma 4L(b) we know $m\notin n$ and $n\notin m$. Thus, trichotomy holds. In the second case, let the set $S=\{n\in\omega \mid \forall m\,[m\in\omega \Rightarrow (n\neq m \Rightarrow [(n\in m \wedge m\notin n) \vee (n\notin m \wedge m\in n)])]\}$ and the set $T=\{m\in\omega \mid (m=0 \wedge 0\notin m) \vee (m\neq 0 \wedge 0\in m)\}$

Let $S = \{n \in \omega \mid \forall m (m \leq n \Rightarrow m \in n)\}$ and $T = \{m \in \omega \mid m \leq 0 \Rightarrow m \in 0\}$. $0 \leq 0 \Rightarrow 0 = 0 \Rightarrow 0 \in 0$. Thus, $0 \in T$.
When $m \in T$; $m \in m^+$ but $m \notin 0$. Thus, $m^+ \notin 0$, which means that the conditional statement is true immediately (that $m^+ \leq 0 \Rightarrow m \in 0$) and hence $m^+ \in T$.

By the Induction Principle for ω, $T = \omega$. $\rightarrow 0 \in S$.

or we could just have said that the only subset of 0 is itself. Therefore, since $0 \leq 0$ implies $0 = 0$ and thus $0 \in 0$, we know $0 \in S$.
Now assume $n \in S$.

$m \in n^+ \Rightarrow m \in n^+$ Know: $m \leq n \Rightarrow m \in n$

$$m \leq n^+ \Rightarrow \forall k (k \in m \Rightarrow k \in n^+)$$
$$\Rightarrow \forall k (k \in m \Rightarrow k \in n)$$

$$\neg(k \in m) \vee (k \in n \vee k = n)$$

$$\forall k [(k \in m \Rightarrow k \in n) \vee k \in n]$$

$\ldots \in k \in n \in m$

~~Let $m \in A$. Then by our assumption, there exists some $n \in A$ with $n \in m$.~~
~~Let $m \in A$ and the set $S = \{$~~
Assume there exists some nonempty subset of ω and for all $m \in A$ there exists some $n \in A$ with $n \in m$.

$\ldots \in k \in n \in m$

$n \in m$
$k \in \ldots$

$$\{n \in \omega \mid \forall m (n \in n + m)\}$$
$$\{n \in \omega \mid$$

$f(n^{++}) \in f(n^{+}) \in f(n)$

$f(n^{++}) \in f(n)$

$f(n+m) \in f(n)$

For all natural m, $f(m) \in f(0)$ $f(m) \subset f(0)$

$f(n) = f(m)$

$n = m$

exists ... ~~with~~ ~~...~~ for every n in ω, if every number less than n is in A, then $n \in A$.

Let the set $S = \{n \in \omega \mid f_1(n) = f_2(n)\}$.

If every number less than n is in A: then either $n=0$ or $n \neq 0$.

when $n = 0$:

$$f_1(0) = G(f_1 \restriction 0) = G(\emptyset) = G(f_2 \restriction 0)$$

$\implies n \in S$.

+ info needed: $f_1 \restriction 0 = f_2 \restriction 0$

$\{\langle x,y\rangle \mid x \in 0 \wedge y = f_1(x)\} = \{\langle x,y\rangle \mid x \in 0 \wedge y \in f_2(x)\}$

$\emptyset = \emptyset$ ✓

$\forall x \forall y \left([x \in 0 \wedge y = f_1(x)] \iff [x \in 0 \wedge y = f_2(x)]\right)$

$z \in f_1 \restriction 0 \iff \exists x \exists y$

$z \in f_1 \restriction 0 \iff [z \in f \wedge \underbrace{\exists x \exists y (x \in 0 \wedge z = \langle x,y\rangle)}_{\text{This is false, so } z \in f \restriction 0 \text{ is false}}]$

~~When $n=0$~~,

If every number less than n is in A, then for all $k \in n$. $f_1(k) = f_2(k)$.

Hence, $f_1 \restriction n = \{\langle k, f_1(k)\rangle \mid k \in n\}$
$= \{\langle k, f_2(k)\rangle \mid k \in n\}$
$= f_2 \restriction n$

$f_1 \restriction \tilde{k}^+ = \{\langle x,y\rangle \mid x \in \tilde{k}^+ \wedge y = f_1(x)\}$
$= \{\langle x,y\rangle \mid x \in k \wedge y = f_1(x)\}$
$= \{\langle x,y\rangle \mid x = \tilde{k} \wedge y = f_1(x)\} \cup \{\langle x,y\rangle \mid x \in \tilde{k} \wedge y = f_1(x)\}$
$= (f_1 \restriction \{\tilde{k}\}) \cup (f_1 \restriction \tilde{k})$

$f_1(\tilde{k})$

$\{\langle \tilde{k}, G(f_1 \restriction \tilde{k})\rangle\}$

Show: $f_1 \restriction n = f_2 \restriction n$

$k \in n \implies f_1(k) = f_2(k)$

$f_1(\tilde{k}) = f_2(\tilde{k})$

Since for all $k \in n$, $k \in A$, this means that
$f_1(k) = f_2(k)$
$G(f_1 \restriction k) = G(f_2 \restriction k)$

$f_1 \restriction n = \{\langle k, G(f_1 \restriction k)\rangle \mid k \in n\}$
$= \{\langle k, G(f_2 \restriction k)\rangle \mid k \in n\}$
$= f_2 \restriction n$

27. G is a function, $f_1 : \omega \to A$, $f_2 : \omega \to A$.

$$\forall n \left(n \in \omega \Rightarrow \left[\underbrace{(f_1 \restriction n)}_{\text{subset of } \omega \times A} \in \operatorname{dom} G \wedge (f_2 \restriction n) \in \operatorname{dom} G \right] \right)$$

$$f_1(n) = G(f_1 \restriction n) \text{ and } f_2(n) = G(f_2 \restriction n)$$

Infer that ω and $\operatorname{ran} G \subseteq A$

$(\operatorname{dom} G) \cap \mathcal{P}(\omega \times A) \neq \emptyset$

For all G, there exists a function functions such that $f : \omega \to A$ with $(f_1 \restriction n) \in \operatorname{dom} G$ and $f_1(n) = G(f_1 \restriction n)$,

f_1 is unique.

$f_1(n) = \{ \langle n, G(f_1 \restriction n) \rangle \mid n \in \omega \}$.

Show that $f_1 = f_2$.

$\Rightarrow$ Prove that for all $n \in \omega$, $f_1(n) = f_2(n)$

Assume that $f_1 \neq f_2$, i.e. there exists some $n \in \omega$ with $f_1(k) \neq f_2(k)$.
Let the set of all such n be S.
By the Well Ordering of ω, there exists some least element m of S.

$$f_1 \restriction m \neq f_2 \restriction m$$

20

$\operatorname{ran} f \cup \bigcup \operatorname{ran} f$ is the set of all natural numbers smaller than or equal to some element of $\operatorname{ran} f$.
By the Well Ordering of ω, there exists some least element m of $\omega \setminus \operatorname{ran} f \cup \bigcup \operatorname{ran} f$

$$k \in \operatorname{ran} f \cup \bigcup \operatorname{ran} f$$
$$k \in \operatorname{ran} f \vee \exists \hat{k} (k \in \hat{k} \in \operatorname{ran} f)$$
$$m \in k \in \operatorname{ran} f \vee \exists \hat{k} (m \in k \in \hat{k} \in \operatorname{ran} f)$$
$$m \in \bigcup \operatorname{ran} f \vee \exists \hat{k} (m \in \hat{k} \in \operatorname{ran} f)$$
$$m \in \bigcup \operatorname{ran} f$$

$$\omega \setminus \left(\operatorname{ran} f \cup \bigcup \operatorname{ran} f\right)$$

least element m

claim: For all $k \in \operatorname{ran} f$, $m \notin k$, lest $m \in k \in \operatorname{ran} f$ and thus $m \in \bigcup \operatorname{ran} f$.

We also know $\operatorname{ran} f$, and hence $\operatorname{ran} f \cup \bigcup \operatorname{ran} f$ too, is nonempty; because ...

$$n^+ \neq 0$$
$$\forall k \left[k \in \underset{\neq \emptyset}{n^+} \Rightarrow \exists k (k \in \operatorname{ran} f) \right]$$

There exists $\tilde{n} \in \operatorname{ran} f$ with $\tilde{n}^+ = m$.

Since $\operatorname{ran} f \neq \emptyset$, ... $m \neq 0$.
There exists $\tilde{n} \in \omega$ with $\tilde{n}^+ = m$. In fact, more specifically, $\tilde{n} \in \operatorname{ran} f$ $\cancel{\bigcup \operatorname{ran} f}$.

$$\exists k (n \in k \in \operatorname{ran} f) \Rightarrow n \in \operatorname{ran} f$$

Claim: $\tilde{n}$ is the largest element of $\operatorname{ran} f$, i.e. for all $k \in \operatorname{ran} f$, $k \subseteq \tilde{n}$.

$k \in \tilde{n}^+$

Proof:

$$\forall k (k \in \operatorname{ran} f \Rightarrow k \subseteq \tilde{n})$$

Assume otherwise: $\exists k (k \in \operatorname{ran} f \wedge \hat{n} \in k)$

$\Rightarrow$

$\tilde{n} \in \bigcup \operatorname{ran} f$

$\exists k (\tilde{n} \in k \in \operatorname{ran} f)$

$k \notin \tilde{n}^+$

$\tilde{n}^+ \subseteq k$

$\tilde{n}^+ \in k$

$m \in k$

Cont

Assume that the Strong Induction Principle for ω is false, i.e. there exists some $A \subseteq \omega$ ~~with~~ ~~$0 \in A$ and~~ so that for every n in ω, if every number less than n is in A, then $n \in A$. But suppose that however, $A \neq \omega$.

$$\forall k \forall n ([k, n \in \omega \wedge (k \in n \Rightarrow k \in A)] \Rightarrow n \in A)$$

By the well ordering of ω, we know that there exists some (natural) least element m of $\omega \setminus A$. Thus, for all natural $k \in m$, $k \notin \omega \setminus A$, lest there exist some $k \in \omega \setminus A$ with $k \in m$ which would mean m is not the least element of $\omega \setminus A$. So, for all natural $k \in m$, $k \in A$. However, by the Strong Induction Hypothesis — for every n in ω, if every number less than n is in A, then $n \in A$ — we know that $m \in A$. This contradicts our previous fact that $m \in \omega \setminus A$.

$$0 \in \ldots \in \overbrace{m \in \ldots}^{A} \in \ldots$$

$$S = \{m \in \omega \mid m \in A\}$$

$0 \in A$ by def.

$\forall k \; k \in \omega \setminus A \Rightarrow k \notin m$

$\exists k \; k \in \omega \setminus A \wedge k \in m$

If $m \in \omega$,

$$k \in \omega \wedge k \in m \Rightarrow k \in \omega \wedge k \notin \omega \setminus A$$
$$\Rightarrow k \in A$$

$m \in A$. Cont..

20.

$\bigcup A = A$

$y \in \bigcup A \Leftrightarrow y \in A$

$\exists x(x \in A \wedge y \in x) \Leftrightarrow y \in A$

$\exists x(y \in x \in A) \Rightarrow y \in A$

$\Rightarrow \forall x[(y \in x \in A) \Rightarrow y \in A]$

$\neg(\exists x(y \in x \in A)) \vee y \in A$

$(\forall x[\neg(y \in x \in A)]) \vee y \in A$

$\forall x[\neg(y \in x \in A) \vee y \in A]$

$y \in$

$[y \in A \Rightarrow \exists x(y \in x \in A)] \Rightarrow$

Assume there exists some nonempty subset A of ω so that $\bigcup A = A$ but $A \neq \omega$. By the Well Ordering of ω, there exists some least element m of $\omega \setminus A$.

$\forall n \forall x \; n, m, x \in \omega \wedge n^+ \in m \wedge n \in x \Rightarrow m \in x$

$n^+ \in m^+ \wedge n^+ \in x \Rightarrow n^+ \in m \wedge n \in x$

$\Rightarrow (n^+ \in m \vee n^+ = m) \wedge n \in x$

$\Rightarrow (n^+ \in m \wedge n \in x) \vee (n^+ = m \wedge n \in x)$

$\Rightarrow m \in x \quad \vee$

$\forall n(n \in \omega \setminus A \Rightarrow m \in n)$

$0 \cup 1 \cup 2 \cup 3 \cup \cdots = 0 \cup \{0\} \cup \{0,1\} \cup \{0,1,2\} \cup \cdots$

$\Rightarrow$

If there exists some $k \in A$ with $m \in k$

$\exists k(m \in k \in A) \Leftrightarrow m \in \bigcup A$

$\Leftrightarrow m \in A$

$\in \bigcup A$ $\quad \omega \setminus A$

A

If there does not exist $k \in A$ with $m \in k$

(for all $k \in A$, $m \notin k$)

$m \in k \Rightarrow k \in \omega \setminus A$

$\forall k(k \in A \Rightarrow m \notin k) \Leftrightarrow \forall k(m \in k \Rightarrow k \notin A)$

$\Rightarrow \forall k(m \in k \wedge k \in \omega \Rightarrow k \in \omega \setminus A$

$m \neq 0$

There exists some n with $n^+ = m$

$(n \in A)$

$\neg(\hat{k} \in n) \quad \neg(\hat{k} \in n^+)$

$n \in x \Rightarrow m \in x$

$n \; x \in \omega$

$n \in x^+$

$\forall k(n \notin k \vee k \notin A)$

$\forall \hat{k}(\hat{k} \in A \Rightarrow n \notin \hat{k})$

$\forall \hat{k}(\hat{k} \in A \Rightarrow n \notin \hat{k})$

Proof of Lemma 3ZB

Need to show that: $m+p+n'+q' = m'+p'+n+q$

Assume that $\langle m,n\rangle \sim \langle m',n'\rangle$ and $\langle p,q\rangle \sim \langle p',q'\rangle$. Thus, we know that $m+n'=m'+n$ and $p+q'=p'+q$. Summing them up, we get that

$$m+n'+p+q' = m'+n+p'+q$$
$$m+p+n'+q' = m'+p'+n+q$$

Consequently, $\langle m+p, n+q\rangle \sim \langle m'+p', n'+q'\rangle$ by definition.

Q.E.D. $\boxed{}$

Recap: Proof of trichotomy

$S = \{n\in\omega \mid \forall m(n=m \vee n\in m \vee m\in n)\}$

$n\in m$ | $m\in n$

$n^+\in m^+$ | $m\in n^+$

$n^+\subseteq m$

Proof of Lemma 3ZE

Need to show that $mp+nq+m'q'+n'p' = m'p'+n'q'+mq+np$

Assume that $\langle m,n\rangle \sim \langle m',n'\rangle$ and $\langle p,q\rangle \sim \langle p',q'\rangle$. Accordingly, we know that $m+n'=m'+n$ and $p+q'=p'+q$.

$$(m+n')(p'+q) = (m'+n)(p+q') \qquad (m+n')(p+q') = (m'+n)(p'+q)$$

$$mp'+n'p'+mq+n'q = m'p+np+m'q'+nq' \qquad mp+n'p+mq'+n'q' = m'p'+np'+m'q+nq$$

$$mp+n'p+mq'+n'q'+mp'+n'p'+mq+n'q = m'p'+np'+m'q+nq+m'p+np+m'q'+nq'$$

Proof of Theorem 3ZG (c)

Let $a\neq 0_{\mathbb{Z}}$ and $b\neq 0_{\mathbb{Z}}$. Then, there exists nonzero $m,n,p,q\in\omega$ so that $a=[\langle m,n\rangle]$ and $b=[\langle p,q\rangle]$. Thus,

$$a\cdot_{\mathbb{Z}} b = [\langle m,n\rangle]\cdot_{\mathbb{Z}}[\langle p,q\rangle]$$
$$= [\langle mp+nq, mq+np\rangle].$$

And by exercise 13 of chapter 4, we know that mp, nq, mq, np are all nonzero. Hence, we first deduce that there exists $\alpha,\beta\in\omega$ with $\alpha^+=nq$ and $\beta^+=np$, by Theorem 4C. Therefore, we know that $[\langle mp+\alpha^+, mq+\beta^+\rangle] = [\langle (mp+\alpha)^+, (mq+\beta)^+\rangle]$ (by (A2)); where $(mp+\alpha)^+$ and $(mq+\beta)^+$ are certainly nonzero by Theorem 4D. Consequently, we conclude that for all integers a,b: $a\cdot_{\mathbb{Z}} b\neq[\langle 0,0\rangle]=0_{\mathbb{Z}}$. Taking the contrapositive of this conditional statement, we know that for all m,n: X
$a\cdot_{\mathbb{Z}} b=0$ implies $a=0_{\mathbb{Z}}$ or $b=0_{\mathbb{Z}}$. ~~In my defense its 11pm~~

Oops not needed lmao: Conversely, we ~~ca~~ see that whenever $a=0_{\mathbb{Z}}$, $a\cdot_{\mathbb{Z}} b = [\langle 0,0\rangle]\cdot_{\mathbb{Z}}[\langle p,q\rangle] = [\langle 0\cdot p+0\cdot q, 0\cdot q+0\cdot p\rangle] = [\langle 0,0\rangle] = 0_{\mathbb{Z}}$ (where $b=[\langle p,q\rangle]$ for some $p,q\in\omega$ as before). Similarly, when $b=0_{\mathbb{Z}}$, $a\cdot_{\mathbb{Z}} b = [\langle m,n\rangle]\cdot_{\mathbb{Z}}[\langle 0,0\rangle] = [\langle m\cdot 0+n\cdot 0, m\cdot 0+n\cdot 0\rangle] = 0_{\mathbb{Z}}$ once again (where $a=[\langle m,n\rangle]$ for some $m,n\in\omega$ as usual). Hence, for all integers a,b: If $a=0_{\mathbb{Z}}$ or $b=0_{\mathbb{Z}}$, then $a\cdot_{\mathbb{Z}} b=0_{\mathbb{Z}}$.

One at most: $a = b \Rightarrow (a \not<_{\mathbb{Z}} b \wedge b \not<_{\mathbb{Z}} a)$ $\quad a <_{\mathbb{Z}} b \Rightarrow (a \neq b \wedge b \not<_{\mathbb{Z}} a)$ by the trichotomy law for ω

At least one: $m+q$ and $p+n$ are natural numbers, so $m+q = p+n$ or $m+q \in p+n$ or $p+n \in m+q$

$\Rightarrow a = b \qquad \Rightarrow a <_{\mathbb{Z}} b \qquad \Rightarrow b <_{\mathbb{Z}} a$

$\Rightarrow <_{\mathbb{Z}}$ satisfies trichotomy on $\mathbb{Z}$

$$a <_{\mathbb{Z}} b \wedge b <_{\mathbb{Z}} c$$

$$m+q \in p+n \qquad p+s \in r+q$$

$$m+q+s \in p+n+s \qquad p+s+n \in r+q+n$$

$$m+q+s \in r+q+n$$

$$m+s \in r+n$$

$$[\langle m,n\rangle] <_{\mathbb{Z}} [\langle r,s\rangle]$$

$\Rightarrow <_{\mathbb{Z}}$ satisfies transitivity on $\mathbb{Z}$

$\Rightarrow <_{\mathbb{Z}}$ is a linear ordering on $\mathbb{Z}$.

Rational numbers

Theorem 5QA

Reflexivity on $\mathbb{Z}\times\mathbb{Z}'$: Assume that $\langle a,b\rangle \in \mathbb{Z}\times\mathbb{Z}'$. Then, it clearly holds that $a\cdot b = a\cdot b$. Which now means that $\langle a,b\rangle \sim \langle a,b\rangle$; i.e. $\sim$ is indeed reflexive on $\mathbb{Z}\times$

Symmetry: Let $\langle a,b\rangle \sim \langle c,d\rangle$. It follows that $a\cdot d = c\cdot b$. Immediately, $c\cdot b = a\cdot d$ must hold true too. Thus, $\langle c,d\rangle \sim \langle a,b\rangle$ by definition; and so $\sim$ is symmetric.

Transitivity: Suppose that $\langle a,b\rangle \sim \langle c,d\rangle$ and $\langle c,d\rangle \sim \langle e,f\rangle$. This means that $a\cdot d = c\cdot b$ and $c\cdot f = e\cdot d$. Accordingly, $a\cdot d\cdot f = c\cdot b\cdot f$ and as d is non-zero $c\cdot f\cdot b = e\cdot d\cdot b$. By Theorem 5ZF, $c\cdot f\cdot b = c\cdot b\cdot f$. Consequently, $a\cdot d\cdot f = e\cdot d\cdot b$, which is the same as $a\cdot f\cdot d = e\cdot b\cdot d$. By Corollary 5ZK, $a\cdot f = e\cdot b$. Therefore, $\langle a,b\rangle \sim \langle e,f\rangle$. We can conclude that $\sim$ is transitive.

Wherefore, since the relation $\sim$ is reflexive on $\mathbb{Z}\times\mathbb{Z}'$, symmetric, and transitive, it is an equivalence relation on $\mathbb{Z}\times\mathbb{Z}'$.

$$+_{\mathbb{Q}} = \{\langle\langle[\langle a,b\rangle],[\langle c,d\rangle]\rangle,[\langle ad+cb, bd\rangle]\rangle \mid a,c\in\mathbb{Z} \wedge b,d\in\mathbb{Z}'\}$$

For a relation R on a set A to be well defined; if $a=c$ and $b=d$, then aRb iff cRd. Normally we don't have to check for this since its rather trivial. But, in the case of A being some set of equivalence classes, things get more complicated and we need to check that we have a relation.

Self-Exercise: Try defining a relation that is not well defined.

$$X \in <_{\mathbb{Q}} \iff [X\in\mathbb{Q}\times\mathbb{Q} \wedge \exists a\exists b\exists c\exists d\,(a,b,c,d\in\mathbb{Z} \wedge \langle[\langle a,b\rangle],[\langle c,d\rangle]\rangle = X \wedge ad<cb)]$$

$\frac{-1}{2}$ $\frac{1}{1}$

$-1\cdot 1 = -1$ $1\cdot 2 = 2$

$-1<2$

$\overset{1}{a}, \overset{-2}{b}$ $\overset{-1}{a'}, \overset{2}{b'}$

$\langle a,b\rangle \sim \langle a',b'\rangle$

$ab' = a'b$

$1\cdot 2 = (-1)\cdot(-2) = 2$ ✓

$\frac{1}{-2}$ $\frac{1}{1}$

$1\cdot 1 = 1$ $1\cdot(-2) = -2$

$<_\mathbb{Q}$ is a linear ordering on $\mathbb{Q}$:

Trichotomy

By Theorem 5ZI, exactly one of the below are true:

$$ad = cb, \quad ad < cb, \quad \text{or} \quad cb < ad.$$

Which means that one and only one of the below are true

$$r = s, \quad r < s, \quad \text{or} \quad s < r.$$

Thus, $<_\mathbb{Q}$ satisfies trichotomy on $\mathbb{Q}$.

$af = eb$

Transitivity

Assume that $p <_\mathbb{Q} q$ and $q <_\mathbb{Q} r$. By definition, there exists the integers a, b, c, d, e, and f (~~nonzero~~ positive) with $p = [\langle a, b\rangle]$, $q = [\langle c, d\rangle]$, and $r = [\langle e, f\rangle]$; so that $ad < cb$ and $cf < ed$.

$adf < cbf \quad cbf < edb$

Consequently, we know that $adf < cbf$ and $cfb < edb$ by Theorem 5ZJ(b) (since b and f are nonzero). By Theorem 5ZF, we can rearrange the above (equivalently) as $afd < cbf$ and $cbf < ebd$. Now, by Theorem 5ZI (transitivity of $<$), $afd < ebd$. Finally, utilising Theorem 5ZJ (b) (as d is nonzero), we conclude that $af < eb$. As a result, $p <_\mathbb{Q} r$ and transitivity holds.

Wherefore, we clearly see that $<_\mathbb{Q}$ must be a linear ordering on $\mathbb{Q}$:

$a+c \quad b+d$

$a,b \quad e,f \quad c,d$

$af < eb \quad ed < cf$

$a(b+d) < (a+c)b \quad (a+c)d < c(b+d)$

$ab + ad < ab + cb \quad ad + cd < cb + cd$

$ad < cb \quad b, d > 0$

Assume otherwise, that there exists some rational numbers p and s so that $p <_\mathbb{Q} s$ but for all rationals r, $p \not<_\mathbb{Q} r$ or $r \not<_\mathbb{Q} s$.

$p = r$

$r <_\mathbb{Q} p$

$r = s$

$s <_\mathbb{Q} r$

$$\neg \exists r (p <_\mathbb{Q} r <_\mathbb{Q} s)$$

$$\forall r (p \not<_\mathbb{Q} r \lor r \not<_\mathbb{Q} s)$$

Proof of Theorem 5Q5 :

(a) Assume $r <_{\mathbb{Q}} s$ and t is a rational number. Then there exists $a, c, e \in \mathbb{Z}$ and $b, d, f \in \mathbb{Z}'$ with $r = [\langle a,b \rangle]$, $s = [\langle c,d \rangle]$ and $t = [\langle e,f \rangle]$ so that $ad < cb$.

$$adff < cbff$$
$$adff + ebdf < cbff + ebdf$$
$$afdf + ebdf < cfbf + debf$$
$$\Rightarrow (af+eb) \cdot df < (cf+de) \cdot bf$$
$$\Rightarrow [\langle a,b \rangle] +_{\mathbb{Q}} [\langle e,f \rangle] <_{\mathbb{Q}} [\langle c,d \rangle] +_{\mathbb{Q}} [\langle e,f \rangle]$$
$$r + t <_{\mathbb{Q}} s + t$$

$$r+t = [\langle a,b \rangle] +_{\mathbb{Q}} [\langle e,f \rangle] = [\langle af+eb, bf \rangle]$$
$$s+t = [\langle c,d \rangle] +_{\mathbb{Q}} [\langle e,f \rangle] = [\langle cf+de, df \rangle]$$

$$(af+eb) \cdot df < (cf+de) \cdot bf$$
$$afdf + ebdf < cfbf + debf$$

Self-Exercise 7

Prove that

$(\forall$ positive ε in $\mathbb{Q})(\exists k \in \omega)(\forall n > k)\,|s_{n+1} - s_n| < \varepsilon$ iff $(\forall$ positive ε in $\mathbb{Q})(\exists k' > \omega)(\forall m' > k')(\forall n' > k')\,|s_m - s_n| < \varepsilon$.

Let ε be a positive rational number. $\varepsilon' = \varepsilon$ try $k' = k + \varepsilon$

Self Proof of Theorem SRA

Let $x <_{\mathbb{R}} y$, and $y <_{\mathbb{R}} z$. We then know that $x \subset y$ and $y \subset z$ by definition. Thus, all elements of x are in y, and hence in z. In addition, there exists an element of z not in y, and so not in x as well. Consequently, $x \subset z$, i.e. $<_{\mathbb{R}}$ is transitive.

Assume x and y are real numbers.

$r \in x \quad q \in y$

$q < r \implies q \in x \quad x \leq y$

$r < q \implies r \in y \quad y \leq x$

$r = q \implies r, q \in x$ and $r, q \in y$

$\exists r \exists q \left([(r \in x \wedge r \notin y) \vee (q \notin x \wedge q \in y)] \vee [(r \notin x \vee r \in y) \wedge (q \in x \vee q \in y)] \right)$

$r \in x \wedge q \in y \implies q < r \vee q = r \vee r < q$

$\implies q \in x \vee q = r \in x \wedge y \vee r \in y$

$\implies$

Field Proof of Theorem SRB

Let S be a bounded nonempty subset of $\mathbb{R}$, and U be the set of all upper bounds of S. We claim that $\bigcap U$ is a least upper bound in $\mathbb{R}$.

$r \in \bigcap U \iff \forall b (r \in b \in U)$

$x \in S$

$x < b$

$q \in x \implies q \in b$

$b \neq \emptyset$

(a) ✓

$x \subset b$ for all $x \in S$ and all $b \in U$

$\implies$ For all $q \in x$, $q \in \bigcap U$.

$(\bigcap U \neq \emptyset)$ (a) ✓

$q \in \bigcap U \wedge p < q \implies \forall x (q \in x \wedge p < q)$ $(\in S)$

$\implies p \in x$

$\implies p \in \bigcap U$

(b) ✓

$r \in b \wedge s < r \implies \exists x \; x \in S \wedge r \in x$

$\forall b (b \in U \implies b \geq \bigcap U)$

$\bigcap U \leq b$

$(\forall x \in S)(\forall q \in x)(\exists p \in x)(q < p)$

$(\forall q \in \bigcap U)(\exists p \in \bigcap U)(r < p)$

$r \in \bigcap U \implies \forall b (r \in b \in U)$

$\implies \forall b \exists q (r \in b \wedge q \in b \wedge b \in U \wedge$

$\implies$

$\forall \tilde{r} \exists r (\tilde{r} \leq r)$

$\forall r (r \in \bigcap U \implies r < \tilde{r})$ $\quad \forall r ([\forall b (r \in b \in U)] \implies r < \tilde{r})$

$\forall r [r \in \bigcap U \implies \exists q (q \in \bigcap U \wedge r < q)]$

$\neg \forall b (r \in b \in U) \vee r < \tilde{r}$

$\forall r \forall b [r \in b \in U \implies \exists q (q \in b \wedge r < q)]$

$\exists b \neg (r \in b \in U) \vee r < \tilde{r}$

$\forall r \exists b (r \in b \in U \implies r < \tilde{r})$

Proof of Lemma SRC

Let x and y be real numbers.

(a) By definition, they are nonempty ~~proper subsets of~~ $\mathbb{Q}$. So, let $q \in x$ and $r \in y$, of which there exists at least one such q and r, respectively. Then, $r+q \in x+_{\mathbb{R}} y$ by definition, implying that $x +_{\mathbb{R}} y \neq \emptyset$. We also know that x and y are proper subsets of $\mathbb{Q}$, meaning there exists some rational numbers s_1 not in x and s_2 not not y. Consequently, the rational number $s_1 + s_2$ is also not in $x +_{\mathbb{R}} y$. By definition, $x +_{\mathbb{R}} y$ contains only rational numbers. As desired, $x +_{\mathbb{R}} y \subset \mathbb{Q}$.

Thus, $\emptyset \neq x+_{\mathbb{R}} y \neq \mathbb{Q}$, satisfying condition (a) (of Dedekind cuts).

(b)

$q+r \in x+_{\mathbb{R}} y$ & $b < q+r \Rightarrow b \in x+_{\mathbb{R}}$

$b-r < q$

$b-r \in x$

$b-r+r \in x+_{\mathbb{R}} y$

$b \in x+_{\mathbb{R}} y$

$q \in x+_{\mathbb{R}} y$ & $b \leq q$

$b-r < q$

$b<q$ $b=q$ $q<b$

$b<r$ $b=r$ $r<b$

A2 + B2

$\forall q [q \in x \Rightarrow \exists \tilde{q} (\tilde{q} \in x \wedge q < \tilde{q})]$

$\forall r [r \in y \Rightarrow \exists \tilde{r} (\tilde{r} \in y \wedge r < \tilde{r})]$

$(\forall b \in x+_{\mathbb{R}} y)(\exists \tilde{b} \in x+_{\mathbb{R}} y)$

$(b < \tilde{b}$

$q < \tilde{q}$

(c) ✓

~~For any $q \in x$, there exists $q' \in x$ with $q < q'$. Thus, $q - q' < 0$.~~

$-x = \{ r \in \mathbb{Q} \mid (\exists s > r)\ -s \notin x \}$

$-s \notin x$

Let $-s \in \mathbb{Q} - x$ of which there exists at least one since $x \subset \mathbb{Q}$. Then, $s-1 < s$ ~~$s-1 < s$~~, and thus, $-s-1 \in -x$. As a result, $-x \neq \emptyset$.

$-s+1 > -s$

$s > s-1$

$$\exists q \left[q \in \mathbb{Q} \wedge \forall s \left(s \not> q \vee -s \in x \right) \right]$$

$$\forall s \left(s > q \implies -s \notin x \right)$$

Consider $r' \in x$. As the real number x is nonempty, we have that there again is at least one such $r' \in x$.
Subsequently, for any $\tilde{s} > -r'$, we see that $-\tilde{s} < r'$.
Correspondingly, $-\tilde{s} \in x$. Resultantly, this means that $-r' \notin -x$
because there exists no $\tilde{s} > -r'$ so that $-\tilde{s} \notin x$. In other words, $-x \neq \mathbb{Q}$.
In sum, it has been proven that $\emptyset \neq -x \subset \mathbb{Q}$.

$-s' \in x \implies s' \notin -x$

$s' \in x \implies -s' \notin -x$

$(\exists \tilde{s} > s')(-\tilde{s} \notin x)$

$-\tilde{s} > -s'$ $\quad \tilde{s} > -s'$

Show: $-s' \notin -x$

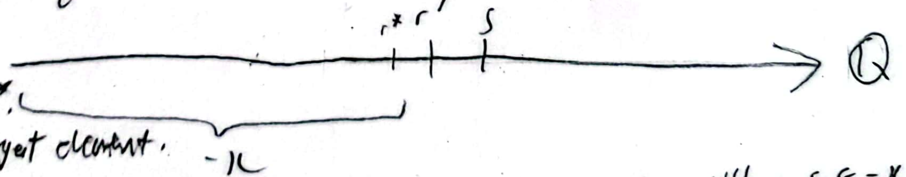
$s' > \tilde{s}$ $\quad s' > -\tilde{s}$

$\tilde{s} \in x$ $\quad -\tilde{s} \in x$

$$r \in -x \ \& \ q < r \implies q \in -x$$

Assume $r \in -x$ and the rational number q is less than r. So, there must exist some $s > r$ with $-s \notin x$. Hence, $q < r < s$, and by transitivity, $s > q$. Thence, $q \in -x$. We see that $-x$ is closed downwards.

V1 (cont.) Suppose $-x$ has a largest element r^*. It follows from definition that there exists some $s > r^*$ with $-s \notin x$. Now, s must not be in $-x$; lest $s \in -x$ and $s > r^*$, which would mean r^* is not the largest member of $-x$. However, we see that a contradiction is inevitable; because by Exercise 14 of this chapter, there exists q^* with $r^* < q^* < s$ (as the ordering of rationals is dense). Therefore, it is clear that $q^* \in -x$ and $r^* < q$ simultaenously; Whence, contradicting our assertion ~~presumption~~ that $-x$ has a largest element r^*.
Consequently, it must be that $-x$ has no largest element.



V2 Dir Suppose $r \in -x$ again. Again we know there exists some $s > r$ with $-s \notin x$ by definition. Now, either $s \in -x$ or $s \notin -x$. Consider the former case of $s \in -x$; immediately we see that $s > r$ and $s \in -x$ simultaenously. In the latter case of $-s \notin x$, by Exercise 14 of this chapter (that proves that the ordering of rationals is dense), there exists some rational q with $r < q < s$. Therefore, $q \in x$ and $q > r$ again. In any case, it is clear that for any element of $-x$, there exists another larger one (in $-x$). Whence, $-x$ has no largest element.

$a < b$ & $c < d$ $\qquad$ $c < d$ $\qquad$ $0 < d - c$

$ac < bd$ $\qquad$ $c < c + (d-c)$

$b + c < b + d$

$a + c < b + c$

$a + c < b + c < b + d$

$a + c < b + d$

$q + r < \tilde{q} + r$ $\qquad$ $r + \tilde{q} < \tilde{r} + \tilde{q}$

(b)

$x +_{\mathbb{R}} (-x) = \{q + r \mid q \in x \ \& \ r \in -x\}$

$= \{q + r \mid q \in x \ \& \ (\exists s > r) - s \notin x\}$

$(\forall q \in x)(\forall r \in -x)(q + r < 0)$

$\underline{x < 0_{\mathbb{R}}}$ $(q \in x \Rightarrow x < 0)$ $\qquad q < 0 \qquad q = 0 \qquad q > 0$

For any $-s \in 0_{\mathbb{R}} \setminus x$ and $q \in x$,

$(q < r \Rightarrow q \in x) \Rightarrow -s > q$

$q + s < 0$

$\underline{x > 0_{\mathbb{R}}}$ $\quad 0_{\mathbb{R}} \subset x$

$s < 0 \quad -s \notin x$

for any $q \in x$, $\quad -s > q$

$s + q < 0$

Let $-s \in \mathbb{Q} \setminus x$.

$-s > -\tilde{s}$

$r < 0$

$s \in -x \setminus 0$ $\qquad (\exists \tilde{s} > s) - \tilde{s} \notin x$

$q = r - s < 0$ $\qquad \tilde{s} > s > 0$

$\tilde{s} > 0$

$-\tilde{s} < 0$

$x < 0_{\mathbb{R}}$

$x \subset 0_{\mathbb{R}}$ $\quad -x$

Let $-s \in 0_{\mathbb{R}} \setminus x$.

$-s < 0$

$s > 0$

$s \in -x$

$-q > s > 0$ for all $q \in x$.

$s + q < 0$

$s - 0 = 0 - (-s)$

$-x = \{r \in \mathbb{Q} \mid (\exists s > r) - s \notin x\}$

$x = 0_{\mathbb{R}}$ ✓

$-x \setminus x$

$x <_{\mathbb{R}} 0_{\mathbb{R}}$

$q < 0$

$-q > 0$

If $x \leq_{\mathbb{R}} 0_{\mathbb{R}}$

$p < 0$

$p + q < q < 0$

$p + q < 0$

$-p - q > 0$

$q + s = p$

$\varepsilon =$

Assume $x <_{\mathbb{R}} y$. Immediately, $x \subset y$ by definition. Then, ∨ a (rational) upper bound b of x, lest there exist some rational $q \in x$ with $b \leq q$ but $b \notin y$

any member of $y \setminus x$ is

When $x <_{\mathbb{R}} 0_{\mathbb{R}}$, $x \subset 0_{\mathbb{R}}$.

Show $x \subset -x$

Let $-s \in 0_{\mathbb{R}} \setminus x$.

$\Longrightarrow -s < 0$

$\Longrightarrow s > 0$

for any $q \in x$, $s > q$. So, $q \in -x$. $\Longrightarrow x \subset -x$ ✓

When $x = 0_{\mathbb{R}}$, $0_{\mathbb{R}} + (-x) = 0_{\mathbb{R}}$

$-x = 0_{\mathbb{R}}$.

$\mathbb{Q} \setminus 0_{\mathbb{R}} \neq \emptyset$

When $x >_{\mathbb{R}} 0_{\mathbb{R}}$, $x + (-x) >_{\mathbb{R}} 0_{\mathbb{R}} + (-x)$

$-x <_{\mathbb{R}} 0_{\mathbb{R}}$

$\Longrightarrow -x \subset -(-x) = x$

$x \in \bigcup P(A) \iff \exists a (x \in a \subseteq A)$

$\implies x \in A$

$x \in A \implies x \in \{x\} \in PA$

$\implies x \in \bigcup P(A)$

Let $x \in A$ and $y \in x$

$x \in P\bigcup A \iff x \subseteq \bigcup A$

$\iff \forall y (y \in x \implies y \in \bigcup A)$

$\iff \forall y (y \in x \implies \exists z (y \in z \in A))$

$(y \in x \in A \implies y \in \bigcup A) \implies x \subseteq \bigcup A$

$\implies x \in P\bigcup A$

~~$x \in C \setminus (A \cup B) \iff x \in C \wedge (x \notin A \wedge x \notin$~~

~~$\iff (x \in C \wedge x \notin A)$~~

$c \in P(a) \Rightarrow c \subseteq a$

$x \in c \Rightarrow (x \in a \in b) \implies x \in \bigcup B$

$c \subseteq \bigcup B$

$c \in P \bigcup B$

1. brackets $(\wedge)$
2. Phrasing less clean
3. ~~brackets again ($\implies$ and $\iff$)~~
3. Use EL instead of $\iff \implies \wedge$

3rd line inline mm wrong

$[\neg(t \in C) \vee t \in A] \wedge [\neg(t \in C) \vee t \in B)]$

$\iff [\neg(t \in C) \vee (t \in A \wedge t \in B)]$

$\iff [t \in C \implies (t \in A \wedge t \in B)]$

$\neg(t \in A \vee t \in B) \vee t \in C$

Exitantial quantifier missing

$\iff [\neg(t \in A) \wedge \neg(t \in B)] \vee t \in C$

brackets

$\iff ([\neg(t \in A) \vee t \in C] \wedge [\neg(t \in B) \vee t \in C])$

$\iff [(t \in A \implies t \in C) \wedge (t \in B \implies t \in C)]$

$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$

$\{ \ \{x\} \ \}$

$(\langle x, y_1\rangle \in f \wedge g \ \& \ \langle x, y_2\rangle \in f \wedge g) \Longrightarrow (\langle x, y_1\rangle \in f \ \& \ \langle x, y_1\rangle \in g \ \wedge \ \langle x, y_2\rangle \in f \ \wedge \ \langle x, y_2\rangle \in g)$

$\Longrightarrow y_1 = y_2$

$\langle x, y\rangle \in f \cup g \iff \langle x, y\rangle \in f$ or $\langle x, y\rangle \in g$

$\iff y = f(x)$ or $y = g(x)$

$x \in \mathrm{dom}(f \cup g) \iff \exists y \, (\langle x, y\rangle \in f \cup g)$

$\iff x \in \mathrm{dom} f$ or $x \in \mathrm{dom}\, g$

$(f \cup g) \setminus (f \wedge g)$ a/s single value

$(x \in A \wedge x \in B) \Longrightarrow (x \in A \vee x \in B)$

$B : \omega_{\mathbb{R}} \longrightarrow \mathbb{R}$

Let $x \in \mathbb{R}$

$h(0) = \inf x$

$h(n^+) = (x - h(n))$

$p(-2) = \inf x$

$p(-1) = \inf \left(10(x - \inf x)\right)$

Let $\star = \bigcup\{\bar{S} - \bar{\bar{S}} \mid \bar{S} \in \underset{\text{ran} f}{\cancel{\mathcal{S}}} \;\&\; \bar{\bar{S}} \in \underset{\text{ran} f}{\cancel{\mathcal{S}}}\}$

If ~~there exists~~ $\bar{S} \in \text{ran} f$ with

$$\bar{S} - \star \neq \emptyset \quad \text{or} \quad \star - \bar{S} \neq \emptyset$$

~~then $\bar{S} - \star \subseteq \star$, implying $\bar{S} - \star = \emptyset$ actually.~~

~~and~~

If $\star \in \text{ran} f$, then for all $\bar{S}$,

$$\bar{S} - \star \subseteq \star \text{ and } \star - \bar{S} \subseteq \star$$

$$\bar{S} \subseteq \star$$

$$\Rightarrow \star = S$$

Let $\star = \bigcap\{ \qquad \}$.

If $\star' = \bigcap\{\text{ran} f - \{\emptyset\}\} \in \text{ran} f$,

$x \in \star' \Leftrightarrow (\forall X \in \text{ran} f - \{\emptyset\})(x \in X)$

$\in \text{ran} f$

Let $\star = \bigcup\{\bar{S} \cap \bar{\bar{S}} \mid \bar{S} \in \text{ran} f \;\&\; \bar{\bar{S}} \in \text{ran} f\}$,

If $\star \in \text{ran} f$, then for all $\bar{S}$,

$$\bar{S} \cap \star \subseteq \star$$

$$S \subseteq \star$$

$$\Rightarrow \star = S$$

Let $\bar{\star} = \{s \in S \mid s \notin f(s)\}$, $\quad f(s) \neq C$

If $\bar{\star} \in \text{ran} f$, then $f(s) = \bar{\star}$ for some s

$s \in S \mid s \notin f(s)$

$$\bar{\star} = S$$

$s \notin f(s)$ for all $s \in S$.

Now assume that $H(B) = H(B')$. Then, we immediately see that they are functions mapping from A into 2 so

$$[H(B)](x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in A - B \end{cases} = \begin{cases} 1 & \text{if } x \in B' \\ 0 & \text{if } x \in A - B' \end{cases} = [H(B')](x).$$

,

$$B = \{x \in A \mid [H(B)](x) = 1\} = \{x \in A \mid [H(B')](x) = 1\}$$

Therefore, it is clear that $[H(B)](x) = 1$ iff $[H(B')](x) = 1$; i.e. $\{x \in A \mid [H(B)](x) = 1\} = \{x \in A \mid [H(B')](x) = 1\}$. since both functions have domain A. However, by definition, the former is just B, while the latter is simply B'. That is:

$$B = \{x \in A \mid [H(B)](x) = 1\} = \{x \in A \mid [H(B')](x) = 1\} = B'.$$

Thence, the function H is injective.

Wherefore, it now follows that H is a bijective function from $\wp A$ to ${}^A 2$. Which means that $\wp A \approx {}^A 2$.

Q.E.D. $\boxed{2}$

Find the bijection from $\omega \times \omega$ to ω or the other way around

$f : \omega \times \omega \to \omega$

$f(m, n) =$

$$\begin{array}{cccc} (0,0) & (0,1) & (0,2) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots \\ (2,0) & (2,1) & (2,2) & \\ \vdots & \vdots & & \end{array}$$

0
1
2
3
4
5
6
7

Let $g \in {}^{A}2$

$$D_0 = \{x \in A \mid g(x) = 0\}$$
$$D_1 = \{x \in A \mid g(x) = 1\}$$

The sets D_0 and D_1 are disjoint, lest $g(x) = 0$ and $g(x) = 1$ simultaneously, which would violate the fact that the function g is single-valued.

This combined with the fact that $D_0 \cup D_1 = A$ — because g has domain A in which $g(x) = 0$ or $g(x) = 1$ by definition — means that $D_0 = A - D_1$. Consequently, we can write the mapping of g as

$$g(x) = \begin{cases} 1 & \text{if } x \in D_1 \\ 0 & \text{if } x \in A - D_1 \end{cases}$$

Accordingly, we notice that $H(D_1) = g(x)$. Hence, H is a surjective function.

Self - Proof of theorem 4B

Assume that the set ω is equinumerous to the set $\mathbb{R}$ of real numbers. Then, there exists a bijective function $f : \omega \to \mathbb{R}$.

Surjectivity $\Rightarrow \neg$ single-valuedness

$n_1 = n_2 \Rightarrow f(n_1) = f(n_2)$

$x \in \mathbb{R}$ exists $n \in \omega$ so that $f(n) = x$.

False SV: $n_1 = n_2$ is true but $f(n_1) \neq f(n_2)$

$[\mathbb{Q} \setminus f(n_1)] \setminus f(n_2) \setminus \cdots$

Let the function $g : \omega \to \mathcal{P}\mathbb{Q}$ be defined by

$$g(0) = \mathbb{Q} \setminus f(0)$$

$$g(n^+) = \mathbb{Q} \setminus g(n)$$

Show ~~$\bigcup \mathbb{R} = \mathbb{Q}$~~

Show ~~$\operatorname{ran} g \neq \mathbb{Q}$~~.

$f(n)$ $f(m)$

$f(n) \neq f(m)$

$\wp S \simeq {}^{S}2$

Assume that there exists some set S that is equinumerous to its power set. By the prior fact / example, given we know that

$\neg$ surjectivity / surj $\Rightarrow \neg\neg S V$

$f : S \longrightarrow {}^{S}2$ / $f : S \longrightarrow \wp S$

$\wp\omega$

$\{0\}\ \{1\}\ \{2\}\ \ldots$

$\{0,1\}\ \{1,2\}\ \{3,4\}\ \ldots$

$\{0,2\}\ \{1,3\}\ \{2,4\}\ \ldots$

$\{0\}\ \{1\}\ \{2\}\ \ldots$

$\{0,1\}, \{0,2\} \ldots$ $\{1,0\}, \{1,2\} \ldots$ $\{2,0\}, \{2,1\} \ldots$

$\{0,1,2\}, \{0,1,3\}, \{0,1,4\} \ldots$

$\{0,2,3\}, \{0,2,4\}, \ldots$ $\star \cap \smile = \emptyset$

$\vdots$

$\underbrace{\star \cap \smile}_{\subseteq \text{ new } \star} \neq \emptyset$

$S \longmapsto \{S, S', S'', \ldots\} = \bar{S}$

$S \longmapsto \{\grave{S}, \grave{S}, \grave{S}, \ldots\} = \bar{\bar{S}}$

$S \longmapsto \{\ \ \}$

Either $\bar{S} \cap \bar{\bar{S}} = \emptyset$

$\bar{\bar{S}} \subset \bar{S}$

$\bar{S} \subset \bar{\bar{S}}$

or $\bar{S} \cap \bar{\bar{S}} \neq \emptyset$

$\star = \{\bar{S} - \bar{\bar{S}} \mid \bar{S} \subseteq S \ \&\ \bar{\bar{S}} \subseteq S\}$

For all $\bar{S}^{\wedge}$, $\bar{S}^{\wedge} - \star = \emptyset$ $\quad \bar{S} \subseteq \star$

Cont.: $S - \star = \emptyset$ $\quad \star - \bar{S}^{\wedge} = \emptyset$ $\quad \star \subseteq \bar{S}^{\wedge}$

$\star - S = \emptyset$ $\quad \bar{S}^{\wedge} = \star$

$\Rightarrow S = \star$

but $S - \emptyset = \emptyset$

$\emptyset - S = \emptyset$

$\Rightarrow S = \emptyset$ $\quad S \longmapsto \star$ $\square$

$\Rightarrow$ exists $\bar{S} - \star \neq \emptyset$

or $\star - \bar{S} \neq \emptyset$

$\bar{S} \quad \bar{\bar{S}}$

$(\bar{S} \cap \bar{\bar{S}}) \cup (\bar{S} \cap \bar{\bar{S}})$

$\star = \bigcup \{\bar{S} \cap \bar{\bar{S}} \mid \bar{S} \subseteq S + \bar{\bar{S}} \subseteq$

If $\bar{S} \cap \star = \emptyset$

Theorem 6A

(a) Clearly, the identity map I_A provides us with such a bijection from A into A. When $a \in A$, $I_A(a) = a$. Thus, the identity map . surjective. Similarly, if $I_A(a) = I_A(a')$, then by definition, $I_A(a) = a$ and $I_A(a') = a'$. Accordingly, $a = a'$. We conclude that the identity map is injective. Whence, A is equinumerous to itself. $\boxed{\text{h}}$

(b) Assume that $A \approx B$. In other words, there exists a bijection $f : A \to B$. By Theorem 3F, f^{-1} is a function mapping from B into A because f is injective. Given any $a \in A$, $f(a)$ exists. Therefore, using Theorem 3G, we have that $f^{-1}(f(a)) = a$. Again, from the same theorem, for any b and b' in B such that $f(b) = f(b')$ — meaning $f(f^{-1}(b)) = f(f^{-1}(b))$ — we observe that $f(f^{-1}(b)) = b = f(f^{-1}(b')) = b'$. As a result, f^{-1} is injective. Thence, f^{-1} is a bijection from B into A; B is equinumerous to A. $\boxed{\text{h}}$

(c) Now let $A \approx B$ and $B \approx C$. Immediately, there must exist some bijections $G_{AB} : A \to B$ and $\tilde{G}_{BC} : B \to C$. Thus, we now construct the bijection $\overline{G_{AC}} : A \to C$ with $\overline{G_{AC}}(a) = \tilde{G}_{BC}(G_{AB}(a))$.

Suppose $a = a'$. Hence, owing to the fact that G_{AB} is a function, $G_{AB}(a) = G_{AB}(a')$. By reason of $\tilde{G}_{BC}$ being a function, $G_{BC}(G_{AB}(a)) = G_{BC}(G_{AB}$ Consequently, $\overline{G_{AC}}(a) = \overline{G_{AC}}(a')$. i.e. $\overline{G_{AC}}$ is a function.

If $c \in C$, then there is some $b \in B$ so $c = \tilde{G}_{BC}(b)$ because G_{BC} is surjective. Similarly, as G_{AB} is surjective, $b = G_{AB}(a)$ for some $a \in A$. In sum, there exists an $a \in A$ such that $c = \tilde{G}_{BC}(G_{AB}(a))$. By definition, $\overline{G_{AC}}(a) = c$. In other words, surjectivity is proven.

Whenever $\overline{G_{AC}}(a) = \overline{G_{AC}}(a')$, $\tilde{G}_{BC}(G_{AB}(a)) = \tilde{G}_{BC}(G_{AB}(a'))$ by definition. Hence, since $\tilde{G}_{BC}$ is injective, $G_{AB}(a) = G_{AB}(a')$. Repeating this once more, due to G_{AB} being injective, it must be that $a = a'$. We see that $\overline{G_{AC}}$ is injective.

Wherefore, $\overline{G_{AC}}$ is indeed a bijection from A into C. Which means that $A \approx C$.

$m \neq m'$ $\quad n \neq n'$ $\qquad 2^m(2n+1) = 2^{m'}(2n'+1)$

$m > m'$ $\quad n > n'$

$m' > m$ $\quad n' > n$ $\quad$ supp. $2^m \neq 2^{m'}$ and $2n+1 \neq 2n'+1$

$2^m < 2^{m'}$ $\qquad 2n+1 < 2n'+1$

$m > m'$ & $n > n'$ $\qquad 2n'+1 < 2n+1$

$2^{m'} < 2^m$

$a \cdot b = c \cdot d$

$\Rightarrow (a = c$ & $b = d)$ or $(a = d$ & $b = c)$

1. $2^m < 2^{m'}$ & $2n+1 < 2n'+1$: $2^m(2n+1) < 2^{m'}(2n'+$

2. $2^m < 2^{m'}$ & $2n'+1 < 2n+1$:

$S = \{a \in \omega \mid \forall b \forall c \forall d \ldots\}$

$M_m(n) = M_{m'}(n')$

Let $S = \{$

$m \neq m' \Rightarrow 2^m \neq 2^{m'}$

$n \neq n' \Rightarrow 2n+1 \neq 2n'+1$

$S = \{n \in \omega \mid \forall n' (m \neq m'$ & $n \neq n')$ implies $M_m(n) \neq M_{m'}(n')\}$

Ass. $m \neq m'$ and $n \neq n'$ $\quad M_m(0) = 0$

$m = 0$

$\left.\begin{matrix} 0 < m' \\ 0 < n' \end{matrix}\right\} 0 < M_{m'}(n')$

Let k be a natural number. By exercise 14 of chapter 4, any natural number is either even or odd. Hence, we consider this case wise. First consider the much simpler case of k being even, which means $k = 2n$ for some natural n. Thus, $f(0,n) = 2^0(2n+1)-1 = 2n = k$ for the above n. As for the latter case where k is odd, there is some natural n with $k = 2n+1$. In other words, $k+1 = 2n+2 = 2(n+1)$ is even. Now let the ~~set~~ Lemma 4.1.1 set $T = \{n \in \omega \mid$ If $2n$ is nonzero, it can be written as 2^i or $2^i(2j+1)$ for some nonzero naturals i and j less than $n\}$. Assume that for all $m \in n$, $m \in T$. If $n=0$, $n \in T$ immediately holds. Suppose that n is nonzero. Clearly, $n \in 2n$ and n is either even or odd. When n is even, there exists nonzero naturals i and j so either $n = 2^i$ — in which case $2n = 2(2^i) = 2^{i+1}$ — or $n = 2^i(2j+1)$, implying $2n = 2^{i+1}(2j+1)$. In any case, we see that $n \in T$.

~~Consequently~~, By the strong induction principle for ω, $T = \omega$.

By the above Lemma 4.1.1, there is some nonzero natural numbers i and j with $2n+2 = 2^i(2j+1)$. It follows that $k = 2n+1 = 2^i(2j+1)-1 = f(i,j)$.

Consequently, irregardless of whether k is even or odd, k is always in the range of f. Which means that f is surjective.

Suppose that $f(m,n) = f(m',n')$. In other words, $2^m(2n+1) = 2^{m'}(2n'+1)$.

1. $m = m'$ and $n = n'$

2. $n = n'$ and $n = n$

3. m

Let S be the set of natural numbers n that, if nonzero, have unique prime factorisations. That is, $n = k_1 k_2 k_3 \cdots k_n$ for primes k_1 to k_n, and, for any natural m, $k_1 k_2 \cdots k_n = \bar{k}_1 \bar{k}_2 \cdots \bar{k}_m$ implies $n = m$ and, for all $i \in n$, there exists j with $k_i = \bar{k}_j$.

$0 \in S$

Assume $n \in S$. Then,

$n^+ = n+1 = k_1 k_2 \cdots k_n + 1$

$=$

$m \neq m'$ $n \neq n'$

$m > m'$ $n > n'$

$m' > m$ $n' > n$

$2^m(2n+1) = 2^{m'}(2n'+1)$

Supp. $2^m \neq 2^{m'}$ and $2n+1 \neq 2n'+1$

$2^m < 2^{m'}$ $2n+1 < 2n'+1$

$2^{m'} < 2^m$ $2n'+1 < 2n+1$

$m > m'$ & $n > n'$

$a \cdot b = c \cdot d$

$\Rightarrow (a=c$ & $b=d)$ or $(a=d$ & $b=c)$

$S = \{a \in \omega \mid \forall b \forall c \forall d \ldots\}$

1. $2^m < 2^{m'}$ & $2n+1 < 2n'+1$: $2^m(2n+1) < 2^{m'}(2n'+1)$

2. $2^m < 2^{m'}$ & $2n'+1 < 2n+1$:

$M_m(n) = M_{m'}(n')$

Let $S = \{$

$\forall n$

$S = \{n \in \omega \mid (m \neq m'$ & $n \neq n')$ implies $M_m(n) \neq M_{m'}(n')$

$m \neq m' \Rightarrow 2^m \neq 2^{m'}$

$n \neq n' \Rightarrow 2n+1 \neq 2n'+1$

Ass. $m \neq m'$ and $n \neq n'$

$M_m(0) = 0$

$m = 0$

$0 \in m'$
$0 \in n'$ $\}$ $0 \in M_{m'}(n')$

Suppose that $f(m,n) = f(m',n')$. i.e. $2^m(2^n+1)-1 = 2^{m'}(2^{n'}+1)-1$

$$2^m(2^n+1) = 2^{m'}(2^{n'}+1)$$

$$\neg(2^m(2^n+1) < 2^{m'}(2^{n'}+1)) \qquad \neg(2^{m'}(2^{n'}+1) < 2^m(2^n+1))$$

1. Let $k \in \omega$. Then,

$$2^m(2n+1)-1 = k$$
$$2^m(2n+1) = k+1$$

if k even, there exists $\tilde{k}$ with $2\tilde{k} = k$
$\Rightarrow k+1$ odd $\Rightarrow m = 0$ ...

If k odd, there exists $\tilde{k}$ so $2\tilde{k}+1 = k$.
$\Rightarrow k+1$ even $\Rightarrow m \geq 1$ ...

$2^m \cdot (2n+1)$

Either $k+1$ is divisible by some odd number or it is not. $n = \tilde{k}$

If not: $k+1 = 2^{k^*}$ If yes: Let

$k = 2^{k^*}-1$

$m = k^*$

Random unrelated thing

If $k+n=m$, then $m-n=k$

$- = \{\langle\langle m,n\rangle, k\rangle \in \omega^3 \mid k+n=m\}$

$$2^m(2n+1)-1 = 2^{m'}(2n'+1)-1$$
$$2^m(2n+1) = 2^{m'}(2n'+1)$$

$k+1 = 2^m(2n+1)$
$= 2[(2^m)(n)+2^m]$ $(+1 \text{ to } m)$: $4n+2$
$= 2[$ $(+1 \text{ to } n)$: 2^m

$$k+1 = 2^m(2n+1)-1$$
$$k+2 = 2^m(2n+1) \quad \text{odd}$$
$$k+2 = 2k^*+1$$
$$k+1 = 2k^*$$

let the set $S = \{k \in \omega \mid k \in \text{ran } f\}$

$f(0,0) = 0 \Rightarrow 0 \in S$ $\quad f(1,0)$

If $k \in S$, then $f(m,n) = k$ for some natural m and n.

$$2^m(2n+1)-1 = k$$
$$2^m(2n+1)-1+1$$
$$= 2^m(2n+1)+1-1$$
$$= 2^m\left(2n+1+\frac{1}{2^m}\right)-1 \quad (2^{m+1})(n)+2^m+1-1$$
$$=$$

Assume that for all $\tilde{k} \in k$, $\tilde{k} \in S$.

If k even; $k+1$ odd

$\Rightarrow m=0$, lest $m \geq 1$ so $k+1 = 2[2^{m-1}(2n+1)]$.

$$k = 2^0(2n+1)-1 \qquad k+1 = 2n+1$$
$$k = 2n+1-1 \qquad = 2(n+1)-1$$
$$k = 2n \qquad = 2^0[2(n+1)]-1$$

If k odd, $2\tilde{k}+1 = k$ for some $\tilde{k}$

$k+1$ even

$m \geq 1$, lest $m=0 \Rightarrow k+1 = 2^0(2n+1) = 2n+1$ which is odd

$$f(m+1, n) = 2^{m+1}(2n+1) - 1$$

$$= 2^m(2n+1) + \underline{\underline{4n+2}} - 1$$

$$f(m, n+1) = 2^m(2(n+1)+1) - 1$$

$$= 2^m(2n+2+1) - 1$$

$$= 2^m(2n+1) + \underline{\underline{2^{m+1}}} - 1$$

$$2^m(2n+1) = 2^{m'}(2n'+1)$$

$$2^m(2n+1) \div 2^m = 2n+1$$

$$2^{m'}(2n'+1) = 2^m(2n'+1)$$

$$\div = \{\langle\langle m,n\rangle, k\rangle \mid m,n,k \in \omega \ \& \ m = kn\}$$

~~Let $T = \{i \in \omega \mid$ If i is even, it can be written as 2^m or $2^m(2n+1)$ for some integer ... and m or it is zero~~

~~$0 \in T$ by def.~~

~~When $i \in T$, i even~~

Let $T = \{ i \in \omega \mid$ If $2i$ is nonzero, can be written as 2^m or $2^m(2n+1)$ for some nonzero ~~integers~~ naturals m and $n \}$.

$0 \in T$ by def.

If $i \in T$ either $i = 0$ or $i \neq 0$. In the first case, $2i^+ = 2$, which is the same as 2^1. So, the claim holds. As for the second case, $2i = 2^m$ or $2i = 2^m(2n+1)$ for nonzero integers m and n.

When $n = 2^m$,
$$\begin{aligned} 2i^+ &= 2^m(2n+1) + 2 \\ &= 2m'(2n+1) + 2 \\ &= 4m'n + 2m' + 2 \\ &= 2[2m'n + m' + 1] \\ &= 2[(2^{m''+1})(n) + (2^{m''}) + 1] \\ &\cancel{= 2^{m''+1}[2n+1]} \\ &\cancel{= f(m'', n) + 1} \end{aligned}$$

$(2^{m+1})(n) + 2^m + 2$

2^{m} must be even and smaller than $2i$

When $n = 2^m$,

$$\begin{aligned} 2i^+ &= 2^m + 2 \\ &= 2(2^{m'} + 1) \end{aligned}$$

let $m=m'$ and $n=n'$ so that $m=kn$ and $m'=k'n'$ for some naturals k and k'.

$$m'=kn=k'n'$$

$$k=k' \quad \text{cancellation laws}$$

$$m=kn=m'=k'n'$$

$$m \div n = m' \div n'$$

Suppose that $f(m,n) = f(m',n')$. i.e. $2^m(2n+1) = 2^{m'}(2n'+1)$. We first define the division operator $\div$ on ω so that $m \div n$ iff there exists natural k with $m = kn$. Since $2^m(2n+1) = (2n+1)(2^m)$, $2^m(2n+1)$ can be divided by 2^m, which gives $2n+1$. By equality we supposed, $2^{m'}(2n'+1)$ must also be divisible by 2^m. That is, $2^{m'}(2n'+1) = \underline{(2n'+1)(2^m)}$. By the cancellation law from Corollary 4P, $2^{m'} = 2^m$. Thence, $m' = m$ (Proof). Resultantly, by the cancellation laws again:

$$2n+1 = 2n'+1$$
$$2n = 2n'$$
$$\therefore n = n'.$$

$$2^m(2n+1) = 2^{m'}(2n'+1)$$
$$2^{m^+}(2n+1) = 2^{m'}(2n'+1)$$
$$2[2^m(2n+1)] =$$

We see that the function f is an injection.

Lastly, it is clear that f is a function. Presume $m = m'$ and $n = n'$. Thereupon, $2^m(2n+1) = 2^{m'}(2n'+1)$ because addition, multiplication and exponentiation are functions.

Wherefore, the bijection $f : \omega \times \omega \to \omega$ defined by $f(m,n) = 2^m(2n+1)$ is indeed a one-to-one correspondence between $\omega \times \omega$ and ω.

Q.E.D. $\boxed{\checkmark}$

Self-Exercise: Prove that $S^n \approx S$ for any set S.

Prove or disprove: $\underbrace{\prod_{i \in I} S}_{{}^I S}$ is equinumerous to S for any sets I and S.

Noo; we misremembered: its supposed to be $S_2 \approx PA$.

Counter example: ~~We see that~~ $\cancel{\prod_{i \in I} S = \{f : I \to S \mid (\forall i \in I)(f(i) \in S)\} = {}^I S}$. Also, notice that $S \not\approx PS$ by Theorem 6B(b). Then, by the contrapositive of Theorem 6A(c), $S \not\approx PS$ implies that either $S \not\approx S_2$ or ${}^2 S \not\approx PA$ consequently. Since we know that ${}^2 S \approx PA$, it must be that $S \not\approx {}^2 S$. Wherefore,

6. Assume that the set K of cardinality κ dominates all members of $\mathcal{A}$, so for all elements A of $\mathcal{A}$, there exists an injection $f_A : A \to K$.

for all $a \in \bigcup\mathcal{A}$, there is the nonempty set B_a containing all $A \in \mathcal{A}$ that has $a \in A$.

$$H : \bigcup\mathcal{A} \longrightarrow \bigcup_{a \in \bigcup\mathcal{A}} \{B_a\} \subseteq \mathcal{P}\mathcal{A}$$

$$H(a) = B_a$$

By AC, $e(a) \in B_a \subseteq \mathcal{A}$

$$G : \bigcup\mathcal{A} \longrightarrow \mathcal{A} \times K$$

$$G(a) = \langle e(a), f_{e(a)}(a) \rangle$$ (If we want full rigor we could use AC to choose a specific injection f_A for each $A \in \mathcal{A}$)

$$G(a) = G(a')$$

$$\langle e(a), f_{e(a)}(a) \rangle = \langle e(a'), f_{e(a')}(a) \rangle$$

$$e(a) = e(a') \quad \& \quad f_{e(a)}(a) = f_{e(a')}(a')$$ (which then tells us G is well-defined, and more importa… $f_{e(a)} = f_{e(a')}$ since $e(a) = e(a')$)

$a = a'$ by the injectivity of f $\square$

Consequently, $\operatorname{card} \bigcup\mathcal{A} = (\operatorname{card} \mathcal{A}) \cdot \kappa$.

Sketch (

Self Proof that $Sq(\omega) \approx \omega$.

Lemma A $\omega^n \approx \omega$ for any n greater than or equal to 1.

Suppose that S is the set of natural numbers n with n being zero or $\omega^n \approx \omega$. Clearly, $0 \in S$ by definition. And $1 \in S$ as $\omega^1 = \omega$. Supposing $n \in S$ is at least 1, i.e. $\omega^n \approx \omega$, we have that $\omega^{n^+} = \omega^n \times \omega$. By the well-definedness of cardinal arithmetic (Theorem 6H) and our supposition, $\omega^{n^+} \approx \omega \times \omega \approx \omega$. So, n^+ is in S. By induction, $S = \omega$. $\diamond$

As such, $\omega^n \approx \omega$ as long as n is at least 1.

Notice that $Sq(\omega) = \bigcup_{n \in \omega} {}^n\omega$. We just have to show ${}^n\omega \preccurlyeq \omega$ before we can apply Theorem 6Q. First see that given any sequence $f \in {}^n\omega$, we have a similar function $f' : \omega \to \omega$ defined by $f' = f \cup \{\langle m, 0\rangle \mid m \in \omega - n\}$. Therefore, we apply the recursion theorem for ω to create the function $H_f : \omega \to \omega$ with

$$H_f(0) = \{f(0)\},$$

$$H_f(m^+) = G(m) \times \{f(m^+)\}.$$

Observe that $H(n)$ is an n^+-tuple and there is exactly one such H_f for each $f \in {}^n\omega$. Define the function $G : {}^n\omega \to \omega^{n^+}$ using the map $G(f) = H_f(n)$. Injectivity of G is now easily proven because when proof is left as a trivial exercise to the reader. Accordingly, we have shown ${}^n\omega \preccurlyeq \omega^{n^+}$ as long as n is a natural number. Consequently, ${}^n\omega$ is countable by Lemma A. All criteria to utilise Theorem 6Q are now satisfied. Consequently, $Sq(\omega) \preccurlyeq \omega$. Furthermore, $\omega \preccurlyeq Sq(\omega)$ is certain since the map $g(n) = \{\langle m, m\rangle \mid m \in n\}$ is such an injection.

Wherefore, $Sq(\omega) \approx \omega$ by the Schröder–Bernstein Theorem. $\square$

If $f \neq \bar{f}$, there is some natural $m \in n$ ~~select least natural m so~~ $f(m) \neq \bar{f}(m)$.

Then $H_f(m) \neq H_{\bar{f}}(m)$. By induction we can show $H_f(n) \neq H_{\bar{f}}(n)$ rigorous

$\Rightarrow G$ is injective.

21. Let S be a chain in $\mathcal{A}$. We claim that $\bigcup S \in \mathcal{A}$. If $S=\emptyset$, $\bigcup S=\emptyset \subseteq B$ for any $B\in\mathcal{A}$ that must exist by the nonemptiness of $\mathcal{A}$. Then $\bigcup S\in\mathcal{A}$ follows immediately.

Now consider $S\neq\emptyset$. Suppose N is the set of all $n\in\omega$ such that given any finite subset $T_n\approx n$, there exists $B\in S$ with $T_n\subseteq B$. Clearly, $0\in N$ because $T_0=\emptyset\subseteq B$ for some $B\in S$ whose existance is ensured since S is nonempty in this case. Assume $n\in N$. Fix any $t\in T_{n+1}$. Notice that $T_{n+1}-\{t\}\approx n$ so there exists a $B\in S$ with $T_{n+1}-\{t\}\subseteq B$ (from our assumption) and a $C\in S$ with $\{t\}\subseteq C$. By the construction of S, either $B\subseteq C$ or $C\subseteq B$, implying $T_{n+1}\subseteq C$ and $T_{n+1}\subseteq B$ in each respective case. Hence, $T_{n+1}\in N$ and $N=\omega$ by induction. In other words, every finite subset T_n of $\bigcup S$ is a member of $\mathcal{A}$: because $T_n\subseteq B$ for some $B\in\mathcal{A}$, which means $T_n\in\mathcal{A}$. Consequently, $\bigcup S\in\mathcal{A}$.

By (6) Zorn's Lemma, there exists a maximal element of $\mathcal{A}$. $\square$

... that there exists ... Tukey Lemma

21. Let S be a chain in $\mathcal{A}$; ~~and N contain all $n \in \omega$ such that given any finite subset $T_n \approx n$ of $\bigcup S$, $T_n \in \mathcal{A}$.~~
~~We see that $T_0 = \emptyset$ is a finite subset of each $B \in \mathcal{A}$ that must exist by the nonemptiness of $\mathcal{A}$. Hence, $T_0 \in \mathcal{A}$ by the construction of $\mathcal{A}$~~
~~so $0 \in N$. Assume $n \in N$. Fix any $t \in T_{n+1} \neq \emptyset$.~~

Idea

$T_{n+1} - \{t\} \in \mathcal{A}$

$\mathcal{A} = \{\{a\}, \{b\}, \{a,b\}\}$

$\{\{a\}, \{a,b\}\}$ is a chain

Let S be a chain in $\mathcal{A}$.

Show $\bigcup S \in \mathcal{A}$

Say T_n is a subset of $\bigcup S$ with cardinality n.

$N := \{n \in \omega \mid (\exists B \in S)(T_n \subseteq B)\}$

$0 \in N$ ✓

Assume $n \in N$. $T_{n+1} - \{t\} \subseteq B$ and $t \in C$ for some B and C in S.

Either $B \subseteq C$ or $C \subseteq B$.

$\Downarrow$ $\Downarrow$

$T_{n+1} \subseteq C$ $T_{n+1} \subseteq B$

$N = \omega$.

$T_n \subseteq \bigcup S$

$\Rightarrow T_n \subseteq B$ always $\Rightarrow T_n \in \mathcal{A} \Rightarrow \bigcup S \in \mathcal{A}$

Zorn's implies maximal element exists

I deas

21 · Teichmuller - Tukey Lemma

Want to find $M \in \mathcal{A}$ (i.e. $m \subseteq M \Rightarrow m \in \mathcal{A}$ & $m \approx n$) so $M \not\subseteq B \in \mathcal{A}$.

At least one such B exists

When $\mathcal{A} = \{\emptyset\}$, then trivially $\emptyset$ is our maximal element

$\mathcal{A} = \{\{\emptyset\}, \emptyset\} = 2$

$\mathcal{A} = \{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}, \{\emptyset\}, \emptyset\}$

$\mathcal{A} \neq \{\{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$

Construct set M with all subsets of M being in $\mathcal{A}$.

Intuition: This Lemma essentially says any set $\mathcal{A}$ with that property can have its elements arranged as

$$M \supseteq m_1 \supseteq m_2 \supseteq m_3 \supseteq \cdots$$

$B \subseteq$; $\cup$ I ; b

This has its own max, maybe. → B, where $B \cap M = \emptyset$ (possibly)

Given any element B of $\mathcal{A}$, take one element from each finite subset b of B and place it into M.

Apply (4), there is a choice function F ~~with its domain~~ being the set of nonempty subsets of $\mathcal{A}$ and $F(b) \in b$ for any $b \in \mathcal{X}$.

Let A be the set ~~of nonempty~~ finite subsets of $\mathcal{A}$. Take $F[\![A]\!]$ $\shortparallel$ $F[\![A_B]\!]$ X — $F(b) \in b$

Partition $\mathcal{A}$ into $\{A_B \mid B \in \mathcal{A}\}$

where $A_B = \{b \in \mathcal{A} \mid b$ is a nonempty sub/superset of $B\}$

$b, b' \in A_B \Rightarrow$ Either $b \subseteq b'$ or $b' \subseteq b$

$b \subseteq B$ ~~$b \supseteq B$~~

Let S be a finite subset of $\bigcup A_B$

for all $s \in S \Rightarrow$ exists $b \in A_B$ ...

~~that $S \in \mathcal{A}$~~ / $S \subseteq b$ for some ...

$S \approx n$

bijection $f : n \to S$

$f : n \xrightarrow{\text{inject}} \bigcup A_B$

~~Either $f(1) \in b' \subseteq B$~~

or

$f(n) \in b \supseteq B$

Suppose that T is the subset of ω containing only the ~~naturals n~~ so if $S \approx n$, then ~~$S \subseteq b$ for some $b \in A_B$~~

When $n = 0$, we know $S = \emptyset$. Hence, $S \subseteq B$. This means $0 \in T$

Now presume $n \in T$ and that there is a bijection $f : n^+ \to S$

21) Assume, for the sake of contradiction, that $\mathcal{A}$ is a nonempty set with the property that for every set B, $B \in \mathcal{A}$ iff every finite subset of B is a member of $\mathcal{A}$; however, $\mathcal{A}$ has no maximal element. In other words, for all elements B of $\mathcal{A}$, there is some other member B' of $\mathcal{A}$ with $B \subset B'$. Hence, there must be a nonempty subset C_B of $\mathcal{A}$ containing all proper supersets of B, for every set B in $\mathcal{A}$. Therefore, we now define the function $H : \mathcal{A} \to \mathcal{P}\mathcal{A}$ wi

$$H(B) = C_B \ (\neq \emptyset).$$

Now, utilizing the second form of the Axiom of Choice, there exists a function $f : \mathcal{A} \to \mathcal{A}$ so that $f(B) \in C_B$. That is, $B \subset f(B)$. Therefore, we also have the function $h : \omega \to \mathcal{A}$ in the way that

(since $\mathcal{A} \neq \emptyset$, there is at least one such $\overline{B} \in \mathcal{A}$)

$$h(0) = \overline{B},$$
$$h(n^+) = f(h(n)).$$

(Intuitively, what we are trying to do is to create a 'chain' of proper supersets ($h(n) \subset h(n^+)$) such that ran h is a maximal element, thus contradicting our assumption.)

We claim that if B is a finite subset of ran h, then $B \in \mathcal{A}$. To show this, first let S be the set of natural numbers n with $B \subseteq$ ran f and $B \approx n$ implying $B \subseteq h(k) \in \mathcal{A}$ for some natural number k. Clearly, $\emptyset \subseteq \overline{B}$, so $\emptyset \in \mathcal{A}$ is guaranteed by the construction of $\mathcal{A}$. In other words, $0 \in S$. Suppose that $n \in S$ and that $B \approx n^+$ is a subset of ran h. Notice that $B = g[\![n]\!] \cup \{\langle n, g(n)\rangle\}$, where $g[\![n]\!] \subseteq h(k)$ for some k by our supposition, and g being the bijection from n^+ to B. Since $B \subseteq$ ran h, $g(n) \in h(m)$ given some natural number m. Either $m \in k$ or $k \in m$. In the former, it is easy to see that $h(m) \subset h(k)$, and therefore, $g(n) \in h(k)$. Similarly in the latter, we have $h(k) \subset h(m)$, informing us that $g[\![n]\!] \subseteq h(m)$. In any case, B is either a subset of $h(k)$ or $h(m)$. As a result, $n^+ \in S$. By induction, $S = \omega$. That means, any finite subset B of ran h is a subset of some $h(k)$ (which is, in turn, itself an element of $\mathcal{A}$!). As a consequence of the definition of the set $\mathcal{A}$, $B \in \mathcal{A}$ as (the finite subset B of
it is an element of the existing member $h(k)$ of $\mathcal{A}$. Which tells us that ran $h \in \mathcal{A}$ as every finite subset of it has been shown above to be in $\mathcal{A}$.
No element A of $\mathcal{A}$ can be a proper superset of ran h, lest

X Good try! :)

Counterexample the set $\overline{\omega}$ containing all subsets of $\omega \cup \{\omega\}$ in which case ran $h = \omega \subset$
h could be such that $h(n) = n$,

ch / Ideas

21. Teichmüller – Tukey Lemma

$$S_B = \{\overline{B} \in \mathcal{A} \mid B \subseteq \overline{B} \text{ or } \overline{B} \subseteq B\}$$

$B \in S_B$, so $S_B \neq \emptyset$.

$\bigcup S_B$

Let $T = \{n \in \omega \mid (B' \subseteq \bigcup S_B \;\&\; B' \approx n) \Rightarrow B' \subseteq \text{some } B_n \text{ in } S_B\}$. Since $\emptyset \subseteq B$, $0 \in T$. Suppose $n \in T$, and B' is a subset of S_B that is equinumerous to n^+, where f is such a bijection from n^+ into B'. So, $f \upharpoonright n$ is a bijection from n into $f[\![n]\!]$, meaning $f[\![n]\!] \approx n$. Due to our supposition that $n \in T$, there certainly exists some $B_n \in S_B$ in the way that makes $f[\![n]\!] \subseteq B_n$.

Notice that $f(n)$ must be in some $B_{n^+} \in S_B$. With this in mind, we have that $B_{n^+} \subseteq B$ or $B \subseteq B_{n^+}$.

Assume that there exists some increasing sequence S of sets and B is a subset of the union $\bigcup_{n \in \omega} S(n)$ such that for every infinite subset B' of B there is some n for which $B' \cap S(n)$ is infinite, but B is not a subset of any $S(n)$. First notice that ~~the~~ this means that for every $n \in \omega$, there exists some b in B that is not in $S(n)$. In other words, for each $n \in \omega$, there is the nonempty subset N_n of ω containing just the natural numbers m so that there exists a $b \in B - S(n)$ that is in $S(m)$. By the well-ordering principle for ω, there is some (unique) least element ℓ_n of N_n. By recursion, we define the function $h : \omega \to \omega$ with

$$h(0) = \ell_0,$$

$$h(n^+) = \ell_{h(n)}.$$

As a result, we can now define our function H of domain ω using $H(n) = B \cap S(\overbrace{h(n^+)}^{\ell_{h(n)}}) - S(h(n))$. Applying the second form of the Axiom of Choice, there is the function f so its domain is also ω and $f(n) \in H(n)$, because $H(n)$ is always nonempty by the definition of our ℓ_n.

Now let T be the set containing the natural numbers n so $m \in n$ implies $h(m) \in h(n)$. Immediately, $0 \in T$ as no natural number is smaller than 0. Presume $n \in T$. If $m \in n^+$, then either $m \in n$ or $m = n$. In the former we know $h(m) \in h(n)$. Hence, $h(m) \in h(n) \in h(n^+)$, lest $\ell_{h(n)} \in h(n)$, which means $S(\ell_{h(n)}) \subseteq S(h(n))$ by Lemma 25A. But this would contradict the fact that there is some $b \in S(\ell_{h(n)})$ that is not in $S(h(n))$ by definition. Thus, $h(m) \in h(n) \in h(n^+)$. As for the latter, $h(m) = h(n)$ holds easily as h is a function. Therefore, $n^+ \in T$ and $T = \omega$ by induction. In other words, if $m \in n$, then $h(m) \in h(n)$ always holds. We want to show $H(n)$ is disjoint from $H(m)$. Suppose that $n \in m$ without loss of generality. Accordingly, $n^+ \in m$ and $h(n^+) \in h(m)$. Using Lemma 25A, $S(h(n^+)) \subseteq S(h(m))$. Consequently, $H(n) \cap H(m) = \varnothing$. This tells us that f is injective. As a result, $\omega \approx \operatorname{ran} f$ as f is such a bijection already. That is, $\operatorname{ran} f$ is an infinite subset of B. Consider $f(m) \in S(n)$, meaning $m \in n$ or $n \in m$. In the latter, $S(n) \subseteq S(m)$. Thence, $f(m) \notin S(n)$ (as $f(m) \notin S(m)$ by definition). Clearly, this is not possible because $f(m) \in S(n)$. This means that $m \in n$ must be true. That is, $f(m) \in S(n)$ implies $m \in n$. Let $I = (\operatorname{ran} f) \cap S(n)$. Notice that f^{-1} provides us with a bijection from $\operatorname{ran} f$ into ω. Thus, $I \approx f^{-1}[\![I]\!] \subseteq n$ because given $m \in f^{-1}[\![I]\!]$, $m \in n$. Utilising Lemma 6F, $f^{-1}[\![I]\!]$ must be finite. And resultantly, $I = (\operatorname{ran} f) \cap S(n)$ is also finite for all n. However, this contradicts our initial assumption that there exists some n for which $(\operatorname{ran} f) \cap S(n)$ is infinite, since $\operatorname{ran} f$ is an infinite subset of B. Wherefore, this must mean that our initial assumption was wrong, and that the statement in the question must be true. $\square$

Lemma 25A $n \in m$ implies $S(n) \subseteq S(m)$.

Proof Fix M to be the set of natural numbers m so $S(n) \subseteq S(m)$ for every $n \in m$. Instantly, $0 \in M$ as no natural number is smaller than it. Suppose $m \in M$ is true and $n \in m^+$. This means that $n \in m$ or $n = m$. In the case of the former, it is easily true by our supposition. As for the latter, recall that $S(m) \subseteq S(m^+)$ from definition — that is, S is an increasing sequence of sets. So, we have that $m^+ \in M$ and $M = \omega$ via induction. In other words, as long as $n \in m$ is true, then so must $S(n) \subseteq S(m)$ be as well. $\diamond$

$H(n) \cap H(m) = \varnothing$

$[S(h(n^+)) - S(h(n))] \cap [S(h(m^+)) - S(h(m))] = \varnothing$

if $n \neq m$:
$n \in m$ wlog

$n^+ \in m$

Show $n \in m$ implies $S(h(n^+)) \subseteq S(h(m))$.

boils down to showing $h(n) \subseteq h(m)$.

$n \in \ell_n$ lest $\ell_n \in n$
$\Rightarrow S(\ell_n) \subseteq S(n)$ Lemma 25A
contr.

$T = \{n \in \omega \mid \forall m \in \omega (m \in n \Rightarrow h(m) \subseteq h(n)\}$

$0 \in T$ ✓

$n \in T$ $\quad m \in n^+ \Rightarrow m \in n$ or $m = n$

Lemma 25A
$n \in m \Rightarrow S(n) \subseteq S(m)$

$T = \{ m \in \omega \mid (m \in n^+ \Rightarrow h(m) \in h(n^+)) \ \& \ (n^+ \in m \rightarrow h(n^+) \in h(m)) \}$

$0 \in n^+ \Rightarrow 0 \in T$ ✓

Assume $m \in n^+$,

$m^+ \in n^+$

$n^+ \in m$

$n^+ \in m^+$

$h(n^+) \in h(m) \in h(m^+)$

Ideas

Suppose $n \in m$

$h(n) \in h(m^+)$

Select least element l_n — unique $\Rightarrow H$ is single-valued

$H(0) = l$ of ran f

$H(n^+) = l_{H(n)}$

$F : \text{ran } f \rightarrow \text{ran } f$

$F(n) = l_n$

where

$l_n = l_m$

$n, m \in l_n = l_m$

$n \in l_m$

$n \in m \in l_m$ OR $m \in n \in l_m$

Not possible since l_m would not be the least element of S_m

Similarly for m,

$m \in n$

So, $m = n$.

$F : \mathcal{P}(\text{ran } f) \rightarrow \mathcal{P}(\text{ran } f)$

with $F(S_n) = \text{ran } f - S_n$

$S_{(S_n)} \subset S_n$

$S = \{ n \in \omega \mid (\forall m \in \omega)(n \in m \Rightarrow h(n) \in h(m)) \}$

$T = \{ m \in \omega \mid \ldots \quad 0 \in m \Rightarrow \quad \}$

$0 \in T$ ✓

Let $m \in T$, then $h(n) \in h(m) \in h(m^+)$

$h(n) \in h(m^+)$

Assume $n \in S$,

$T' = \{ m \in \omega \mid (\forall m \in \omega)(n^+ \in m \Rightarrow h(n^+) \in h(m^+)) \}$

for any $n \in m$, $h(n) \in h(m)$

$0 \in T$ ✓

Suppose $m \in T$ i.e. $h(n^+) \in h(m) \in h(m^+)$

$h(n^+) \in h(m^+)$

Therefore, $S = \omega$. That is, for all natural m and n, if $n \in m$, then $h(n) \in h(m)$. Hence, $h(n) \neq h(m)$.

For any naturals $n \neq m$, either $n \in m$ or $m \in n$

$\Longrightarrow$ So, $h(n) \neq h(m)$.

$\text{ran } f \preceq \omega$

trivially by the identity map into ω

~~$\#$~~

$n \in m$,

$n \in m, \in m_{(m,)}$

$n \in m_{(m)}$

So, h is not a function

Ideas

$N \subseteq \omega$ & $N \not\approx \omega$ tells us that N is finite (?)

~~Corollary 6P~~

$A \approx \omega \Rightarrow A \approx \omega \approx w$ ~~where~~ $w \subset \omega$) (in proof of Corollary 6D(b))

$\Rightarrow A \xrightarrow[\text{biject}]{f} \omega \xrightarrow[\text{biject}]{g} w$

$\Rightarrow w \xrightarrow[\cancel{\text{biject}}]{f^{-1}\upharpoonright w} \operatorname{ran}(f^{-1}\upharpoonright w)$

$\Rightarrow w \approx \operatorname{ran}(f^{-1}\upharpoonright w)$ & $w \approx A$

$\Rightarrow A \approx \operatorname{ran}(f^{-1}\upharpoonright w)$

$A - \operatorname{ran} g \subset A$

$h(0) =$

$A - \operatorname{ran} g \preccurlyeq A$ trivially

$f : A - \operatorname{ran} g \to A$

$f' : A \to A - \operatorname{ran} g$?

$g : \omega \to A$ (Not surjective)

No $g' : A \to \omega$ exists

~~$h : \omega \to A - \operatorname{ran} g \to A$~~

$h(0) = \quad \in A - \operatorname{ran} g$

$T = \{ m \in \omega \mid (m \in n^+ \Rightarrow h(m) \in h(n^+)) \;\&\; (n^+ \in m \Rightarrow h(n^+) \in h(m)) \}$

$0 \in n^+ \Rightarrow 0 \in T$ ✓

Assume $m \in n^+$,

$m^+ \in n^+$

$n^+ \in m$

$n^+ \in m^+$

$h(n^+) \in h(m) \in h(m^+)$

I deas

Suppose $n \in m$

$h(n) \in h(m^+)$

Select least element l_n — unique $\Rightarrow H$ is single-valued

$H(0) = l$ of ran f

$H(n^+) = l_{H(n)}$

$F : \text{ran } f \to \text{ran } f$

$F(n) = l_n$

where

$l_n = l_m$

$n, m \in l_n = l_m$

$n \in l_m$

$n \in m \in l_m$ OR $m \in n \in l_m$

Similarly for m,

$m \in n$

So, $m = n$.

Not possible since l_m would not be the least element of S_m.

$F : \mathcal{P}(\text{ran } f) \to \mathcal{P}(\text{ran } f)$

with $F(S_n) = \text{ran } f - S_n$

$S_{F(S_n)} \subset S_n$

$S = \{ n \in \omega \mid (\forall m \in \omega)(n \in m \Rightarrow h(n) \in h(m)) \}$

$T = \{ m \in \omega \mid \cdots \quad 0 \in m \Rightarrow \cdots \}$

$0 \in T$ ✓

Let $m \in T$, then $h(n) \in h(m) \in h(m^+)$

$h(n) \in h(m^+)$

Assume $n \in S$,

for any $n \in m$, $h(n) \in h(m)$

$T' = \{ m \in \omega \mid (\forall m \in \omega)(n^+ \in m \Rightarrow h(n^+) \in h(m^+)) \}$

$0 \in T$ ✓

Suppose $m \in T$. i.e. $h(n^+) \in h(m) \in h(m^+)$

$h(n^+) \in h(m^+)$

Therefore, $S = \omega$. That is, for all natural m and n, if $n \in m$, then $h(n) \in h(m)$. Hence, $h(n) \neq h(m)$.

For any naturals $n \neq m$, either $n \in m$ or $m \in n$

$\Rightarrow$ So, $h(n) \neq h(m)$.

$\text{ran } f \preceq \omega$

trivially by the identity map into ω

~~ht~~

$n \in m$,

$n \in m, \in m_{(m,n)}$

$n \in m_{(m,n)}$

So, h is not a function

Now let T' another subset of ω, but this time with the property that the natural numbers n it contains are such that $(\operatorname{ran} f) \cap S(n) \preceq n$.

Since $H(0)$ is disjoint from $S(0)$ by defn., $f(0) \notin S(0)$. $0 \in \overline{T}$ Presume $n \in \overline{T}$. $f(n^+) \notin S(n^+)$, so $f(n^+) \notin S(0)$ — which is a subset of $S(n^+)$ $(\operatorname{ran} f) \cap S(n) = \varnothing$

$n^+ \in \overline{T} \implies \overline{T} = \omega$, $0 \in T'$. When $n \in T'$, $(\operatorname{ran} f) \cap S(n) \preceq n$.

$\to n$

$(\operatorname{ran} f) \cap S(n^+)$ $\qquad \{n \in \omega \mid \text{exists } m \text{ with } f(m) \in S(n)\}$

$n \in l_n$

$n^+ \in l_n$

$f(m) \in S(n)$ for all $m \in n$.

If $n \in m$, then $f(m) \notin S(n)$ $\quad$ as $(f(m) \notin S(m)$ where $S(n) \subseteq S(m))$

$n \in m$

$S(n) \subseteq S(m)$

$f(m) \in S(n) \implies m \in n$

$\{m \in \omega \mid f(m) \in S(n)\} \subseteq n$

$\approx \overline{n} \subseteq n$

$(\operatorname{ran} f) \cap S(n) \approx \overline{n} \implies$ finite

never infinite $\qquad$ Contradiction..

15. Ideas

Sequence S so $S(n) \subseteq S(n^+)$ ☆ (increasing sequence of sets)

Then $\underbrace{B \subseteq \bigcup_{n\in\omega} S(n)}_{\text{ran } S}$ & (for any infinite $B' \subseteq B$, there exists n so $B' \cap S(n)$ infinite) implies $B \subseteq S(n)$.

☆

Assume that S is an increasing sequence of sets with domain ω, as well that B is a subset of the union $\bigcup_{n\in\omega} S(n)$ such that for every infinite subset B' of B, there is some n for which $B' \cap S(n)$ is infinite. (Show $B \subseteq S(n)$!) $\Rightarrow$ If B is infinite, $B \cap S(n)$ is infinite.

$\nearrow B \not\subseteq$ any $S(n)$ / for any $n \in \omega$, there exists some b in B that is not in $S(n)$ (but which is in some other $m \in \omega$, since $B \subseteq \bigcup_{n\in\omega} S(n)$)

Suppose otherwise.

$$T(n) = B - S(n) \neq \emptyset$$

ω & 2ω

Let T be the subset of ω so that it contains only the natural numbers n so ran $f \neq n$.

ran $f \neq \emptyset$ hence ran $f \neq \emptyset$ (i.e. ran $f \neq 0$).

Suppose $n \in T$. This tells us that ran $f \neq n$.

ran $f > n$

is injective g mapping n into ran f that is never surjective. $\Rightarrow$ (ran f) $-$ (ran g) $\neq \emptyset$ e'

$g \cup \{\langle n, e\rangle\}$; $e = f(k) \in B \cap S(l_k)$

$f(l_k) \in B \cap S(l_{l_k})$

$e' \in B - S(l_k)$

$\Rightarrow e \neq e'$

$T'(n) = \{ b \in B - S(n) \mid \forall m \forall k [(b \in S(m) \ \& \ k \in m) \Rightarrow b \notin S(k)]\}$

$N_n = \{ m \in \omega \mid (\exists b \in B - S(n))\ b \in S(m)\} \neq \emptyset$

By Well-Ordering Principle, there is some unique least element l_n of N_n:

(For this least l_n, there is some $b_n \in S(l_n)$ that is in $B - S(n)$.)

$h : \omega \to \mathcal{P}B$

$h(n) = B \cap S(l_n) \neq \emptyset$ / $h_n = B \cap S(l_n)$

By second form (1) of the Axiom of Choice,

there exists a function f of domain ω so $f(n) \in h(n)$.

ran $f > n$

We see that ran $g \subset$ ran f.

$n^+ \in T$ By induction, $T = \omega$.

☆ Change to $h(n) = (B - S(n)) \cap S(l_n)$

Since ran f is not equinumerous to any natural number, it is infinite.

Theorem 6N

$\omega \approx A$ trivial $\omega \not\approx A \Rightarrow \omega \prec A$ or $A \prec \omega$

IDEAS

$f : A \to \omega$ injective but not surjective $\operatorname{ran}(f) \prec \omega$ $\operatorname{ran} f \not\approx \omega$

$\operatorname{card}(A) < \operatorname{card}(\omega)$ lest there is a bijection g from $\operatorname{ran} f$ to ω, and hence, $g \circ f$ is a bijection from A to $\omega \Rightarrow$ Contradiction!

Self-Proof

(a) Let A be an infinite set. In the case that $\omega \approx A$, $\omega \preceq A$ trivially holds. Therefore, now consider $\omega \not\approx A$. Now, by the fifth form of the Axiom of Choice, we have that either $\omega \prec A$ and $A \prec \omega$. When the latter is true, there must exist an injection f mapping A into ω, that is never surjective. We see that $\operatorname{ran} f$ is never equinumerous to ω, lest there is a bijection g from $\operatorname{ran} f$ into ω. Which implies $g \circ f$ is a bijection that maps A to ω thus contradicting our prior assumption that $\omega \not\approx A$. Therefore, there is some largest member k of $\operatorname{ran} f$. Otherwise, for every $n \in \operatorname{ran} f$, there is the nonempty subset S_n of $\operatorname{ran} f$ containing just the members (of $\operatorname{ran} f$) strictly larger than n. By the well-ordering of ω, there is some least element l of $\operatorname{ran} f$ and l_n for each S_n. Notice that these least elements are unique. Hence, we define the function $H : \omega \to \operatorname{ran} f$ given by

$$H(0) = l,$$

$$H(n^+) = l_{H(n)}.$$

We claim that H is injective. First suppose that $l_n = l_m$. Then, $n \in l_m$. It follows that either $n \in m \in l_m$ or $m \in n \in l_m$. The latter is impossible since it contradicts l_m being the smallest member of $\operatorname{ran} f$ that is larger than m. So, it must be that $n \in m$. Repeating the same procedure, we have that $m \in n$. Resultantly, $m = n$, just like we wanted. By induction, it is now easy to show that $H(n) = H(m)$ implies $n = m$. Indeed, we find that H is an injection from ω to $\operatorname{ran} f$, i.e. $\omega \preceq \operatorname{ran} f$.

Proof Continues on The Next Page.

$l = l_m \Rightarrow 0 = m$

If $n \in S$, $l_{H(n)} = l_{H(m)}$, $H(n) = H(m)$, $n = m$

$1 \rangle 3 — 5 — \dots$
 2

Ideas

~~$W \preceq \omega$~~

$W \prec \omega$ $W \not\approx \omega$

~~Show $A \approx n$: Must exist largest member k of $\operatorname{ran} f$, lest for every $n \in \operatorname{ran} f$, there exists a $M_n \in \operatorname{ran} f$ so that / with $n \in M_n$. However, then we have the function $h : \omega \to \operatorname{ran} f$ given by~~

(i.e. $n \in k$ for all $n \in \operatorname{ran} f$)

$S = \{m \in \omega \mid 0 \in m \to h(0) \in h(m)\}$

$0 \checkmark$ If $m \in S$, $h(0) \in h(m) \in h(m^+)$.

$T = \{m \in \omega \mid n^+ \neq m \Rightarrow h(n^+) \neq h(m)\}$

$0 \in n^+$ $h(n^+) \neq h(0)$ by previous part

$m \in n^+$ $m^+ = n^+$ or $m^+ \in n^+$

$h : \omega \to \operatorname{ran} f$

$h(0) = m$

$h(n^+) = M_n$

$h(0) = N$

$h(n^+) = \operatorname{ran} f - \{h(n)\}$

$F : \operatorname{ran} f \to \mathcal{P}(\operatorname{ran} f)$

~~$F(n) = \operatorname{ran} f - \{n\}$~~

In addition, there is also the identity map from $\operatorname{ran} f$ into ω, which is injective. Thence, as $\omega \preccurlyeq \operatorname{ran} f$ and $\operatorname{ran} f \preccurlyeq \omega$ simultaenously, we have that $\omega \approx \operatorname{ran} f$ by the Schröder-Beinstein Theorem. However, this is in clear contradiction with the fact we established earlier, that $\omega \not\approx \operatorname{ran} f$. This entails that our earlier assumption of there not existing a largest element of $\operatorname{ran} f$ must have been false. That is, there certainly exists a largest element k of $\operatorname{ran} f$. Accordingly, $\operatorname{ran} f \subseteq k^+$ because $n \leq k \in k^+$ for any $n \in \operatorname{ran} f$. By Lemma 6F, $\operatorname{ran} f \approx \bar{m}$ for some $\bar{m} \in k$. In other words, $\operatorname{ran} f$, and whence A, is finite. But this creates a contradiction with the assumption given at the beginning of the proof, that A is an infinite set. Consequently, $A < \omega$ must be impossible. Which means $\omega < A$ for any infinite set A not equinumerous to ω. Wherefore, $\omega \preccurlyeq A$ is guarenteed as long as A is infinite. $\square$

$\square$

(b) This is simply a rephrasing of (a) in terms of cardinal numbers.

Exercise / Proof:

19\. Let the subset S of ω contain (only) the natural numbers n so that for any function H with finite domain I equinumerous to n and $H(i)$ nonempty for each $i \in I$, there exists the function f with domain I such that $f(i) \in H(i)$ for every $i \in I$. First consider $I \approx 0$, then $I = \varnothing$ and any function H with finite domain I must be the empty function as well. Vacuously, $\varnothing$ is also the function with domain $I = \varnothing$ in so that $f(i) \in H(i)$. Hence, $0 \in S$. Now consider $n \in S$ being true. Given some function H having a finite domain $I \approx n^+$ and $H(i) \neq \varnothing$ for each $i \in I$, we first notice that there is a bijection g from n^+ into I. And by our assumption that $n \in S$, there is the function f of domain $g[\![n]\!]$ in the way that $f(i) \in H(i)$ for all $i \in g[\![n]\!] \subseteq I$. Since $H(g(n)) \neq \varnothing$, select[1] $h \in H(g(n))$, of which must exist at least one. Then take $F = f \cup \{\langle g(n), h \rangle\}$. Clearly, F has domain I and $F(i) \in H(i)$ so long as $i \in I$. Therefore, $n^+ \in S$ (provided that $n \in S$). Consequently, by induction, $S = \omega$. Wherefore, H being a function of finite domain I and $H(i)$ being nonempty for each $i \in I$ implies the existence of some function f having domain I together with the property that $f(i) \in H(i)$ given $i \in I$. $\square$

Oops accidentally wrote out the formal proof here instead of on the other paper

[1] The choice of h is insignificant because for distinct h, we will simply get multiple distinct functions F. However, the key is that these F are still functions

---

Ideas (19.)

Assume $n \in S$.

H has domain $I \overset{g}{\approx} n^+$ & $H(i) \neq \varnothing$ for each $i \in I$

is function f of domain $g[\![n]\!]$ in the way that $f(i) \in H(i)$ for each $i \in g[\![n]\!] \subseteq I$

Since $H(g(n)) \neq \varnothing$, select $h \in H(g(n))$ (the choice of h doesn't matter)[1]. By ~~Axiom of Pairing, there is the set~~ Take $f \cup \{\langle g(n), h \rangle\}$.

[1] As for distinct h, we will simply get multiple different resultant functions. BUT, each is still a function!

Ideas

18. (3) ⟹ (7)

X is a nonempty subset of $\bigcup \mathcal{A}$.

Let A be the set of nonempty subsets of $\bigcup \mathcal{A}$.

By (3), there is a function f with domain A so that $F(X) \in X$ for any $X \in A$.

Now take $F \restriction \mathcal{A}$.

Clearly, $F \restriction \mathcal{A}$ is a function of domain $\mathcal{A}$ such that $f(X) \in X$ for all $X \in \mathcal{A}$.

(7) ⟹ (3) Find function F with domain A' and $F(B) \in B \subseteq A$.

Let A be a set, and A' be the set of nonempty subsets of A.

Apply (7) on A',

is function f with domain A' such that $f(X) \in X$ for all $X \in A'$. Doing a simple rephrasing, we see that the domain of f is the set of nonempty subsets of A, with $f(B) \in B$ for every nonempty $B \subseteq A$.

Ideas

20. $f : \omega \to A$ $\quad f(n^+)$ is the y so that $y R f(n)$

Find some $F : A \to A$ so $\underbrace{F(x) = y}$ for every $x \in A$

$F : A \to \operatorname{ran} R \quad y R x$

Wait. A may be finite. In which case $f(m) = f(k)$ for some naturals $m \neq k$.
$A \approx n$

$f(0) =$

$f(n^+) = F(f(n))$

Subset Axiom on A

for all $x \in A$, there is the nonempty set Y_x containing all $y \in A$ so $y R x$.

$H : A \to \mathcal{P}A$

$H(x) = Y_x$

By form (2) of the Axiom of Choice,
There exists the function h with domain A and $h(x) \in H(x)$.
$x R h(x)$

h helps us to pick a single y for every x.
Notice $h(x) \in A$ (Since $h(x) \in Y_x \subseteq A$) / $h : A \to A$

Pick any element a of A (which must exist since A is nonempty)

By recursion then, there is the func $f : \omega \to A$

with $\quad f(0) = a$

$f(n^+) = h(f(n))$.

- ω

the p

per

cti

~~# Self Proof of Corollary 6P~~

~~Let A be an infinite set. First notice that in the proof of Corollary 6D (b), we have shown ω to be equinumerous to the proper subset $\omega - \{0\}$ of itself. In addition, suppose that $A \approx \omega$; meaning a bijection f from ω into A exists. Therefore, notice that $f \restriction (\omega - \{0\})$ bijects $\omega - \{0\}$ into the proper subset $f[\![\omega - \{0\}]\!]$ of A; Furthermore, since $A \approx \omega \approx \omega - \{0\}$, we know that $\omega - \{0\} \approx A$. So, $f[\![\omega - \{0\}]\!]$ is equinumerous to (its proper superset) A. In other words, $f[\![\omega - \{0\}]\!] \approx \omega - \{0\}$~~

~~Now consider the other case where $A \not\approx \omega$. That is, $\omega \prec A$. Thus, there is the injective map $g : \omega \to A$ that never is surjective.~~

Exercises

18. First assume that $\mathscr{A}$ is a set whose members are nonempty sets and A be the set of nonempty subsets of $\bigcup \mathscr{A}$. By the third form of the Axiom of Choice, there is a function F with domain A so that $F(X) \in X$ for any nonempty $X \subseteq \bigcup \mathscr{A}$ (i.e. $X \in A$). Now take $F \restriction \mathscr{A}$. Clearly, $F \restriction \mathscr{A}$ is a function of domain $\mathscr{A}$ such that $f(X) \in X$ for all $X \in \mathscr{A}$. (Notice that $X \in \mathscr{A}$ is always a nonempty subset of $\bigcup \mathscr{A}$.)
In other words, we have shown that if form (3) of the Axiom of Choice is true, then so is the statement mentioned in 18.

Conversely, suppose that A is a set and $\mathscr{A}$ the set of nonempty subsets of A. Presuming the statement mentioned in the question (from here on referred to as (7)) is true, we apply it on $\mathscr{A}$. Thus, there is the function f with domain $\mathscr{A}$ in such a way that $f(X) \in X$ for all $X \in \mathscr{A}$. Doing a simple rephrasing, we see that the domain of f is the set of nonempty subsets of A, with $f(B) \in B$ for every nonempty $B \subseteq A$. Consequently, the validity of (3) is guaranteed by (7).

Wherefore, (7) is an equivalent statement of the Axiom of Choice. $\square$

20. Suppose that A is a nonempty set and R is a relation such that $(\forall x \in A)(\exists x \in A)\, yRx$. Then, for all $x \in A$, there is the nonempty subset Y_x of A containing all $y \in A$ so yRx (and those elements only). Thus, there is the function $H : A \to \mathcal{P}A$ defined by $H(x) = Y_x$. Now, utilizing the second form of the Axiom of Choice, there exits the function h with domain A and $h(x) \in H(x)$ as long as $x \in A$. Notice that $h(x) \in A$ since $Y_x \subseteq A$, i.e. h is a function mapping A into A. Select[1] any element a of A (of which there is at least one, as $A \neq \varnothing$). By the recursion theorem for ω, we have the function $f : \omega \to A$ with both

$$f(0) = a, \text{ and}$$
$$f(n^+) = h(f(n)).$$

$\square$

By definition, it now holds that the function $f : \omega \to A$ has the property that $f(n^+) R f(n)$, because $f(n^+) \in Y_{f(n)}$.

[1] Every distinct choice of $a \in A$ gives us a unique function f_a with $f_a(0) = a$. However, each of these f_a are, notably, still functions.

(6) $\Rightarrow$ (2)

Define the function $h_J : J \to \bigcup_{i \in I} H(i)$ by $h(i) \in H(i)$ for any $i \in J \subseteq I$. Let S be the set of all such functions h_J and T be a chain in S. It must be that $\bigcup T \in S$, lest at least one of the three below cases hold:

i) $\mathrm{dom}(\bigcup T) \not\subseteq I$. This is not possible as this is a union of subsets of I.

ii) $(\bigcup T)(i) \notin H(i)$. So, $h_J(i) \notin H(i)$ for some $i \in J \subseteq S$. Clearly, this contradicts the construction of such functions h_J.

iii) $\bigcup T$ is not single-valued. We see that there must be h_J and h_K in T such that $h_J(i) \neq h_K(i)$ for some $i \in J \cap K$. But since T is a chain, $h_J \subseteq h_K$ (or the other way around), which tells us that $h_J(i) = h_K(i)$ is certain.

Now, $\bigcup T \in S$ indeed. By Zorn's Lemma (6), there is a maximal element $f \in S$. All's that is left is to show $\mathrm{dom}\, f = I$.

Suppose for the sake of contradiction that, instead, $\mathrm{dom}\, f - I = \emptyset$ and let $k \in \mathrm{ran}\, f - I$. But then $f \cup \{\langle k, e\rangle\} \in S$ for every $e \in H(k)$, so that $f \subset f \cup \{\langle k, e\rangle\}$. Contradicting the fact that f is maximal in S. Hence, $\mathrm{dom}\, f = I$ holds true. By the construction of S, $f(i) \in H(i)$ for all $i \in I$.

Exercise 21 of Ch 7 $\Rightarrow$ (6)

This is simply a special case of exercise 21 where the partial ordering $<$ is given by the subset relation $\subset$.

Consequently, since (2) was used to prove the validity of exercise 21, (2) suffices to show (6).

Wherefore, (2) is indeed equivalent to (6).

(5) $\Rightarrow$ The Well-ordering Theorem / Numeration Theorem

By Hartogs' we know there exists some ordinal α not dominated by A. By cardinal comparability (5), there must hence be an injection $f : A \to \alpha$ so that $\alpha \not\preceq A$. Let β be the ordinal number of $\langle \operatorname{ran} f, \in_\alpha \rangle$. Then $A \approx \beta$ $\Diamond$

Consequently, we have proved that the Numeration Theorem is equivalent to Cardinal Comparability.

The Well-ordering Theorem / Numeration Theorem $\Rightarrow$ (3)

By the Numeration Theorem, $A \approx \alpha$ for some ordinal number α, with the bijection provided by a $f : A \to \alpha$. Now define the function G from the set of nonempty sublets B of A to A with

$$G(B) = f^{-1}(\text{the least element of } f[\![B]\!]).$$

Since $F[\![B]\!] \neq \emptyset$ for any nonempty subset B of A and least elements are unique, G is certainly well-defined. Indeed, $G(B) \in B$ because $f(G(B)) = f(\text{the least element of } f[\![B]\!]) \in f[\![B]\!]$, so $f(G(B)) = f(b)$ for some $b \in B$, and by the injectivity of f, $G(B) = b \in$

Notice that we used (2) to prove the Well-Ordering Theorem, and hence, the Numeration Theorem. That is, (2) $\Rightarrow$ the Numeration Theorem. Furthermore, the Numeration Theorem suffices to show (3) is true. Wherefore, since

1. (2) is equivalent to (3),

2. The Numeration Theorem is equivalent to Cardinal Comparability,

so Cardinal Comparability is equivalent to (2).

The Well-Ordering Theorem / Numeration Theorem $\Rightarrow$ (5)

If $C \preccurlyeq D$, the result is immediate. Here consider $\gamma \approx C \not\preccurlyeq D \approx \delta$ where the existance of such ordinals γ and δ are guaranteed by the Numeration Theorem. Now, Theorem 7K tells us that there are two cases worth considering

Case 1

$\langle \gamma, \in_\gamma \rangle \cong \langle \text{seg } \eta, \mathring{\in}_\delta \rangle$. Then $\gamma \approx \text{seg } \eta \preccurlyeq \delta$, but this contradicts our initial assumption. In other words, this is not possible.

Case 2

$\langle \text{seg } \lambda, \mathring{\in}_\lambda \rangle \cong \langle \delta, \mathring{\in}_\delta \rangle$. We see that $\gamma \succcurlyeq \text{seg } \lambda \approx \delta$. So, $C \succcurlyeq D$ indeed. $\square$

(6) $\Rightarrow$ (1)

Rough idea Idea

$h_J : J \to \bigcup_{i \in I} H(i)$, $J \subseteq I$, $h(i) \in H(i)$ for any $i \in J$

We claim the set S of all such h_J satisfies the hypothesis of (5); assume $T \subseteq S$ is a chain.

$\bigcup T \in S$, lest $\underbrace{\mathrm{dom}(\bigcup T) \not\subseteq I}_{\text{Not possible as this is a 'union' of subsets of } I.}$ or $\underbrace{(\bigcup T)(i) \notin H(i)}$ or $\underbrace{\bigcup T \text{ not single-valued}}$

$h_J(i) \notin H(i)$ for some $i \in J \subseteq I$. Not possible by construction of S and hence T.

h_J and h_K in T s.t. $h_J(i) \neq h_K(i)$ for some $i \in J \cap K$. But $h_J \subseteq h_K$ (or the other way around).

$\bigcup T \in S$ indeed

By (6), there is a maximal element $f \in S$. All that's left is to show $\mathrm{dom}\, f = I$. Suppose ttsoc that $\mathrm{dom}\, f - I =$ and $k \in \mathrm{dom}\, f - I$. But then $f \cup \{\langle k, e \rangle\}$ $\in S$ for any $e \in H(k)$ is a superset of f. Contradicting the fact that f is maximal S. Hence, $\mathrm{dom}\, f = I$. By the construction of S, $f(i) \in H(i)$ for all $i \in I$.

Continuation of self proof of theorem 7M

Ideas

$AC \Rightarrow (5)$

the well-ordering Theorem!

By AC, $C \approx \gamma$ and $D \approx \delta$

If $C \not\preccurlyeq D$, no injective map $f: C \to D$ exists

$g: \gamma \to \delta$

Theorem 7K, tells us that

Either $\langle \gamma, \in_\gamma \rangle \cong \langle \text{seg}\, \eta, \in_\delta^\circ \rangle \Rightarrow \gamma \approx \text{seg}\, \eta \preccurlyeq \delta \Rightarrow C \preccurlyeq D$ cont.

or $\langle \text{seg}\, \lambda, \in_\gamma^\circ \rangle \cong \langle \delta, \in_\delta \rangle \Rightarrow \gamma \succcurlyeq \text{seg}\, \lambda \approx \delta \Rightarrow D \preccurlyeq C$

$AC \Rightarrow (6)$

Special case of ZL exercise in Ch 7.

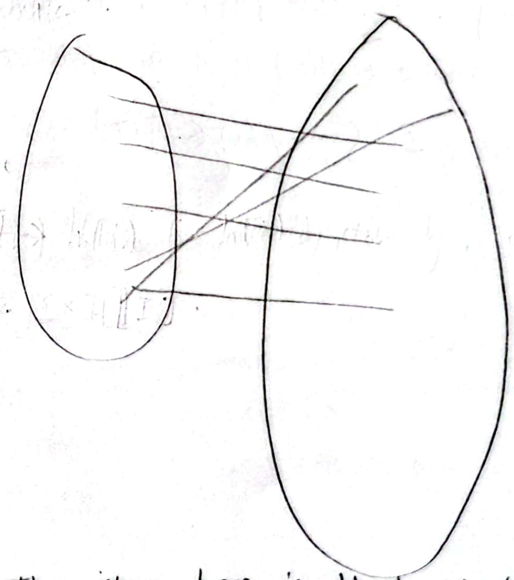
$(5) \Rightarrow AC$

Either exists ① injection $f: \mathscr{A} \to \bigcup\mathscr{A}$, or

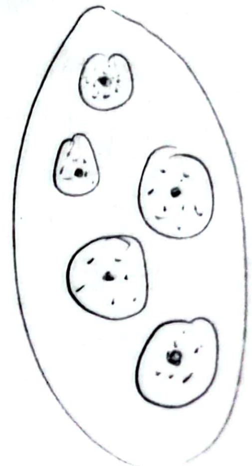
② injection $g: \bigcup\mathscr{A} \to \mathscr{A}$

Case 2: If g is not bijective, then

(5)



(3)



The issue here is that injections (specified by cardinal comparability) doesn't (immediately) allow us to specify our elements chosen.

Proof

(1) $\Leftrightarrow$ (2)

Assume that H is a function with domain I with $H(i) \neq \emptyset$ for any $i \in I$. Now, let the relation R be the subset of $I \times \bigcup_{i \in I} H(i)$ defined by iRj implies $j \in H(i)$. Notice that since $H(i)$ is always nonempty, so there always exists a $j \in H(i)$ with iRj for any $i \in I$. In other words, R has domain I. Clearly, by applying form 1 of the Axiom of Choice, we have that there exists a function F with domain I and $F(i) \in H(i)$ — because $j = F(i)$ means iRj, and hence $F(i) \in H(i)$. Consequently, we have shown that the first form of the Axiom of Choice implies the second.

Conversely, suppose that R is a relation. We observe that for any $i \in \text{dom } R$, there exists the subset $H(i)$ of the range of R containing (only) the elements j with iRj. It is certain that $H(i)$ must be nonempty, because by definition; for all $i \in \text{dom } R$, there exists some j with iRj. Now, we can define a function H with equivalent domain as R and with $H(i)$ as described above. Then, utillising the second form of the Axiom of Choice, we deduce that there is a function f with $\text{dom } f = \text{dom } R$, specifically one where $f(i) \in H(i)$. Therefore, the function f is a subset of R — as if $j = f(i) \in H(i)$, then iRj by definition. As a result, it is proven that if the second form of the Axiom of Choice is true, then so is the first form.

Wherefore, we conclude that the first form of the Axiom of Choice is, in fact, equivalent to its second form.

(2) $\Leftrightarrow$ (3)

Let A be a set and A' be the subset of A containing (only) the nonempty subsets of A. We see that $I_{A'}$ is a function with domain A' where $I_{A'}(B)$ is always nonempty. Hence, we apply the second form of the Axiom of Choice, from which we know that there exists a function f with domain A' and $f(B) \in I_{A'}(B) = B$ for any $B \in A'$. In other words, any nonempty $B \subseteq A$. Consequently, we see that f is our choice function for A, and so, the third form of the Axiom of Choice is guarenteed to hold as long as the second does.

Conversely, presume that H is a function with domain I in such a way that $H(i) \neq \emptyset$ for any $i \in I$. By form 3 of the Axiom of Choice, we know that for the set $\bigcup_{i \in I} H(i)$, there is some choice function F with its domain being the set of all nonempty subsets of $\bigcup_{i \in I} H(i)$ and $F(\bar{H}) \in \bar{H}$. Notice that $(F \circ H)(i) = F(H(i)) \in H(i)$ for any $i \in I$. Whence, we see that $F \circ H$ is the function we are looking for to satisfy form 2 of the Axiom of Choice. In other words, the third form of the Axiom of Choice, being true, tells us that the second must also be true.

Wherefore, the second and third forms of the Axiom of Choice are equivalent.

(3) $\Longleftrightarrow$ (4)

(3) $\Rightarrow$ (4)

Assume that $\mathcal{A}$ is a nonempty set whose elements are also nonempty and disjoint. By use of the third form of the Axiom of Choice, there is the function F with its domain being the set of nonempty subsets of $\bigcup\mathcal{A}$ and $F(B) \in B$ for every nonempty $B \subseteq A$. Then, we see that $\operatorname{ran}(F \restriction \mathcal{A}) \cap B = \{F(B)\}$ for any $B \in \mathcal{A}$, because for any $B' \neq B$ that is in $\mathcal{A}$, $F(B') \notin B$ by virtue of the members of $\mathcal{A}$ being pairwise disjoint. Thus, we see that $\operatorname{ran}(F \restriction \mathcal{A})$ is the set containing exactly one element from each member of $\mathcal{A}$ that we are looking for. Therefore, if the third form of the Axiom of Choice is true, then so must the fourth be.

(4) $\Rightarrow$ (3)

Let A be a set. ~~In the case that $A = \varnothing$, A itself is already such a function trivially. Hence, we shall now consider A being nonempty~~ Thus, we apply form (4) of the Axiom of Choice on $\mathcal{A} = \{\{B\} \times B \mid B \subseteq A \ \& \ B \neq \varnothing\}$, the set of all $\{B\} \times B$ where B is a nonempty subset of A, to form the set C with $C \cap (\{B\} \times B)$ being a singleton set. Which is allowed since if B and B' are distinct nonempty subsets of A, then $(\{B\} \times B) \cap (\{B'\} \times B') = \varnothing$ because the 'first coordinates' are always the distinct sets B and B'. Take the set $C' = C \cap \mathcal{A}$ now. Clearly, C' is a function with the property that $C'(B) \in B$. And the domain of C' must be the set A' of nonempty subsets of A, by virtue of $C' \cap (\{B\} \times B)$ being nonempty (i.e. $A' \subseteq \operatorname{dom} C'$) and $C' = C \cap \mathcal{A}$ (i.e. $\operatorname{dom} C' \subseteq A'$). Consequently, C' is the function we are looking for to satisfy the (3)rd form of the Axiom of Choice. In other words, we have shown that (3) is a consequence of (4).

Wherefore, statements (3) and (4) must be equivalent varients of the Axiom of Choice.

Oh oops! Should be $C \cap (\bigcup\mathcal{A})$.

Proof of Theorem 6M

Ideas (1)$\Leftrightarrow$(2)

($\Rightarrow$) let $R = I \times \bigcup_{i \in I} X(i)$

$R' \subseteq R$ with iRx implies $x \in X(i)$

($\Leftarrow$) If $R = \emptyset$, then R itself is clearly already a function with dom R.

$$\prod_{i \in I} X(i) := \left\{ f : I \to \bigcup_{i \in I} X(i) \;\middle|\; (\forall i \in I)[f(i) \in X(i)] \right\}$$

$R' \subseteq R$ defined by $R' = \{$

Define the function $H = \text{dom } R \times \{\text{ran } f$

$H' \subseteq H$

s.t.

$\cancel{H'(x) = y}$ iff

x

$R = I \times \bigcup_{i \in I} X(i)$

$R = \text{dom } R \times \bigcup_{i \in \text{dom } R} \text{ran } R$

$=$

$\bigcup_{i \in I} H(i) = \bigcup_{i \in I} \text{ran} =$

$= \text{ran } f$

$\prod_{i \in \text{dom } R} H(i) = \{ f : \text{dom } R \to \text{ran } f \mid$

$(\forall i \in \text{dom } R)(f(i) \in H(i) = \text{ran}$

$f(i) \in R$

For all $i \in \text{dom } R$, of which there exists at least one since $\text{dom } R \neq \emptyset$, there exists the set

$$H(i) = \{ y \in \text{ran } R \mid iRy \}$$

I Uets

Equivalency with (3)

dom $F = \{x \in \mathcal{P}A \mid x \neq \varnothing\} \subseteq \mathcal{P}A$

$(2) \Rightarrow (3)$

Let A be a set and A' be the subset of A containing (only) the nonempty subsets of A.

function H with domain A' defined by $H(B) = B$ $\quad H = \{\langle x, y\rangle \in A' \times A' \mid x = y\}$

~~H be the subset of $A' \times A'$ with BHB / $H(B) = B$.~~ $\quad I_{A'}$ //

There exists a function f with domain A' and $f(B) \in \underset{I_{A'}(B)}{H(B)} = B \subseteq A$

$(3) \Rightarrow (2)$

H with domain I and $\underset{B}{H(i)}$ always nonempty

$\bigcup_{i \in I} H(i)$ is

$i \in I \Rightarrow i \neq \varnothing$ & $i \subseteq A$

By (3), for the set $\bigcup_{i \in I} H(i)$, there exists a choice function F with its domain being the set of nonempty subsets of A and $F(\overline{H}) = \overline{H}$.

$\therefore F \circ H$ is the f we are looking for.

$(F \circ H)(i) = F(H(i)) \in H(i)$

(1) Ideas

(3) ⇒ (4)

Assume the set A is such that all its elements are nonempty and pairwise disjoint. $\bigcup A$

By third form, there is a choice func. F with dom $F = \cdots$ & $F(B) \in B$

$F(B) \in ran(F \upharpoonright A) \cap B$

$F(B') \notin ran(F \upharpoonright A) \cap B$

$ran(F \upharpoonright A)$

Since pairwise disjoint, for any $B \neq B'$ (in A), $F(B) \in B$ is not in B'. i.e. $ran(F \upharpoonright A) \cap B = \{F(B)\}$.

(4) ⇒ (3)

Let A be a set. A' set of nonempty subsets of A

By fourth form, exists $\underbrace{C}_{ran\ F}$ with one element from each member of A

Must be function because, if $B = B'$, then $B \cap C = B' \cap C = \{x\}$. So, $F(B) = \{x\} = F(B')$.

Take $F \subseteq A' \times C$ with $F(B) \in B$ / $BFB' \Rightarrow B' \in B$

Since for any $B \in A'$, $B \cap C = \{x\}$

~~$C \in B \in A$~~

$C = \{c\}$

$F \subseteq A' \times \bigcup A'$ with $\underbrace{F \upharpoonright \{B\} = S_B}_{\text{OR: } F(B) = c_B}$?

~~All~~ $S_B = \langle B, c_B \rangle \in A' \times \bigcup A'$

For any nonempty $B' \subseteq A$, there exists the set $\{B'\}$ by the Axiom of Pairing. By fourth form of the Axiom of Choice, there exists the set $\{c\}$ with $c \in B$ (as long as $A \neq \emptyset$).

Take the ~~function~~ relation R ~~F~~ $\subseteq A' \times \bigcup A'$ defined by BRc implying $c \in B'$ ~~$F(B') \in B$~~.

Clearly, for any B', since we know there exists a c with $F(B') = c \in B'$, so $B' \in \text{dom } F$.

Not necessarily a function yet.

~~$A' \to A$~~

To apply (4) to a ~~set~~ A, we need $A \neq \emptyset$ and all elements of A to be nonempty and pairwise disjoint.

C with $C \cap B = \{x\} \rightarrow x \notin B \cap B' = \emptyset$

dom $F = A'$ (set of nonempty subsets of A) & $\underbrace{F(B) \in B}_{\text{Possible for } F(B) \in B \cap B'}$

Construct a partition of A $\quad (A/=_A) = \{\{a\} \mid a \in A\}$

C is just A.

$$\{\underset{\underset{\mathcal{P}A}{\cap}}{\overset{\mathcal{P}\mathcal{P}A}{\cup}}{\{B\}} \times \overset{\mathcal{P}A}{\cup}{B} \mid B \subseteq A \ \& \ B \neq \emptyset\} \neq \emptyset \checkmark$$

Pairwise disjoint ✓

$B \neq \emptyset$ ✓

By (4);

(6)

Result: exists $m \in M$ with $M \not\leq a$ for any $a \in \mathcal{A}$

~~If there exists some element $M \in \mathcal{A}$ that is not in any chain $\underbrace{\mathcal{B} \subseteq A}$, then M is immediately the a Maximal element.~~

~~Eg: $\mathcal{B} \cup \{M\}$~~

~~where $\mathcal{B}$ is already a chain such as $\emptyset \subseteq \mathcal{A}$.~~

No AoC needed

For all $a \in \mathcal{A}$, $\{a\}$ is a chain.

i.e. All elements $a \in \mathcal{A}$ is in some chain $\mathcal{B} \subseteq \mathcal{A}$.

The maximal element $\underset{\hat{\mathcal{B}}}{M}$ we are looking for is so that $b \subset M$ for all $b \in \mathcal{B}$.

$b_1 \leq b_2 \leq \cdots$

$b_1 \cup b_2 \cup b_3 \cup \cdots \in \mathcal{A}$

$M \nsubseteq a$

$a \cap M = \emptyset$

or $a \cap M \neq \emptyset \Rightarrow a \subset M$

(5)

(5) $\Rightarrow$ (1)

Suppose R is a relation. We know that $R \subseteq (\text{dom } R) \times (\text{ran } R)$. There is an injective function $f: \text{dom } R \to \text{ran } R$ or $f: \text{ran } R \to \text{dom } R$ (form 5 of AoC). Clearly, $f \subseteq \text{dom } R \times \text{ran } R$.

show $f \subseteq R$

$A \longrightarrow \bigcup A$

If no subset of R is single-rooted, there cannot exist an injective function $F \subseteq R$ with $\text{dom } F = \text{dom } R$. (with domain dom R)

$x R F(x)$

OR $f: \text{dom } R \longrightarrow \text{ran } R$

$f: \text{ran } R \longrightarrow \text{dom } R$

(5) $\Rightarrow$ (2)

Let H be a function with domain I so that for any $i \in I$, $H(i) \neq \emptyset$.

$5) \Rightarrow (4)$

$f : A \longrightarrow \bigcup A$ $\quad \langle B, b\rangle \in \{B\} \times b$

$f(B) \in B \quad = \quad g : \{B\} \longrightarrow B$

ran $f = C$

$f(B) = x$ OR $f(B) = \{x\}$

$A \xrightarrow{f} \bigcup A$

$A \xrightarrow{f} \mathcal{P}(\bigcup A)$

Can't directly apply (5).

$\{\{1,1\}, \{2,1\}\}$ $\left\{\begin{matrix} \{1,1\} & \{1,2\} \\ \{2,1\} & \{2,2\} \\ \{3,1\} & \{3,2\} \end{matrix}\right]$

$\{\{1,1\}, \{1,2\}\}$ $\left\{\begin{matrix} \{1,1\} & \{1,2\} & \{1,3\} \\ \{2,1\} & \{2,2\} & \{2,3\} \end{matrix}\right]$

$(1) \Rightarrow (5)$

$C \times D$ $\qquad D \times C$

$f_{CD} : C \to D$ $\qquad f : D \to C$

Find single-valued subset R of $C \times D$ with domain C or subset R' of $D \times C$ with domain D.

no single-v subset exists : $R \subseteq C \times D \Rightarrow$ exists c, c' and d with cRd and $c'Rd$ but c

$(4) \Rightarrow (5)$

$F : \underbrace{E}_{\text{set of nonempty subsets of } C \cup D} \longrightarrow C \cup D$

If $C \cup D \neq \varnothing$,

$F(C \cup D) \in C \cup D$

for every $e \in C \cup D$, $F(\{e\}) = e$

17. (a) Consider the natural numbers n and m with $n < m$. Then, $n + \aleph_0 = \aleph_0 \nless \aleph_0 = m + \aleph_0$ because $\omega \approx \omega$.

(b) $n \cdot 0 = \operatorname{card}(n \times \varnothing) = \operatorname{card} \varnothing = 0 \nless 0 = \operatorname{card} \varnothing = \operatorname{card}(m \times \varnothing) = m \cdot 0$ since $\varnothing \approx \varnothing$.

(c) For any cardinals $\kappa \le \lambda$, $\kappa^0 = 1 \nless 1 = \lambda^0$.

(d) For any nonzero cardinals $\kappa \le \lambda$, $0^\kappa = 0 \nless 0 = 0^\lambda$.

Consequently, strengthening the Theorem by replacing "$\le$" with "$<$" throughout causes it to no longer be true in general.

Exercise 15.

Ideas: $\varnothing$ not necessarily in $\mathcal{A}$ — let x be any set, exists $y \in \mathcal{A}$ with $x \preceq y$ — Exists the set $\mathcal{B} \in \mathcal{A}$ with $\mathcal{A} \preceq \mathcal{B}$

$\kappa \not\leq \mu \implies \kappa \not\leq \lambda$ or $\lambda \not\leq \mu$ — $f : x \longrightarrow y$

$(K \not\preceq M$ or $K \not\approx M) \implies K \not\preceq L$ or $L \not\preceq M$ or $L \not\approx M$

Sketch: For all $A \in \mathcal{A}$, $A \preceq \bigcup\mathcal{A}$ because $I_A : A \longrightarrow \bigcup\mathcal{A}$ is injective. Clearly, $\bigcup\mathcal{A} \preceq \mathcal{P}\bigcup\mathcal{A}$ because there is the injection $f : \bigcup\mathcal{A} \longrightarrow \mathcal{P}\bigcup\mathcal{A}$ defined by $f(a) = \{a\}$. And since we know that $\bigcup\mathcal{A} \not\approx \mathcal{P}\bigcup\mathcal{A}$ by Theorem 6B, so $A \preceq \mathcal{P}\bigcup\mathcal{A}$ and $A \not\approx \mathcal{P}\bigcup\mathcal{A}$ for any $A \in \mathcal{A}$ (using example 3 on page 148). Consequently, we know that $\mathcal{P}\bigcup\mathcal{A} \not\preceq A$, lest $A \approx \mathcal{P}\bigcup\mathcal{A}$ by the Schröder-Beinstein Theorem, which would contradict the fact that $A \not\approx \mathcal{P}\bigcup\mathcal{A}$ as stated in the previous line. Wherefore, no set $\mathcal{A}$ has the property that, for every set there is some member of $\mathcal{A}$ that dominates it.

16. (Without looking at hint)

Sketch: Clearly the function $\hat{F} : S \to {}^S 2$ where $\hat{F}(s)$ is the function mapping S into 2 with

$$[\hat{F}(s)](\bar{s}) = \begin{cases} 1 & \text{if } \bar{s} = s, \\ 0 & \text{if } \bar{s} \neq s. \end{cases}$$

is injective. Hence, $S \precsim {}^S 2$ is true. Now consider any function $F : S \to {}^S 2$. We can form another function $g : S \to 2$ defined by

$$g(s) = \begin{cases} 1 & \text{if } [F(s)](s) = 0, \\ 0 & \text{if } [F(s)](s) = 1. \end{cases}$$

We see that $g \notin \operatorname{ran} F$, lest $F(s) = g$ for some $s \in S$, which implies $[F(s)](s) = g(s)$, contradicting our construction of g (where $g(s)$ is always the 'opposing value' of $[F(s)](s)$). Consequently, for any function mapping S into ${}^S 2$, its range is never ${}^S 2$. In other words, no surjection (or bijection) exists between the two sets. Wherefore, $S \not\approx {}^S 2$ indeed.

$$\Big[X \subseteq A \;\&\; \forall y \big(y \in A \Rightarrow [\forall x (x < y \Rightarrow x \in X) \Rightarrow y \in X] \big) \Big] \Rightarrow X = A$$

$$(\forall y \in A)(\forall x < y)(x \in X) \Rightarrow y \in X$$

Theorem 6L

(a) $K \preccurlyeq L \Rightarrow K \cup M \preccurlyeq L \cup M$, where $K \cap M = \varnothing$ and $L \cap M = \varnothing$

(b) $\Rightarrow K \times M \preccurlyeq L \times M$

(c) $\Rightarrow {}^M K \preccurlyeq {}^M L$

(d) $\Rightarrow {}^K M \preccurlyeq {}^L M$ where $K \neq 0$ or $M \neq 0$ (otherwise, if $K \approx M \approx 0$ but $L \neq 0$, then ${}^0 0 = \{\varnothing\} \not\approx \varnothing = {}^L 0$)

Sketch

(a) $f \cup I_M$ where f is an injection from K into L

(b) $g : K \times M \to L \times M$ defined by $g(k, m) = (f(k), m)$

(c) $h : {}^M K \to {}^M L$ with $[h(g)](m) = f(g(m))$

$$\begin{tikzcd} m \in M \arrow[r, dashed, "h(g)"] \arrow[d, "g"'] & L \ni l \\ k \in K \arrow[ur, "f"'] & \end{tikzcd}$$

$h(g) = h(g')$

$f(g(m)) = f(g'(m))$ for any $m \in M$

$g(m) = g'(m)$ by the injectivity of f

Hence, $g = g'$.

$\bar{f} = f^{-1} \cup \{\langle l, k \rangle \mid l \in L - \mathrm{ran}$

(d) $h : {}^K M \to {}^L M$ where $[h(g)](l) = g(f^{-1}(l))$

$$\begin{tikzcd} K \arrow[d, "g"'] & \\ M & L \arrow[ul] \arrow[l, dashed, "h(g)"] \end{tikzcd}$$

dom f^{-1} not necessarily L

$h(g) = h(g')$

For all $l \in L$, $[h(g)](l) = [h(g')](l)$

$g(f(l)) = g'(f(l))$

14

Cantor – Schröder – Bernstein Theorem

Self - Proof

Assume that ~~$A \preccurlyeq B$~~ and ~~$B \preccurlyeq A$~~.

<u>Ideas</u>

There exists ~~the injections~~ $f: A \to B$ and $g: B \to A$.

$\overset{\text{bijective}}{f^{-1}: \operatorname{ran} f \to A}$ ~~$\overset{\text{bijective}}{g^{-1}: \operatorname{ran} g \to B}$~~ where $\overset{\operatorname{ran} g}{A'} \subseteq A$ and $B' \subseteq B$.

$f \cap g^{-1}: A'' \longrightarrow B$

where A'' is a subset of both A' and A

$A - A''$

$g \circ f$ clearly injective

$$\left[g \restriction \overbrace{(B - \operatorname{ran} f)}^{\text{values missing from } f} \right]^{-1}$$

range of this is a subset of A.

Abbott Exercise 1.5.11

(a) Naturally, g^{-1} bijects A' into B'. Therefore, $(f \restriction A) \cap (g^{-1} \restriction A')$ be our desired bijection from X into Y, since $A \cap A' = \emptyset$ and $A \cup A' = X$ (the same applies to the range of f and g). As such, we can then immediately conclude $X \sim Y$.

(b)

(b) If $A_1 = \emptyset$, that would mean that $X \subseteq g(Y)$. And since g maps to X, i.e. $g(Y) \subseteq X$, it must be that $g(Y) = X$. As such, the injection g is also surjective. Which means we already found a bijection between X and Y.

Let S be the set of natural numbers m so $A_m \cap A_n = \emptyset$ for all $m \neq n$.

We see that $A_1 \cap A_n = \emptyset$ for all $n \neq 1$, lest there exists some $x \in A_n \subseteq g(Y)$ that is also in A_1. However, this would contradict the fact that $A_1 \cap g(Y) = [X \setminus g(Y)] \cap g(Y)$ must be empty. Thus, $1 \in S$. Assume $m \in S$ and T is the set of natural numbers $n \neq m^+$ with $A_{m^+} \cap A_n = \emptyset$. By the above, $A_{m^+} \cap A_1$ must be empty as it is simply $A_1 \cap A_{m^+}$, and m^+ is always greater than 1. In other words, $1 \in T$. Now suppose $n \in T$. If $m^+ = n^+$ then $n^+ \in T$ by definition. So, presume $m^+ \neq n^+$. That is, $n \neq m$. By our assumption that $m \in S$, we know with certainty that $A_m \cap A_n = \emptyset$. Consequently, $A_{m^+} \cap A_{n^+} = g(f(A_m)) \cap g(f(A_n)) = \emptyset$ — because since $A_m \cap A_n = \emptyset$, if $a \in A_m$ and $b \in A_n$, then $a \neq b$. Hence, as f and g are injective, $g(f(a)) \neq g(f(b))$. It follows that $n^+ \in T$, and as a result, $T = \omega$[1] by induction. Meaning, for all natural $n \neq m^+$, $A_{m^+} \cap A_n = \emptyset$. Thence, $m^+ \in S$ and resultantly, $S = \omega$ by induction. Whence, we have proven that ~~the set of~~ all our A_n's are indeed pairwise disjoint. Wherefore, it is also clear that our $f(A_n)$'s are pairwise disjoint as well, by the same reasoning as we used to show $n^+ \in T$.

(c) To show surjectivity, we assume $b \in \bigcup_{n=1}^{\infty} f(A_n)$[2] first. This b must be an element of some $f(A_n)$, i.e. $b \in f(A_n)$ for some natural n. It follows that $b = f(a)$ for some $a \in A_n$. Of course, this a must also be in $\bigcup_{n=1}^{\infty} A_n$. As such, f surjects A into B.

(d) Let $x \in X \setminus A$ now. Then, x must not be in A_1. So, we can be sure that $x \in g(Y)$, since $x \in X$ by definition. For this reason, we know that $x = g(y)$ for some $y \in Y$. In addition, we know $y \notin B$; lest $y \in f(A_n)$ for some n, which would imply $x = g(y) \in g(f(A_n)) = A_{n+1}$. This would contradict the fact that $x \notin A$. Accordingly, $y \in Y \setminus B$ and g surjects B' into A'.

[1] In this case, we exclude 0 from ω, but by defining A_1 as A_0, it is true that it is ω anyways. So, it doesn't really matter here.

[2] It goes without saying that we assume $B \neq \emptyset$, otherwise A must be empty as well. Then, f mapping A onto B is trivial.

14. Proof

Let $K \approx K'$ and the function $F : \{f \mid f \text{ is a permutation of } K\} \longrightarrow \{f \mid f \text{ is a permutation of } K'\}$ be defined by $F(f) = g \circ (f \circ g^{-1})$, where g is a bijection from K into K' (which exists by the prior assumption that $K \approx K'$). We shall see that F is bijective. First assume $F(f) = F(f')$. In other words we have $g \circ (f \circ g^{-1}) = g \circ (f' \circ g^{-1})$. Thus, utilising the associativity of function composition, we have that

$$\begin{aligned}
g^{-1} \circ [g \circ (f \circ g^{-1})] &= g^{-1} \circ [g \circ (f' \circ g^{-1})] \\
[g^{-1} \circ g] \circ (f \circ g^{-1}) &= [g^{-1} \circ g] \circ (f' \circ g^{-1}) \\
I_K \circ (f \circ g^{-1}) &= I_K \circ (f' \circ g^{-1}) \\
f \circ g^{-1} &= f' \circ g^{-1} \\
(f \circ g^{-1}) \circ g &= (f' \circ g^{-1}) \circ g \\
f \circ (g^{-1} \circ g) &= f' \circ (g^{-1} \circ g) \\
f \circ I_K &= f' \circ I_K \\
f &= f'.
\end{aligned}$$

That is, F is indeed injective. Now let us suppose that f is a permutation of K'. Then, we claim that $g^{-1} \circ (f \circ g)$ is the permutation of K so that $F(g^{-1} \circ (f \circ g)) = f$. Clearly, it must be a permutation of K by virtue of the function f and g being bijective. Observe that:

$$\begin{aligned}
g \circ ([g^{-1} \circ (f \circ g)] \circ g^{-1}) &= g \circ (g^{-1} \circ [(f \circ g) \circ g^{-1}]) \\
&= (g \circ g^{-1}) \circ [f \circ (g \circ g^{-1})] \\
&= I_{K'} \circ (f \circ I_{K'}) \\
&= f,
\end{aligned}$$

as desired. Hence, F is indeed a surjection. Consequently, F is a bijection from the set of permutations of K into the set of permutations of K' as we claimed previously. Wherefore, K being equinumerous to K' implies that $\operatorname{card}\{f \mid f \text{ is a permutation of } K\} = \operatorname{card}\{f \mid f \text{ is a permutation of } K'\}$. Which also means that $\kappa!$ is well-defined, since the value of $\kappa!$ is independent of which set chosen.

13. Let the set S contain only the natural numbers n so that for any set B of cardinality n whose members are themselves finite, $\bigcup B$ is also finite. ~~Assume that for all naturals less than n, p ∈ S~~ Consider $n = 0$. Then, $B = \varnothing$ and $\bigcup B = \varnothing$. Clearly, $\bigcup B$ is of cardinality 0 and must be finite. Hence, $0 \in S$. Assume $n \in S$ and the finite set C of cardinality n^+ contains finite sets exclusively. Since $n^+ \neq 0$, C is certainly nonempty. That is, there exists some element c of C which has cardinality k. Notice that $C - \{c\}$ must have cardinality n, lest card $C \neq n^+$. Consequently, (by Theorems 6I and J)

$\bigcup C = \bigcup[(C-\{c\}) \cup \{c\}] = \bigcup(C-\{c\}) \cup \bigcup\{c\} = \bigcup(C-\{c\}) \cup c$. Therefore, we see that it has cardinality $n+k$, a natural number.

As a result, $\bigcup C$ is finite, meaning $n^+ \in S$. By induction, $C = \omega$. Wherefore, a finite union of finite sets is always finite. $\boxed{4}$

14. Show that if $\mathrm{card}(K) = \mathrm{card}(K') = \kappa$, $\mathrm{card}\{f \mid f \text{ is a permutation of } K\} = \mathrm{card}\{f \mid f \text{ is a permutation of } K'\}$.
$K \approx K'$

Ideas: If f is a bijection from K into K, $g \circ f$ is a bijection from K into K' (where $g: K \to K'$ is a bijection)

If f is a bijection from K' ... $g^{-1} \circ f$ ...

~~$\Rightarrow f$ is a bijection~~

$$F : \{f \mid f \text{ is a permutation of } K\} \longrightarrow \{f \mid f \text{ is a permutation of } K'\}$$

$$F(f) = g \circ (f \circ g^{-1})$$

Injectivity: Assume $F(f) = F(f')$.

$K' \to K'$

$$g \circ (f \circ g^{-1}) = g \circ (f' \circ g^{-1})$$

~~for any $k \in K'$ $[g \circ (f \circ g^{-1})](x) = [g \circ (f' \circ g^{-1})](x)$~~

~~$g((f \circ g^{-1})(x)) = g((f' \circ g^{-1})(x))$~~

~~$g^{-1}(g((f \circ g^{-1})(x))) = g^{-1}(g((f' \circ g^{-1})(x)))$~~

~~$I_K((f \circ g^{-1})(x)) = I_K((f' \circ g^{-1})(x))$~~

~~$(f \circ g^{-1})(x) = (f' \circ g^{-1})(x)$~~

~~$f \circ g^{-1}$~~ $= f' \circ g^{-1}$

$g^{-1} \circ [g \circ (f \circ g^{-1})] = g^{-1} \circ [g \circ (f' \circ g^{-1})]$

$[g^{-1} \circ g] \circ (f \circ g^{-1}) = [g^{-1} \circ g] \circ (f \circ g^{-1})$

$I_K \circ (f \circ g^{-1}) = I_K \circ (f' \circ g^{-1})$

$f \circ g^{-1} = f' \circ g^{-1}$

$(f \circ g^{-1}) \circ g = (f' \circ g^{-1}) \circ g$

$f \circ (g^{-1} \circ g) = f' \circ (g^{-1} \circ g)$

$f \circ I_K = f' \circ I_K$

$f = f'$

Surjectivity: Now suppose that f is a permutation of K'.

$g \circ ([g^{-1} \circ (f \circ g)] \circ g^{-1})$

$= g \circ (g^{-1} \circ [(f \circ g) \circ g^{-1}])$

$= (g \circ g^{-1}) \circ [f \circ (g \circ g^{-1})]$

$= I_{K'} \circ (f \circ I_{K'})$

$= f$

Exercise 6

Assume that there exists some nonzero cardinal number κ and a corresponding set T to which every set K of cardinality κ belongs. Since $\kappa \neq 0$, the set $K \neq \emptyset$. Which also means that no bijection exists between the set K and $\emptyset$. Accordingly, it is certain that K is nonempty, lest there is the bijection $f: S \to \emptyset$, thus contradicting the previous sentence. The word "cardinal" in this context is the set κ that is the cardinality of some set. Thence, we will assume that for any cardinal κ, there exists a set of cardinality κ, further justified by the fact that we only have a loose, informal definition of cardinals. Seemingly making it downright impossible to prove the assumed fact. Anyways, we shall continue the proof with this assumption in mind.

For any set s, s is in some set K of cardinality κ — because if not, ~~we can select an arbitrary~~ we can select any nonempty set K of cardinality κ, which there exists at least one, and remove one of its elements, k. That is, we form the set $K \setminus \{k\}$. Then we adjoint the set s to that. Now, we see that $(K \setminus \{k\}) \cup \{s\}$ is of cardinality κ as well.

Let the function $f: (K \setminus \{k\}) \cup \{s\} \longrightarrow K$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in K \setminus \{k\} \\ k & \text{if } x = s \end{cases}$$

When $f(x) = f(x')$, either $x \in K \setminus \{k\}$ and $x' \in K \setminus \{k\}$, or: $x = s$ or $x' = s$. In the former, $x = x'$ immediately follows. For the latter, since ~~$s \in K \setminus \{k\}$ and~~ $k \notin K \setminus \{k\}$, the other must also be s. Hence, $x = x'$, meaning f is injective.
If $y \in K$, then we have two cases. Case I — $y \in K \setminus \{k\}$: then $f(y) = y$. Case II, $y = k$. Then, $f(s) = k$.
So, f is also surjective. Consequently, f is bijective. Thus, the set $(K \setminus \{k\}) \cup \{s\}$ is also of cardinality κ indeed.
Therefore, $\bigcup T$ is the set of all sets. However, this contradicts Theorem 2A which states that no such set can exist.
Wherefore, there does not exist a set to which every set of nonzero cardinality κ belongs.

$\boxed{\sim}$

$A \approx m$

$B \approx n$

$A \cup B \approx k \supseteq \max(m,n)$

$A \times B \approx mn$

~~$\mathrm{card}(B \setminus (A \cap B)) \in n$~~

$\mathrm{card}(B - A) \in n$

Bijection $f : A \longrightarrow m$

$F : m \longrightarrow A$

~~$f \cup \{\langle b, n^+ \rangle\} \cup$~~

$g : B \to n$

$G : n \to B$

Precisely one of $n \in m$, $n = m$, or $m \in n$ is true.

Without loss of generality, assume that $n \in m$.

~~Let $B' = B \setminus (A \cap B)$~~

Checking the Well-Definedness of Cardinal Arithmetic

Let the sets K and K' be of cardinality κ and the sets L and L' be of cardinality λ. Which also means that there exists the bijections $k : K \longrightarrow K'$ and $\ell : L \longrightarrow L'$.

(a) Further assume that K is disjoint from L and K' is disjoint from L'. Clearly, $u = k \cup \ell$ is a function by Exercise 14 (b) of Chapter 3; because since $K \cap L = \emptyset$ by assumption, it is vacuously true that $k(x) = \ell(x)$ for every x in $K \cap L$. We shall now show that it is a bijection. When $u(x) = u(y)$

(same ol' procedure)

Consequently, we see that $u = k \cup \ell$ is indeed a bijection from $K \cup L$ into $K' \cup L'$. Wherefore, $\operatorname{card}(K \cup L) = \operatorname{card}(K' \cup L')$.
We have shown that cardinal addition is well-defined.

(b) Sketch) Show that the function $f : K \times L \longrightarrow K' \times L'$ defined by $f(a,b) = (k(a), \ell(b))$ is both well-defined and a bijection.

$${}^{L}K \approx {}^{L'}K'$$

$$G(g) = g' \qquad g : L \to K$$

(c) Sketch

$$\text{where } g' = \{\langle \ell(b), k(g(b))\rangle \mid b \in L\}$$

Show that the function $G : {}^{L}K \to {}^{L'}K'$ defined by $G(g) = g' := \underbrace{\{\langle \ell(b), k(g(b))\rangle \mid b \in L\}}_{\text{this can be instead phrased as where } g' \text{ is the function mapping from } L' \text{ into } K' \text{ with } g'(\ell(b)) = k(g(b))}$ is well-defined / single-valued, and bijective.

$k(g(\ell^{-1}(b')))$

Injectivity

If $G(g) = G(\bar{g})$, then for any $b \in L$ $k(g(b)) = k(\bar{g}(b))$ $\therefore$ (Injectivity of k) $g(b) = \bar{g}(b)$. Thus, $g = \bar{g}$. This means that G is injective.

Surjectivity

Whenever $g' \in {}^{L'}K'$, $g = \{\langle \ell^{-1}(b'), k^{-1}(g'(b'))\rangle \mid b' \in L'\}$ is the function so that $G(g) = g'$.
For any $b \in L$, there exists some $b' \in L'$ with $b = \ell^{-1}(b')$. Consequently, for all b, $k(g(b)) = k(g(\ell^{-1}(b'))) = k(k^{-1}(g'(b'))) = g'(\ell(b$

⋮

Cardinal Arithmetic does not conflict with regular Arithmetic on Natural Numbers

Suppose that m and n are natural numbers / finite cardinals ; and also that the set K is of cardinality m while the set L is of cardinality n. Note that, thus, $K \approx m$ and $L \approx n$. That is, there exists the bijections $f : K \to m$ and $g : L \to n$.

(a) First, further presume that K is disjoint from L. Then, let the function $h : K \cup L \to m+n$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in K, \\ m + g(x) & \text{if } x \in L. \end{cases}$$

Now, we see that h is bijective. Assume $h(x) = h(x')$.

$\vdots$ (same thing)

When $k \in m+n$, either $k \in m$ or $m \subseteq k$. In the former case, ...

Similarly for the latter, $m \subseteq k \in m+n$ tells us that $k = m + \bar{n}$ for some natural number $\bar{n}$ smaller than n. (Provable by induction)

$\vdots$

Indeed, h is a bijection from $K \cup L$ into $m+n$. Consequently, $\text{card}(K \cup L) = \text{card}(m+n) = m+n$. $\boxed{2}$

(b)

$$y = \frac{2x-1}{1-x}$$

$$y - xy = 2x - 1$$

$$y + 1 = 2x + xy$$

$$y + 1 = x(2+y)$$

$$x = \frac{y+1}{2+y}$$

$$y = 2 - \frac{1}{x}$$

$$y - 2 = -\frac{1}{x}$$

$$xy - 2x = -1$$

$$(y-2)x = -1$$

$$x = \frac{1}{2-y}$$

$$\frac{1}{2} \le x < 1$$

$\frac{1}{2} \le x$ $\quad$ $x < 1$

$1 - x \le \frac{1}{2}$ $\quad$ $0 < 1 - x$

$2 \le \frac{1}{1-x}$ $\quad$ $0 < \frac{1}{1-x}$

$$0 \le \frac{1}{1-x} - 2$$

$$\frac{\left(\frac{p}{q}+1\right)}{\left(2+\frac{p}{q}\right)} = \frac{p+q}{2+p}$$

$$\frac{1}{2-\left(\frac{p}{q}\right)} = \frac{q}{2q-p}$$

$$-\frac{1}{y+2} < 0$$

$$1 - \frac{1}{y+2} < 1$$

when $y < 0$,

$$f\left(\frac{1}{2-y}\right) = 2 - \frac{1}{\left(\frac{1}{2-y}\right)}$$

$$= 2 - (2-y)$$

$$= y$$

Injectivity

$$0 < x < \frac{1}{2}$$

$0 < x$ $\quad$ $x < \frac{1}{2}$

$0 < \frac{1}{x}$ $\quad$ $2 < \frac{1}{x}$

$-\frac{1}{x} < 0$ $\quad$ $-\frac{1}{x} < -2$

$2 - \frac{1}{x} < 2$ $\quad$ $2 - \frac{1}{x} < 0$

$$\frac{1}{1-x} - 2 = \frac{1}{1-x'} - 2$$

$$\frac{1}{1-x} = \frac{1}{1-x'}$$

$$1 - x' = 1 - x$$

$$x = x'$$

$y \ge 0$ $\quad$ $\frac{y+1}{y+2}$

$y + 2 \ge 2$

$\frac{1}{2} \ge \frac{1}{y+2}$ $\quad$ $= 1 - \frac{1}{y+2}$

$-\frac{1}{y+2} \ge -\frac{1}{2}$

$\frac{1}{2} - \frac{1}{y+2} \ge 0$ $\quad$ $1 - \frac{1}{y+2} \ge \frac{1}{2}$

$$2 - \frac{1}{x} = 2 - \frac{1}{x'}$$

$$\frac{1}{x'} = \frac{1}{x}$$

$$x = x'$$

Surjectivity

Let $y \in \mathbb{R}$, either $y \ge 0$ or $y < 0$ must hold by trichotomy.

If $y \ge 0$, we see that

$$f\left(\frac{y+1}{2+y}\right) = \frac{\left[2\left(\frac{y+1}{2+y}\right) - 1\right]}{\left[1 - \left(\frac{y+1}{2+y}\right)\right]}$$

$$= \frac{2(y+1) - (2+y)}{(2+y) - (y+1)}$$

$$= \frac{2y + 2 - 2 - y}{2 + y - y - 1}$$

$$= \frac{y}{1}$$

$$= y$$

$$y < 0$$

$$2 < 2-y \implies 0 < 2-y$$

$$\frac{1}{2-y} < \frac{1}{2} \qquad \underbrace{2p-2q}\, 0 < \frac{1}{2-y}$$

$$\frac{q-(2q-2p)}{q-p} = \frac{2p-q}{q-p}$$

$$\frac{1}{(1-\frac{p}{q})} - 2 = \frac{q}{q-p} - 2 \qquad 2 - \frac{1}{(\frac{p}{q})} = 2 - \frac{q}{p}$$

$$\frac{(\frac{p}{q}+1)}{(\frac{p}{q}+2)} = \frac{p+q}{p+2q} \qquad \frac{1}{(2-\frac{p}{q})} = \frac{q}{2-p}$$

Let F be a finite set and F' be a proper subset of F.

$F \approx n$

$f : F \to n$

$f \upharpoonright F'$

$\operatorname{ran}(f \upharpoonright F') \subset n$

$\operatorname{ran}(f \upharpoonright F') \approx m \in n$

$F' \approx \operatorname{ran}(f \upharpoonright F') \approx m$

$F' \approx m$

F is finite

By definition, F is equinumerous to some natural n, meaning there exists a bijection $f : F \to n$. So, $f \upharpoonright F'$ bijects F' into its range, $\operatorname{ran}(f \upharpoonright F')$, which must be a proper subset of n — since $F' \subset F$, there exists $x \in F$ with $x \notin F'$. Thus, for this particular x, the element $f(x)$ of n mustn't be in $\operatorname{ran}(f \upharpoonright F')$. Consequently, by Lemma 6F, there exists some natural m less than n such that $\operatorname{ran}(f \upharpoonright F') \approx m$. Concurrently, we also know $F' \approx \operatorname{ran}(f \upharpoonright F')$. Wherefore, F' is equinumerous to the natural number m. As such, F' must be finite by definition. □

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{2x}{(1-x^2)^2}$$

$$y = \int \frac{x}{(1-x^2)^2}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{(1-x^2)^2}\,dx$$

$$= \frac{1}{2(1-x^2)} + C$$

When $x=0$, $y=1$ on the curve. Hence,

$$\frac{1}{2} + C = 1$$

$$C = \frac{1}{2}$$

Therefore, the equation of the curve is given by $y = \frac{1}{2(1-x^2)} + \frac{1}{2}$

Let S be the set of natural numbers n so that if C is a proper subset of n, then $C \approx m$ for some natural m less than n.

Clearly, $0 \in S$ vacuously since there is no proper subset of 0. Now, assume $n \in S$. There are two cases to consider; either the proper subset $C \subset n^+$ is also a subset (not necessarily proper) of n, or $n \in C$. Hence, we consider this casewise.

Case 1. If $C \subset n$, then since $n \in S$, $C \approx m$ for some $m \in n \in n^+$ immediately. Thus, $m \in n^+$ by transitivity. Similarly, when $C = n$, $n \approx n \in n^+$ also.

Case 2. We see that $C - \{n\}$ is a proper subset of n, by virtue of $n \in C$ and $C \subset n^+$. Again as $n \in S$, $C - \{n\} \approx m$ for some m less than n. ~~By transitivity, m must be less than n^+~~. Now utilising Corollary 4M, we observe that $m \subset n$, meaning there exists some k in n that is not in m. From the definition of equinumerosity, there is a bijection $f: C - \{n\} \to m$. As a result, the function $g = f \cup \{\langle n, k\rangle\}$ is a bijection from C into $m \cup \{k\}$. If it is not clear enough to the reader that g is bijective, we shall show it concretely. Given $g(x) = g(x')$, either both are in $C - \{n\}$ or (at least) one is n. In the former, $x = x'$ immediately, because $f: C - \{n\} \to m$ is injective by definition. For the latter, the other must also be n since $k \notin m$ (the codomain of f). Resultantly, g is injective. Whenever $y \in m \cup \{k\}$, $y \in m$ would mean there is some $x \in C - \{n\} \subseteq C$ with $y = f(x) = g(x)$ since f is surjective, and $y \in k$ means $y = g(n)$. Correspondingly, g is surjective. Notice that $m \cup \{k\} \subseteq n \in n^+$ implies $m \cup \{k\} \in n^+$. Consequently, $C \approx m \cup \{k\}$ where $m \cup \{k\}$ is less than n^+.

In any case, for any proper subset C of n^+, there certainly exists some set $\bar{m}$ less than n with $C \approx m$. That is, $n^+ \in S$. By the Induction Principle for ω, $S = \omega$. Wherefore, we have proven that if C is a proper subset of a natural number n, then $C \approx m$ for some m less than n.

Self Proof of Corollary 6C

Let $\underbrace{S \text{ be some finite set}}_{S \approx n}$ and S' be a proper subset of S

Outline : 1. Show $S' \approx m$ for some natural $m \subset n$.
2. use the Pigeonhole Principle.

Since S is a finite set, there exists a bijection $f : S \to n$ for some natural n.

$A \approx B$ & $B \approx C \Rightarrow A \approx C$

$\operatorname{ran}(f \upharpoonright S) \subset n$

$\Longrightarrow S \approx n$ & $S' \approx \operatorname{ran}(f \upharpoonright S')$ & $n \not\approx \operatorname{ran}(f \upharpoonright S')$ (Pigeonhole Principle)

$\Longrightarrow n \not\approx S$ or $S \not\approx \operatorname{ran}(f \upharpoonright S')$

$\Longrightarrow S \not\approx S'$ or $S' \not\approx \operatorname{ran}(f \upharpoonright S')$

$\Longrightarrow n \not\approx S'$

$\Longrightarrow S' \not\approx n$ $\boxed{\checkmark}$

Let S be some finite set and S' be a proper subset of S. It follows from definition that there exists a bijection $f : S \to n$ for some natural n.
Clearly, $\operatorname{ran}(f \upharpoonright S') \subset n$ and $S' \approx \operatorname{ran}(f \upharpoonright S')$. By the Pigeonhole Principle, n is not equinumerous to $\operatorname{ran}(f \upharpoonright S')$ since $\operatorname{ran}(f \upharpoonright S')$
is a proper subset of n. Now utilizing the contrapositive of Theorem 6H(c), it must be that $S \not\approx \operatorname{ran}(f \upharpoonright S')$ because $n \approx S$ by definition.
Consequently, using the same theorem a second time, we observe that as $S' \approx \operatorname{ran}(f \upharpoonright S')$, $S \not\approx S'$ is certainly true. Hence, the finite set S is not
equinumerous to any proper subset S' of itself. $\boxed{\checkmark}$

Let n be a natural number, $\bar{n}$ be some proper subset of n and the function $f: n \to \bar{n}$.

Let S be the set of natural numbers n so that it is equinumerous to none of its proper subsets.

Immediately, $0 \in S$ vacuously because it has no proper subsets.

$\bar{n} \cup \{n\}$

Now, assume $n \in S$ and $\bar{n}$ is a proper subset of n

$g: \bar{n} \to n$

$\{0, 1, 5\}$ $\quad \{2, 3, 4\}$

$\{0, 1, 2, 3, 4, 5\}$

$\bar{n} \not\approx n^+$ $\quad n \setminus \bar{n}$ "has or one element"

$n^+ \setminus \bar{n}$ "has ≥ 2 elements"

Let S be the set of natural numbers n that is not equinumerous to any of its proper subsets.

$n \in \bar{m}$: $n = \bar{m}$ ($n = 0$, $n \neq 0$), $\bar{n} \neq \bar{m}$

Note: $n = 0$.

We immediately see that $0 \in S$ vacuously because 0 has no proper subsets.

Now, we assume $n \in T$. That is, every proper subset $\bar{n}$ of n is not equinumerous to n itself. Then, for any proper subset $\bar{m}$ of n^+, either $n \in \bar{m}$ or $n \notin \bar{m}$. In the former, it is clear that $\bar{m} \setminus \{n\}$ is a proper subset of $\bar{n}$, lest $\bar{m} = n^+$. As for the latter,

Self Proof of Corollary 6D

a) Taking the contrapositive of Corollary 6C, we clearly see that for any set S, if S is equinumerous to some proper subset of itself, then it is infinite.

$\boxed{2}$

b) Let $\mathcal{E}$ be the set of even natural numbers. Then, we see that there exists the bijection $f: \omega \to \mathcal{E}$ defined by $f(n) = 2n$. When $f(n) = f(n')$, $2n = 2n'$, and so, $n = n'$ (Corollary 4P). Hence, f is indeed injective. If $2n \in \mathcal{E}$ for some natural n, then immediately, $f(n) = 2n$, which means that f is surjective. Therefore, f bijects ω into $\mathcal{E}$ as desired. Consequently, ω is equinumerous to $\mathcal{E}$. Wherefore, by part (a), since $\mathcal{E}$ is a proper subset of ω yet $\omega \approx \mathcal{E}$ still, ω must be an infinite set.

$\boxed{2}$

Self Proof of Corollary 6E

Again, suppose that S is a finite set that is equinumerous to the natural numbers n and n'. Which means that there exists the bijections $f: S \to n$ and $f': S \to n'$. Now, let the function $g: n \to n'$ be defined with $g(k) = f'(s)$ where $f(s) = k$. When $g(k) = g(k')$, then we know that $f'(s) = f'(s')$ with $f(s) = k$ and $f(s') = k'$ immediately by definition. Since f' is injective, $s = s'$. Consequently, $f(s) = f(s')$. In other words, $k = k'$. As such, $g(k) = g(k')$, meaning g is injective. Assume that $m \in n'$. By virtue of f' being surjective, there exists some $s \in S$ with $f'(s) = m$. Thus, $g(f(s)) = f'(s)$. Therefore, g is surjective.

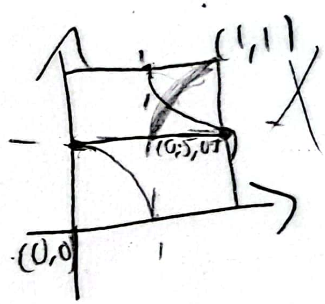
$k = k' \in n$

$f(s) = k = k'$ for a unique $s \in S$ since f is bijective.

$g(k) = f'(s) = g(k')$

4. Let g be the function mapping from $[0,1]$ into $(0,1)$ with

$$g(x) = \begin{cases} \end{cases}$$



3. Let f be the function from $(0,1)$ into $\mathbb{R}$ so that

$$f(x) = \begin{cases} \\ \\ \end{cases}$$

$\neg[(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})(x = \frac{p}{q})]$

$\Longleftrightarrow \neg[(\exists p \in \mathbb{N})(\exists q \in \mathbb{N})(\frac{1}{x} = \frac{q}{p})]$

$\Longleftrightarrow (\forall p \in \mathbb{N})(\forall q \in \mathbb{N})(\frac{1}{x} \neq \frac{q}{p})$ $\frac{1}{n} \longmapsto \frac{1-1}{n} = \frac{0}{n} = 0$

$x = \frac{nx}{n}$ $\frac{q-1}{p}$ $\frac{1}{3} \longmapsto \frac{3-1}{2} = \frac{2}{2} = 1$

p and q relatively prime $\frac{p}{q} \longmapsto$ $\frac{2n-(2n-nx)}{n}$ $\underbrace{p}_{\text{reals greater than or equal to } 0}$

$\frac{n}{2n-nx}$

0.4 0.5 $\frac{2n-(2n-nx)}{n} = \frac{2m-(2m-mx)}{m}$

$\downarrow$ $\downarrow$ $\frac{p-q+x}{p}$?? $\frac{p-q+1}{p}$ Not a function

0 1

$1 \leq p < q$

For all integers n, $n \mid p \Longrightarrow \neg(n \mid q)$ o/R $p = nm$ for some m, $\Longrightarrow q \neq nm$ for all m

$n \mid p' \Longrightarrow \neg(n \mid q')$

$\frac{2}{4} \longmapsto \frac{2-4+1}{2} = \frac{1}{2}$

p even q odd

$\frac{1}{2} \longmapsto \frac{1-2+1}{1} = \frac{0}{1} = 0$

Well-defined function: when $\frac{p}{q} = \frac{p'}{q'}$; $2 - \frac{q}{p} = 2 - \frac{q'}{p'} \Longrightarrow \frac{2p-q}{p} = \frac{2p'-q'}{p'}$ ✓ p odd q even

$\frac{2p-q}{p}$

Injectivity

$2 - \frac{q}{p} = 2 - \frac{q'}{p'}$

$\frac{q}{p} = \frac{q'}{p'}$

Assume $\frac{2p-q}{p} = \frac{2p'-q'}{p'}$

$2 - \frac{q}{p} = 2 - \frac{q'}{p'}$

$\frac{q}{p} = \frac{q'}{p'}$

$\frac{p}{q} = \frac{p'}{q'}$ ✓

Show $\frac{p}{q} = \frac{p'}{q'}$

$pq' = p'q$

p odd q odd

$\frac{2}{4} \longmapsto \frac{2(2)-4}{2} = \frac{0}{2} = 0$

Surjectivity $\frac{2b-(2b-a)}{b}$ ✓

☆ $0 < p < q$ $2p - q = 0$

$2p = q$

$\frac{1}{2}$ $\frac{2(1)-2}{2} = 0$

$\frac{1}{2} \longmapsto \frac{2(1)-2}{1} = \frac{0}{1} = 0$

$r = \frac{a}{b} = \frac{2p-q}{p}$

$\frac{1}{3}$ $\frac{2(3)-5}{3}$ $\frac{1}{3}$

Injectivity:

$$\frac{2x-1}{4x(x-1)} = \frac{2x'-1}{4x'(x'-1)}$$

$$\frac{1}{2(x-1)} - \frac{1}{4x(x-1)} = \frac{1}{2'(x'-1)} - \frac{1}{4x'(x'-1)}$$

$0 < x < x'$ $\quad$ $x < 1 \quad x' < 1$

$0 < 2(x-1) < 2(x'-1) < 1$ $\quad$ $0 < 4x(x-1) < 4x'(x'-1) < 1$

$1 < \frac{1}{2(x'-1)} < \frac{1}{2(x-1)}$ $\quad$ $1 < \frac{1}{4x'(x'-1)} < \frac{1}{4x(x-1)}$

$1 - \frac{1}{\quad}$

Surjectivity:

$$y = \frac{2x-1}{4x(x-1)} \quad 4x^2y - 4xy$$

$$4x^2y - 4xy = 2x - 1$$

$$4x^2y - 4xy - 2x + 1 = 0$$

$$(4y)(x^2) - (4y+2)x + 1 = 0 \longrightarrow \frac{-(4y+2) \pm \sqrt{(4y+2)^2 - 4(4y)(1)}}{2(4y)}$$

$$4y\left[x^2 - \underbrace{\left(\frac{4y+2}{4y}\right)}_{\frac{2y+1}{4y}}(x)\right] + 1 = 0$$

$$4y\left[x - \left(\frac{2y+1}{4y}\right)\right]^2 - 4y\left(\frac{2y+1}{4y}\right)^2 + 1 = 0$$

$$4y\left(\frac{4y^2+4y+1}{16y^2}\right) = \frac{4y^2+4y+1}{4y}$$

$$4y\left(\frac{2y+1}{4y}\right)^2 = 4y$$

$$4y\left[x - \frac{2y+1}{4y}\right]^2 = 4y\left(\frac{2y+1}{4y}\right)^2 - 1$$

$$\left[x - \frac{2y+1}{4y}\right]^2 = \left(\frac{2y+1}{4y}\right)^2 - \frac{1}{4y}$$

$$\frac{4p^2}{q^2} + 1 \qquad \frac{4p^2+q^2}{q^2}$$

$$x = \pm\sqrt{\left(\frac{2y+1}{4y}\right)^2 - \frac{1}{4y}} + \frac{2y+1}{4y}$$

$$= \frac{2y+1 \pm \sqrt{(2y+1)^2 - 4y}}{4y} = \frac{2y+1 \pm \sqrt{4y^2+1}}{4y}$$

Let f be a function mapping $(0,1)$ into $\mathbb{R}$ so that

$$f(x)\begin{cases} \pi - \frac{q}{p} & \text{if there exists integers } p \text{ and } q \text{ with } x = \frac{p}{q}, \\ 2 - \frac{1}{x} & \text{if } x \text{ is irrational.} \end{cases}$$

Any real number is either rational or irrational. Fix $\mathbb{Q}$ to be the set of real rational numbers; then $\mathbb{R}\setminus\mathbb{Q}$ is the set of all irrational reals.
We first check that f is a (well-defined) bijective function 'for rationals' in the following sense.

$2b-a=0$
$a=2b$

$\frac{2b}{b}=2$

Ass. $b>0$

Surjectivity
Suppose that $x \in \mathbb{Q}$. Then, by definition, $x=\frac{a}{b}$ for some integers a and $b \neq 0$. Thus, $f\left(\frac{b}{2b-a}\right) = \frac{2b-(2b-a)}{b} = \frac{a}{b}$.

$-1 < \frac{a}{b} < 1$

$-b<a \quad a<b$

$-b<a<b$

$(b)(2b-a)>0$

$b<0$ or $\frac{1}{2}a>0$

Question: 3

$\frac{a}{b} = \frac{\pi p - q}{p}$

$2-\frac{q}{p}$

$pq=a \implies p=\frac{a}{q} \quad \frac{p}{q}=\frac{\left(\frac{a}{q}\right)}{q} = \frac{a}{q}$

$x=\frac{1}{2},\ f(x)=-\frac{1}{3}$

$\pi b -$

$q=1$
$p=1$

$\frac{x}{x^2-1}$

$\frac{\frac{p}{q}}{\left(\frac{p^2}{q^2}-1\right)} = \left[\frac{\left(\frac{p}{q}\right)}{\left(\frac{p^2-q^2}{q^2}\right)}\right]$

$= \frac{pq}{p^2-q^2}$

$p^2-q^2=b$

$5\times 3=15$

$5^2-3^2=16$

$\left(\frac{a}{q}\right)^2-q^2=b$

$\frac{a^2}{q^2}-q^2=b$

$\frac{p}{q} = \frac{\frac{a}{\sqrt{\text{chunk}}}}{\sqrt{\text{chunk}}} = \frac{a}{\text{chunk}}$

$\frac{a^2}{\sqrt{x^2-\frac{b}{2}}}$

$= \frac{a^2}{\sqrt{x^2-\frac{b}{2}}}$

$=$

$\left(\frac{b}{2}\right)^2$

$(q^2+\frac{b}{2})^2 = a^2+\frac{b^2}{4}$

$a^2-q^4=bq$

$q^4+bq^2-a^2=0$

$q^2+\frac{b}{2} = \sqrt{a^2+\frac{b^2}{4}}$

$q^2 = \sqrt{a^2+\frac{b^2}{4}}-\frac{b}{2}$

$(q^2+\frac{b}{2})^2-\frac{b^2}{4}-a^2=0$

If $a>0$ take $-\sqrt{}$
$a<0$ take $\sqrt{}$

$$\frac{-\sqrt{4y^2+1}+2y+1}{4y} = -\sqrt{\frac{4y^2+1}{16y^2}} + \frac{2y+1}{4y}$$

$$=$$

$4y^2 - 4y + 1$

Let the function $f:(0,1)\to\mathbb{R}$ be so that

$$f(x)=\frac{2x-1}{4x(x-1)}.$$

Injectivity: Assume that $f(x)=f(x')$. In other words,

$$\frac{2x-1}{4x(x-1)}=\frac{2x'-1}{4x'(x'-1)}$$

$$4x'(2x-1)(x'-1)=4x(2x'-1)(x-1)$$

$$4x'(2xx'-2x-x'+1)=4x(2x'x-2x'-x+1)$$

$$8xx'^2-8xx'-4x'^2+4x'=8x'x^2-8x'x-4x^2+4x$$

$$8xx'^2+8x'x-4x'^2+4x'=\underbrace{8x'x^2+8xx'}_{8x'x(x+1)}\underbrace{-4x^2+4x}_{-4x(x-1)}$$

$$8xx'(x'+1)-4x'(x'-1)$$

$$8xx'^2-4x'^2+8x'x+4x'=8x'x^2-4x^2+8xx'+4x$$

$$4x'^2(2x-1)+4x'(2x+1)=4x^2(2x'-1)+4x(2x'+1)$$

$$x'^2(2x-1)+x'(2x+1)=x^2(2x'-1)+x(2x'+1)$$

☆ $0<x<1$

$$\frac{2x-1}{4x(x-1)}=\frac{2x}{4x(x-1)}+\frac{-1}{4x(x-1)}$$

$$=\frac{1}{2(x-1)}-\frac{1}{4x(x-1)}$$

$$=\qquad\frac{1}{4x^2-4x}$$

$$\frac{\left(\frac{2p}{q}-1\right)}{\left(\frac{4p}{q}\right)\left(\frac{p}{q}-1\right)}=\frac{2pq-q^2}{4p(p-1)}$$

$x^2 \Rightarrow \sqrt{x}$.

$$x<x'$$

$$x-1<x'-1$$

$$2(x-1)<2(x'-1)$$

$$\frac{1}{2(x'-1)}<\frac{1}{2(x-1)}$$

$$x^2<x'^2$$

$$4x^2-4x<4x'$$

$$\frac{1}{4x(x-1)}<\frac{1}{4x'}$$

3. Let f be the function mapping $(0,1)$ into $\mathbb{R}$ so that

$$f(x) = \begin{cases} \dfrac{2p-q}{p} & \text{if there exists integers } p \text{ and } q \text{ with } x = \frac{p}{q} \\ \dfrac{2x-1}{x} & \text{if } x \text{ is irrational.} \end{cases}$$

Injectivity

$$\frac{2x-1}{x} = \frac{2x'-1}{x'}$$

$$2 - \frac{1}{x} = 2 - \frac{1}{x'}$$

$$-\frac{1}{x} = -\frac{1}{x'}$$

$$x = x'$$

$$\frac{2x-1}{x} \qquad 2 - \frac{1}{x}$$

$$2^x - \frac{1}{x}$$

$y \in \mathbb{R} \setminus \mathbb{Q}$

$(x > 0.5)$ $x \to 1$; $f(x) \to \infty$ $\frac{1}{2-2x}$

$(x < 0.5)$ $x \to 0$, $f(x) \to -\infty$ $\frac{2x-1}{x}$

$f(0.5) = 1$

$f(0.5) = 0$

Surjectivity

$$\frac{2x-1}{x} = y$$

$$2x - 1 = xy$$

$$-1 = xy - 2x$$

$$-1 = x(y-2)$$

$$x = \frac{1}{2-y} \qquad y \neq 2$$

$x \neq 0$

$$\frac{2\left(\frac{1}{2-y}\right) - 1}{\left(\frac{1}{2-y}\right)} = \frac{\left(\frac{2}{2-y}\right) - 1}{\left(\frac{1}{2-y}\right)}$$

$$= 2 - (2-y)$$

$$= y$$

$\frac{1}{2} < x < 1$

$0 < x < \frac{1}{2}$

$0 < x$ $\qquad$ $x < \frac{1}{2}$

$\frac{1}{2} < x$

$1 < 2x$

$2 < 2x + 1$

$2 - 2x < 1$

$1 < \frac{1}{2-2x}$

$x < 1$

$2x < 2$

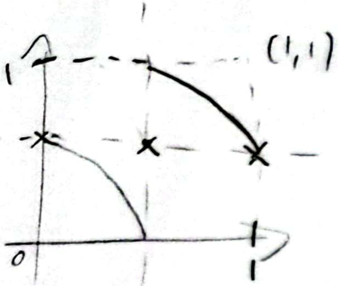
$0 < 2 - 2x$

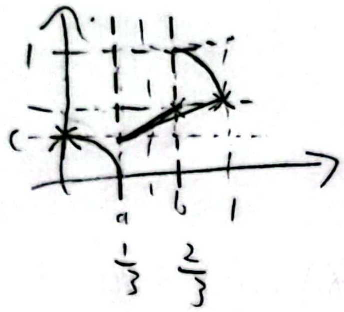
$0 < 1$

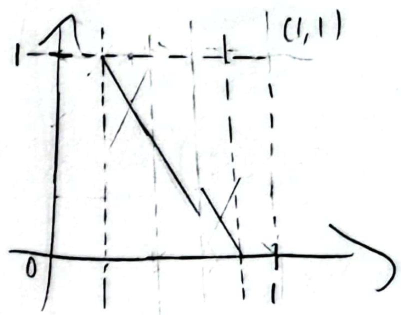
$0 < \frac{1}{2-2x}$

$$\frac{\left(\frac{2p}{q} - 1\right)}{\left(\frac{p}{q}\right)} = \frac{2p-q}{p} = 2 - \frac{q}{p}$$

$0 < x < 1$!!!

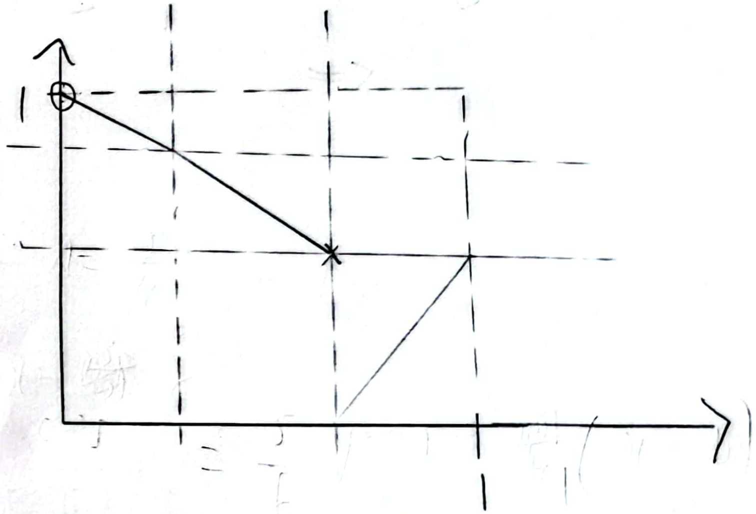


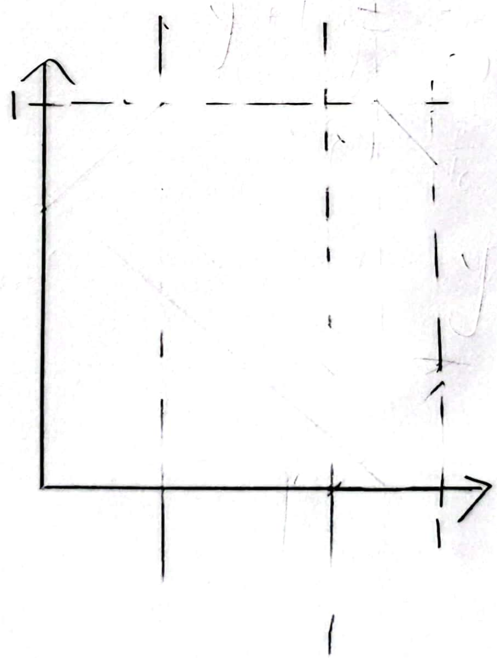


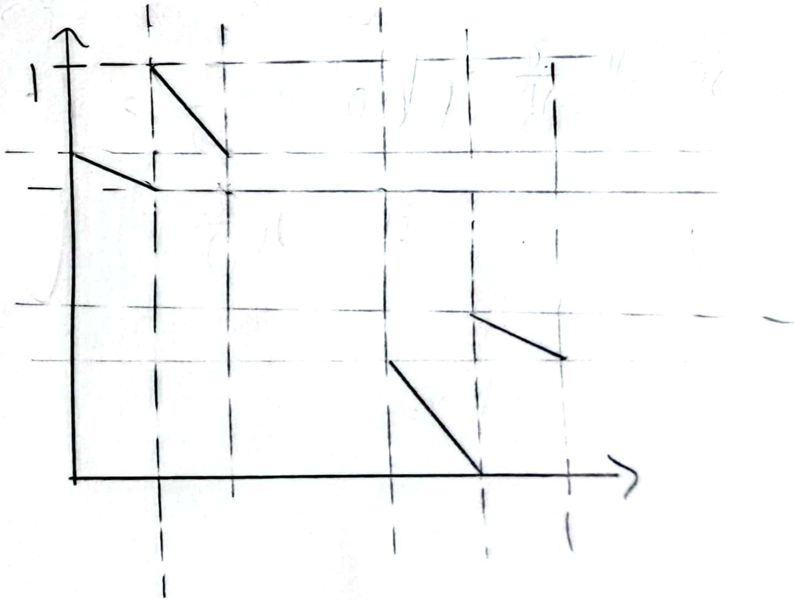


$I : (0,1) \hookrightarrow [0,1]$ $\quad I(x) = x$

$f : [0,1] \hookrightarrow (0,1)$ $\quad f(x) = \frac{1}{2}x + \frac{1}{4}$







Let the bijection $f : [0,1] \longrightarrow (0,1)$ be defined by

$$f(x) = \begin{cases} \frac{1}{x-1} + \frac{5}{4} & \text{if } 0 \leq x < \frac{2}{10}, \\ \frac{5}{6}x + \frac{7}{12} & \text{if } \frac{2}{10} \leq x \leq \frac{1}{2}, \\ \frac{6}{2x-5} + \frac{5}{2} & \text{if } 0.5 < x \leq 1. \end{cases}$$

ω

$\cancel{\{0,0\}}$ $\{0\}$ $\{1\}$ $\{2\}$

$\cancel{\{1,1\}}$ $\cancel{\{2,2\}}$

$\{0,1\}$ $\{0,2\}$ $\{0,3\}\cdots$ $\{1,2\}\{1,3\}\{1,4\}\cdots$ $\{2,3\}, \{2,4\}, \{2,5\}$

$\{0,1,2\}$ $\{0,1,3\}$ $\{0,1,4\}\cdots$ $\{1,2,3\}$ $\{1,2,4\}$ $\{$

$\{0,2,3\}\{0,2,4\}\{0,2,5\}\cdots$

$\{0,3,4\}\{0,3,5\}$ $\{0,3,6\}\cdots$

$L_0 = \{\{n\} \mid n \in \omega\}$

$L_1 = \{\{n,m\} \mid n,m \in \omega \text{ \& } n < m\}$

$L_2 = \{\{n,m,k\} \mid n<m<k\}$

$L_N = \{\{n_1, n_2, n_3, \ldots, n_N\} \mid n_1 < n_2 < n_3 < \cdots$

$\mathcal{P}\omega = \{L_N \mid N \in \omega\}$

Assume that there exists a surjection $f : S \longrightarrow PS$

s'

s' s''

$S = \emptyset$

$PS = \{\emptyset\}$

$S = 1$

$PS = \{\emptyset, \{0\}, \{1\}\}$

$S = \{x_1\}$

$PS = \{\emptyset, \{x_1\}\}$

$S = 2$

$PS = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$

$S = \{x_1, x_2\}$

$PS = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2$

$S = \{x_1, x_2, x_3\}$

$PS = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x$
$\{x_1, x_2, x_3\}\}$.

Let $C = \{x \in S \mid$ exactly one of $x \notin f(x)$ or there exists $y \in S$ with $x \in f(y)\}$

$C \in \operatorname{ran} f$

$\Rightarrow$ exists $s \in S$ with $f(s) = C$

$x \in C \iff x \in f(s)$

$\Rightarrow (x \notin f(x)$ or exists $y \in S$ $x \in f(y)) \iff x \in f(s)$

$C = \bigcup \{\cancel{x \in f(s) \mid \emptyset}$

$\bigcup \{\cancel{x \notin f(s)}$

$\cancel{\{x \in S \mid}$

$s \in f(s) \iff s \in C$

$\iff s \notin f(s)$ or exists $y \in S$, $y \neq s$ such that $s \in f(y)$

$s \in C \iff s \in f(s)$ | exists $y \in S$ $s \in f(y)$

$\Rightarrow \cancel{s \notin f(s)}$

$\Rightarrow s \in C$

$\Rightarrow s \in f(s)$

Assume there exists a surjective function f mapping from S into its powerset. Now, let C be the subset of S so that $x \in C$ iff precisely one of $x \notin f(x)$ or there exists $y \in S$ with $x \in f(y)$ holds. Then, by our assumption, there must exist some $s \in S$ with $f(s) = C$.

It follows that $s \in f(s)$ iff $s \in C$. However, we know that $s \in C$ implies

$$C = \{x \in S \mid x \notin f(x)\}$$

$$s \in f(s) \Longleftrightarrow s \in C$$
$$\Longleftrightarrow s \notin f(s)$$

$$\exists s\, (f(s) = C) \Longleftrightarrow \exists s\, \forall x\, (x \in f(s) \Longleftrightarrow x \in C$$
$$\Longleftrightarrow \exists s$$

Self Proof of Theorem 7V(a)

(a) Assume $a \in A$ where A is grounded. Then $a \in V_{\text{rank } A}$ so that $a \subseteq V_\beta$ for some $\beta \in \text{rank } A$ by Theorem 7S. Consequently, a is grounded by definition and in such a way that $\text{rank } a \in \beta \in \text{rank } A$.

(b) ~~By a replacement axiom we know there exists the set $S := \{\mathcal{P}V_{(\text{rank } a)^+} \mid a \in A\}$. Notice that $S = \{\mathcal{P}V_\beta \mid \beta \in (\text{rank } a)^+ \text{ for some } a \in A\}$ from Theorem 7U(a). Further~~, There must be some least ordinal α with rank a always in α lest a contradiction occurs with the Burali Forti Theorem.

Namely, $\alpha = \bigcup \{(\text{rank } a)^+ \mid a \in A\}$ — can be formed with replacement axiom — which has the property of containing all rank a (thus fulfilling the least upper bound criteria above).

Now, we see that $a \in V_\alpha$ for all $a \in A$ because $a \subseteq V_{\text{rank } a}$ where $\text{rank } a \in \alpha$. Hence, $A \subseteq V_\alpha$ is certain. As such, A is indeed grounded.

Furthermore, given any ordinal $\beta \in \alpha$, $\beta \in \text{rank } a$ for some a. So, $V_\beta \subseteq V_{\text{rank } a}$ where $a \notin V_{\text{rank } a}$ lest $a \subseteq V_\gamma$ for some $\gamma \in \text{rank } a$.

That is, $a \notin V_\beta$ so that $A \nsubseteq V_\beta$. Wherefore, $\text{rank } A = \alpha$. $\square$

Lemma 77

Assume $u \in v \in V_\alpha = F_{\alpha^+}(\alpha)$. We know that v must be a subset of some $F_{\alpha^+}(\beta)$, and $\underbrace{\text{hence a subset of } F_{\alpha^+}(\alpha)}_{v \subseteq \bigcup\{\mathcal{P}F_{\alpha^+}(\gamma) \mid \gamma \in \beta \in \alpha\} \subseteq \bigcup\{\mathcal{P}F_{\alpha^+}(\gamma) \mid \gamma \in \alpha\} \text{ trivially}}$. Consequently, $u \in F_{\alpha^+}(\alpha) = V_\alpha$ $\square$

Self Proof of Theorem 74

(a) Trivial.

(b) Trivial. (Well, $F_1(0) = \bigcup\{\mathcal{P}F_1(\beta) \mid \beta \in 0\} = \bigcup\varnothing = \varnothing$)

(c) Ideation:

$$V_\lambda = F_{\lambda^+}(\lambda) \qquad , V_\beta = \bigcup\{\mathcal{P}F_{\lambda^+}(\gamma) \mid \gamma \in \beta\}$$

$$= \bigcup\{\mathcal{P}F_{\lambda^+}(\mu) \mid \mu \in \lambda\} \qquad = \bigcup\{\mathcal{P}F_{\lambda^+}(\kappa) \mid \kappa \in \gamma \in \lambda\}$$

$$= \bigcup_{\mu\in\lambda} \mathcal{P}F_{\lambda^+}(\mu)$$

$$x \in \bigcup_{\mu\in\lambda} V_\mu \Rightarrow x \in V_\mu \text{ for some } \mu \in \lambda$$
$$\Rightarrow x \in V_\lambda \text{ trivially. e.g. by (a)}$$

$$\bigcup_{\mu\in\lambda} V_\mu \subseteq V_\lambda$$

$$x \in V_\lambda \Rightarrow x \subseteq F_{\lambda^+}(\mu) \text{ for some } \mu \in \lambda$$
$$\Rightarrow x \in F_{\lambda^+}(\mu^+)$$
$$\Rightarrow x \in V_{\mu^+} \text{ (where } \mu^+ \in \lambda \text{ as } \lambda \text{ is a limit ordinal)}$$
$$\Rightarrow x \in \bigcup_{\mu\in\lambda} V_\mu$$
$$\Rightarrow V_\lambda \subseteq \bigcup_{\mu\in\lambda} V_\mu$$

Therefore, $V_\lambda = \bigcup_{\mu\in\lambda} V_\mu$ for limit ordinals λ.

$\square$

2. Since WO $\Rightarrow$ Exercise 21 $\Rightarrow$ (6) $\Rightarrow$ (2) and (2) is equivalent to WO, both shown in the self proof of Theorem 6M, we can safely conclude Exercise 21 is an equivalent form of AC as those shown in Theorem 6M. Consequently, we have already (indirectly) proven the well-ordering theorem from the version of Zorn's Lemma given in Exercise 21. $\square$

3. (i) In the proof of Hartogs, it has been shown that α is an ordinal number. Hence, we take this fact to be granted.
By AC(15) / Cardinal Comparability, $\alpha \not\preccurlyeq A \succcurlyeq \beta$ for every $\beta \in \alpha$, ~~last $\alpha \in \alpha$~~. So, card $\alpha = \alpha$ because no ordinal smaller than α is equinumerous to it.

(ii) Since card $A \approx A$, i.e. card $A \preccurlyeq A$, card $A \in \alpha$. Thus, $\alpha \geq$ card A as shown in (i), so that $\alpha >$ card A.

(iii) Assume, for the sake of contradiction, that α is not the least cardinal greater than card A. So, there exists some $\beta \in \alpha$ with $\beta >$ card A. But $\beta \in \alpha$ implies $A \succcurlyeq \beta$, and hence, card $A \geq \beta$ (by the construction of α). Which is a contradiction. Consequently, it must be that, instead, α is indeed the least cardinal greater than card A. $\square$

4. By exercise 23 above. (Since $\in$ agrees with $<$) $\square$

25. (Trivial Corollary of Transfinite Induction)

Let ord be the set of ordinals $\beta \in \alpha$ with $\varphi(\beta)$ true. Assume $\operatorname{seg} \beta \subseteq \text{ord}$, i.e. for every $\gamma \in \beta \in \alpha$, $\varphi(\gamma)$ is true. Then $\varphi(\beta)$ is true by assumption. Thus, $\text{ord} = \alpha$ by transfinite induction. As such, $\varphi(\alpha)$ (because all $\beta \in \alpha$ are such that $\varphi(\beta)$) again by assumption.

$\square$

Idea for how to define V_α:

By TR we can define a function G_α with domain α given by

$$V_\beta := G_\alpha(\beta) = \bigcup\{\underbrace{\mathcal{P}G(\gamma)}_{\text{well, if we wanted to we could say } (G \restriction \operatorname{seg}\beta)(\gamma) \text{ also.}} \mid \gamma \in \beta\}.$$

Yep! :D

Im fao even gave this same example

So, if we wanna get any V_α, we just need to take $V_\alpha = G_\delta(\alpha)$ for any $\delta \ni \alpha$, e.g. $\delta = \alpha^+$.

lmao even the symbols I used are identical to the author's

Conjecture: $\operatorname{rank}(A) > \operatorname{rank}(B)$ iff the least ordinals α and β s.t. $A \in V_\alpha$ and $B \in V_\beta$ are so that $\alpha \ni \beta$.[1] (asserted in an upcoming exercise)

[1] – interesting the author defines a tiny difference. But basically the same.

Thus, since cardinals are a certain kind of least ordinals, the property that $\underbrace{\text{card } A = \text{card } B \text{ iff } \text{card } A \approx \text{card } B}_{A \approx B \Rightarrow \operatorname{rank}(A) > \operatorname{rank}(B)}$ should hold.

However, $\operatorname{rank}(A) > \operatorname{rank}(B)$ clearly does not imply $A \approx B$. (Within the same rank, two elements can be distinctly big)

Oh oops mis-rmb the gay a bit

Self Proof of Lemma 7R

Suppose, without loss of generality, that $\delta \in \varepsilon$ (i.e. $\delta = \delta \cap \varepsilon$) and let S be the set of $\alpha \in \delta \in \varepsilon$ with $F_\delta(\alpha) = F_\varepsilon(\alpha)$. Now assume $\underbrace{\operatorname{seg} \alpha \subseteq}_{F_\delta(\beta) = F_\varepsilon(\beta)}$

Then,

$$\begin{aligned} F_\delta(\alpha) &= \bigcup\{\mathcal{P}F_\delta(\beta) \mid \beta \in \alpha\} \\ &= \bigcup\{\mathcal{P}F_\varepsilon(\beta) \mid \beta \in \alpha\} \quad \text{since } \operatorname{seg}\alpha \subseteq S \text{ by our assumption.} \\ &= F_\varepsilon(\alpha) \end{aligned}$$

Hence, $\alpha \in S$ and $S = \delta\ (= \delta \cap \varepsilon)$ by transfinite induction.

Self Proof of The Numeration Theorem

This follows trivially from the Well-Ordering Theorem.

Nice my hypothesis is correct, we indeed define cardinal numbers as the least ordinal (cf ...).

Self Proof of Theorem 7P

(a) Assume card A = card B. Then $A \approx$ card A = card $B \approx B$. Conversely, suppose $A \approx B$. In the proof of Hartogs' Theorem, we already found a set α of ordinals dominated by A. So, applying a subset axiom, the least ordinal card A that is equinumerous to A exists. Similarly, such an ordinal card $B \approx B$ must exist. Consequently, card $A \approx A \approx B \approx$ card B.

(b) For a finite set $A \approx n$, $A \approx m$ implies $n \approx m$. Hence, from (a), $\underbrace{n = \text{card } n}_{\text{Follows quite easily from the Pigeonhole Principle}} = \text{card } m = m$.

Self Proof of The Well Ordering Theore

Consider $A \neq \varnothing$ since $\varnothing$ well-orders $A = \varnothing$. By the second form of AC, there is a function f from the set of nonempty proper subsets of A to A defined by $f(B) \in A - B$. Using transfinite recursion on an ordinal number α not dominated by A (whose existance is guarenteed by Hartogs' theorem), we define the function F of domain α with

$$F(\beta) = \begin{cases} a & \text{if } \beta = 0, \\ f(F[\![\beta]\!]) & \text{if } F[\![\beta]\!] \text{ is a nonempty proper subset of } A, \\ & \quad (\text{i.e. } \in \operatorname{dom} f) \\ \{A\} & \text{otherwise.} \end{cases}$$

Where a is any element of $a \neq \varnothing$. Notice that if $F[\![\beta]\!]$ is a nonempty proper subset of A, then for $\gamma \in \beta$, $F[\![\gamma]\!] \subseteq F[\![\beta]\!]$ and in which $F[\![\gamma]\!] \neq \varnothing$ as $a \in F[\![\gamma]\!]$ as long as $\gamma \geq 1$. Furthermore, we see that $F(\gamma) = f(F[\![\gamma]\!]) \notin F[\![\gamma]\!]$ but $F(\gamma) \in F[\![\beta]\!]$. Hence, every $\gamma \in \beta$ is such that $F[\![\gamma]\!] \subset F[\![\beta]\!]$ and $F[\![\gamma]\!] \neq \varnothing$ unless $\gamma = 0$ in which case $F(\gamma) = a$. In other words, note that we have just shown that if $F(\beta) \in A$, then $F(\beta) \neq F(\gamma) \in A$ given $\gamma \in \beta$ (→ injectivity up to β). Now, for the least ordinal ℓ such that $F(\ell) \notin A$ (i.e. $F(\ell) = \{A\}$, which must exist lest A dominates α (bijection would be given by F)), $F \restriction \ell$ must hence be an injection into A. In fact, it must be a bijection, lest $F[\![\ell]\!] \subset A$ but $F(\ell) = \{A\} \notin \operatorname{ran} f$. Consequently, by Lemma 7F we can define the well-ordering $<$ on A with

$$\beta \in \alpha \quad \text{iff} \quad F(\beta) < F(\alpha).$$

since $\in$ well-orders α.

Self Proof of The Well-Ordering Thm

Idea

Let N be the set of nonempty proper subsets of A (proper $B \Rightarrow A - B \neq \emptyset$) and consider $A \neq \emptyset$ since $\emptyset$ well-orders $A = \emptyset$.

By second form of AC, there is a function $f : N \to A$ defined by $f(B) \in A - B$.

using TR on an ordinal number α, — given by Hartogs' Theorem

$$F(\beta) = \begin{cases} a & \text{if } \beta = 0, \\ f(F[\![\operatorname{seg}\beta]\!]) & \text{if } \underset{\neq\emptyset}{F[\![\operatorname{seg}\beta]\!]} \subset A \ (\text{i.e. } \in N) \\ \{A\} & \text{otherwise.} \end{cases}$$

for some $a \in A \neq \emptyset$.

$\underset{\neq\emptyset}{F[\![\operatorname{seg}\beta]\!]} \subset A \Rightarrow F[\![\operatorname{seg}\gamma]\!] \subset F[\![\operatorname{seg}\beta]\!]$ ($\&$ $F[\![\operatorname{seg}\gamma]\!] \neq \emptyset$ if $\gamma \neq 0$)

by TI

$\Rightarrow$ F is an injection into A "up to some point".

Take the least ordinal ℓ of α such that $F(\ell) \notin A$, i.e. $F(\ell) = \{A\}$ which must exist lest A dominates α.

$A = F[\![\ell]\!]$ must be true, lest $F[\![\ell]\!] \subset A$ but $F(\ell) = \{A\}$.
$\Rightarrow$ bijective.

$F \restriction \ell$ is a bijection from ℓ into A. By Lemma 7F, we can define a well-or[dering]

$<$ on A by $\beta \in \alpha$ iff $F(\beta) < F(\alpha)$.

Footnote

on right, $\beta = \operatorname{seg}\beta$ for any ordinal β.

Oh wait we don't even need TI lol

$\Downarrow$

Let Ord be the set of ordinal numbers $\beta \in \alpha$ s.t. if $F[\![\beta]\!]$ is a nonempty subset of A (i.e.

then for any $\gamma \in \beta$ $F[\![\gamma]\!]$ is also a ~~nonempty~~ subset of $F[\![\beta]\!]$

which is nonempty as long as $\gamma \neq 0$.

Assume $\operatorname{seg}\beta \subseteq \text{Ord}$ and $F[\![\operatorname{seg}\beta]\!] \in N$.

Nonemptiness of any $F[\![\gamma)$ is guaranteed for each $\gamma \geq 1$ as $a \in F[\![\gamma]\!]$. We know $\beta \neq 0$ as $F[\![0$

By defn, since $\gamma \in \beta$, $F[\![\gamma]\!] \subseteq F[\![\beta]\!]$ is ce

Also, $F(\gamma) \notin F[\![\gamma]\!]$ as $F(\gamma) = f(F[\![\gamma]\!]) \notin F[\![\gamma]\!]$

Thus, since $F(\gamma) \in F[\![\beta]\!]$, $F[\![\gamma]\!]$ is a nonempty subset of

Self Proof of Hartogs' Theorem

Assume that W is the set of all well-orders $<_X$ on some subset X of A, which exists by a subset axiom on $\mathcal{P}(A \times A)$. Now define the formula φ wit

$$\varphi(x, y) \quad \text{iff there exists } X \subseteq A \text{ so } x \text{ is a well-order on } X \text{ and } y \text{ is the ordinal number of } \langle X, x \rangle, \text{ or}$$
$$\text{for all } X \subseteq A, \quad x \text{ is not a well-order on } X \text{ and } y = \varnothing.$$

By a replacement axiom on W and φ, there exists a set Ord containing all the ordinal numbers α of every well-ordered subset of A.

Further notice that given any ordinal number α such that there is some injection $f : \alpha \to A$, we can define the well-ordering $<_X$ on $X := \operatorname{ran} f$ using

$$\gamma \in_\alpha \beta \quad \text{iff} \quad f(\gamma) <_X f(\beta).$$

This is equivalent to $x <_X x'$ iff $f^{-1}(x) \in_\alpha f^{-1}(x')$, and so, $<_X$ is indeed a well-ordering on X by Lemma 7F(c). As such, Ord certainly contains every ordinal number dominated by A.

We see that $\text{Ord} \subseteq (\bigcup \text{Ord})^+$ where $(\bigcup \text{Ord})^+$ must be an ordinal number by Corollary 7N. Moreover, for any $\alpha \in \text{Ord}$, $\alpha \in (\bigcup \text{Ord})^+$. Consequently, $(\bigcup \text{Ord})^+ \neq \alpha$ follows from trichotomy. In other words, $(\bigcup \text{Ord})^+ \notin \text{Ord}$.

Wherefore, $(\bigcup \text{Ord})^+$ is an ordinal number not dominated by A because Ord contains all ordinal numbers dominated by A. $\square$

$\alpha \sim X$

For any ordinal α so that there exists an injection $f : \alpha \to A$, we can define the wellordering $<_X$ on $X := \operatorname{ran} f$ by

$$\gamma \in_\alpha \beta \quad \text{iff} \quad f(\gamma) <_X f(\beta), \quad \text{i.e.} \quad x <_X x' \quad \text{iff} \quad f^{-1}(x) \in_\alpha f(x').$$

Since $\in_\alpha$ is a wellordering on α, $<_X$ is a wellordering on X as well by Lemma 7F.

Self Proof of Hartogs' Theorem

Let A be any set and $\underbrace{S \text{ the set of all ordinals } \cancel{\neq} \text{ dominated by } A}$. Notice that $S \subseteq (\bigcup S)^+ \subset (\bigcup S)^{++}$. Hence, the ordinal $(\bigcup S)^{++} \notin S$ is an ordinal not dominated by A.

$\Downarrow$

How do you know there exists such a set of ordinal numbers

Ideas Assume, ftsoc, that A dominates all ordinals $\alpha \xrightarrow{inj} A$.
$\llcorner$ No upper bound γ

Ideas

$\underbrace{a^+ \quad a^{++} \quad a^{\cdots}}$
We don't know if this is big enough for A

$H: \alpha \longrightarrow A$

$H(\beta) = f(A - H[\![\text{seg } \beta]\!])$

$f: \mathcal{P}A - \{\varnothing\} \longrightarrow A$

$f(B) \in B$

$i: \alpha \longrightarrow A$

$\text{ran } i \subseteq A$

Suppose $i_\alpha: \alpha \to A$, $i_\beta: \beta \to A$ with $\beta \in \alpha$

$(i_\alpha \restriction \beta): \beta \to A$ (tho $i \restriction \beta \neq i_\beta$ necessarily)

Set W of all well-orders $<_x$ on some subset X of A, exists by subset axiom on $\mathcal{P}(A \times A)$

Well-order $<$ on $X \subseteq A$

~~Set W_x of well-orders $<_x$ on $X \subseteq A$ exists~~ by a subset axiom on $\mathcal{P}(A \times A)$, ~~Set W of all W_x which exists by~~ subset axiom on $\mathcal{P}\mathcal{P}(A$
$\bigcup W$

$\varphi(x, y)$ iff $y = \langle \alpha, \in_\alpha \rangle \cong \langle X, <_x \rangle$
& $x = <_x$ for some $X \subseteq A$, or

$y = \varnothing$ & x is not a well-order on any $X \subseteq A$.

Set Ord of all ordinals corresponding to some well-order $<$ in W.

$\bigcup$Ord is an ordinal, in fact the sup of Ord

Consequently, we now indeed have that each ordinal number α is dominated by L (the injection being provided by F), contradicting with Hartogs' Theorem. So, it mustn't hold true that $U_C - C$ is always nonempty. Wherefore, there indeed exists a maximal element of A (this case has been discussed at the start of the proof). $\square$

21. Let U_C be the set of upper bounds of a subset C of A, that is linearly ordered by $<$, and L the set of all such C. If $U_C - C$ is empty for some $C \in L$, it trivially follows that U_C is the singleton set containing the largest member of C, which is maximal in A. Now consider $U_C - C \neq \varnothing$ for every $C \in L$. Applying the second form of AC, we can define the function $f : L \to \bigcup_{C \in L} U_C$ by $f(C) \in U_C$.

By transfinite recursion, there exists some function F with domain α and

$$F(\beta) = \begin{cases} \bigcup F[\![\operatorname{seg}\beta]\!] \cup \{f(\bigcup F[\![\operatorname{seg}\beta]\!])\} & \text{if } \bigcup F[\![\operatorname{seg}\beta]\!] \text{ is a linearly ordered subset of } A \text{ (i.e. } \in L\text{), or} \\ \varnothing & \text{otherwise.} \end{cases}$$

for each ordinal α. (To achieve this, simply define $\gamma(g, y)$ in a similar fashion)

We want to show F is an injection from any ordinal α to L to contradict Hartogs' Theorem. To do this, we shall utillise transfinite inducti[on]

First suppose B_{ord} is the set of all $\beta \in \alpha$ with $\gamma \in \beta$ implying $F(\gamma) \subset F(\beta)$. Assume $\operatorname{seg}\beta \subseteq B_{ord}$. In other words,

$F(\gamma)$ is always nonempty, lest $F(0) = \{f(\varnothing)\} \subset \varnothing = F(\gamma)$, a contradiction. Hence, $F(\gamma) \in L$ for any $\gamma \in \beta$.

given $\gamma \in \beta$, $\mu \in \gamma$ entails $F(\mu) \subset F(\gamma)$. We see that $\bigcup F[\![\operatorname{seg}\beta]\!]$ is a subset of A linearly ordered by $<$, i.e. $\in L$.

Presume x, y, and z are any three of its elements.

Trichotomy: For some μ and γ, $x = F(\mu)$ and $y = F(\gamma)$. Without loss of generality, we can assert that $\mu \underline{\in} \gamma$. Thus, $F(\mu) \subset F(\gamma)$ so that $x, y \in F(\gamma)$ $(\in L)$.
By the trichotomy of $<^{\circ}$, ... well ... trichotomy must hold.

Transitivity: Again, without loss of generality we assert $x = F(\mu)$, $y = F(\gamma)$, and $z = F(\eta)$ is such that $\mu \underline{\in} \gamma \underline{\in} \eta$, therefore, $x, y, z \in F(\eta)$ $(\in L)$.
As such, transitivity certainly holds.

$\bigcup F[\![\operatorname{seg}\beta]\!]$ must thence be in L, as desired. $\diamond$

Since $f(\bigcup F[\![\operatorname{seg}\beta]\!]) \in U_{\bigcup F[\![\operatorname{seg}\beta]\!]} - \bigcup F[\![\operatorname{seg}\beta]\!]$, $F(\gamma) \subset F(\beta)$ is clearly ensured. Accordingly, $\beta \in B_{ord}$. By transfinite indu[ction]
$B_{ord} = \alpha$.

21. ~~First notice that A must be nonempty, lest $A=\varnothing$ is a linearly ordered subset of itself, and hence, there exists a maximal element in $\varnothing$.~~ Suppose U_C is the set of upper bounds of a well-ordered subset C of A, and that S is the set of all well-ordered subsets of A. Then if $U_C - C$ is empty for some $C \in S$, trivially it follows that U_C is the singleton set containing the largest member of C, which is maximal in A. Otherwise, $U_C - C$ is nonempty for every $C \in S$, allowing us to apply the second form of AC to define the set $f : S \to \bigcup_{C \in S} U_C$ by $f(C) \in U_C$. Notice that A must be nonempty, lest $A = \varnothing$ is a linearly ordered subset of itself without a upper bound. As such, there exists an element $a \in A$. Clearly, $\{a\}$ is a well-ordered set since the only nonempty subset of itself is itself, having the least element a.

Ideas

$\alpha \times A$

$\gamma(g, y) \Longleftrightarrow \bigcup \operatorname{ran} g \cup \{f(\bigcup \operatorname{ran} g)\}$ if $\bigcup \operatorname{ran} g \in S$ or

$\varnothing$ otherwise.

Apply TR on $\omega \times \omega$ with $<_L$

$F(t) = \bigcup F[\![\operatorname{seg} t]\!] \cup \{f(\bigcup F[\![\operatorname{seg} t]\!])\}$

$\bigcup F[\![\omega \times \omega]\!]$ is countable by AC

$\operatorname{card}({}^{\omega}2) = \operatorname{card}(\mathbb{R})$

$\operatorname{card}(\mathbb{R}) < \operatorname{card}({}^{\omega}\omega)$

... Contradicts Hartogs! ...

C s.t. $C \cup \{f(C)\} \subseteq C$ & C

$h(0) = \{a\}$

$h(n^+) = h(n) \cup \{f(h(n))\}$ ✓

$F(h(n))$

$F(x) = x \cup \{f(x)\} \in S$

$x \in S$

$\Rightarrow F : S \to S$ ✓

$C := \bigcup h[\![\omega]\!]$

Let I be the set of $\beta \in \alpha$ with $\gamma \in \beta$ implying $F(\gamma) \subset F(\beta)$.

Assume $\operatorname{seg} \beta \subseteq I$. That is, $\gamma \in \beta$ is s.t. $\mu \in \gamma$ ⇒ $F(\mu) \subset F(\gamma)$.

For all $\gamma \in \beta$, $F(\gamma) \neq \varnothing$, lest there is one such γ w/ $F(\gamma) = \varnothing$. But then $0 \in \gamma$ and $F(0) = \{f(\varnothing)\}$. Thus $F(0) \not\subseteq F(\gamma)$ despite $0 \in \gamma$, contradiction our ass.

Trichotomy & trans. is clear of $\bigcup F[\![\operatorname{seg} \beta]\!]$

$\Rightarrow \ell_\beta := \bigcup F[\![\operatorname{seg} \beta]\!]$ is a linearly ordered set ...

Since $f(\ell_\beta) \in U_{\ell_\beta} - \ell_\beta$,

$F(\gamma) \subset F(\beta)$ is clearly ...

1. Let the set of upper bounds of C be $U_C \neq \emptyset$. Assume for the sake of contradiction that $U := \bigcup_{C \in P(A)} U_C$ has no maximal element, that is, for all $u \in U$ there exists $v \in U$ with $u < v$. Suppose that the set of all such v is $V_u \neq \emptyset$ and $V := \bigcup_{u \in U} V_u$. Now by the second form of AC, there is a function $f : U \to \bigcup_{u \in U} V_u$ such that $f(u) \in V_u$. By the recursion theorem for ω, we know there exists the function $h : \omega \to \bigcup_{u \in U}$ with

$$h(0) = u,$$
$$h(n^+) = h(f(h(n))),$$

for some fixed $u \in U$. From strong induction, we can show that $h[\![\omega]\!]$ is a linearly ordered set.

upper bound u_h of $h[\![\omega]\!]$

$f: P \to A \quad f(L) = u$ (upper bound of L)

Aim: Find maximal element of A

ran f $\Rightarrow$ Either a loset or not

$\Downarrow$

Take P_f (set of all losets that are a subset of ran f)

$$f[\![P_f]\!] \subseteq \text{ran } f$$

$$(\exists u \in A)(\forall v \in A)(u \nless v) \quad \text{O/R} \quad (\forall u \in A)(\exists v \in A)(u < v)$$

Let L be a linearly ordered subset of A and U_L be the set of upper bounds of L: $U_L - L = \emptyset$

$\hookrightarrow m \in U_L$ is maximal

$U_L - L \neq \emptyset$

exists upper bound u not in L.

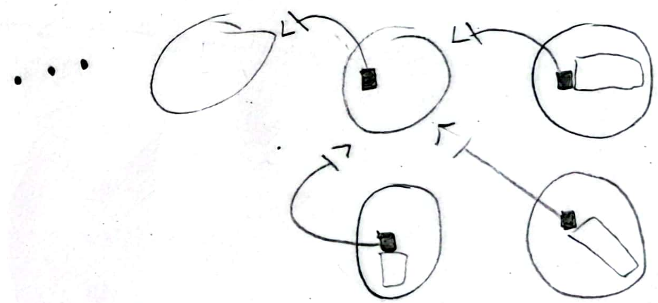
~~$h(0) < h(m) < h(m^+)$~~

~~Assume $h(n) < h(m)$ for all $m \in \omega$ with $n \in m$~~

~~Base case of 0 ✓~~

~~$h(n^+) < h(m)$~~

Show $n \in m \Rightarrow f(n) < f(m)$



Assume for all natural numbers $n \in m$,

$$h(k) < h(n) \text{ if } k \in n$$

For any $k \in n^+$, either $k \in n$ or $k = n$

$k \in n$: $\quad h(k) < h(n) < h(n^+)$

$k = n$: $\quad h(n) < h(n^+)$

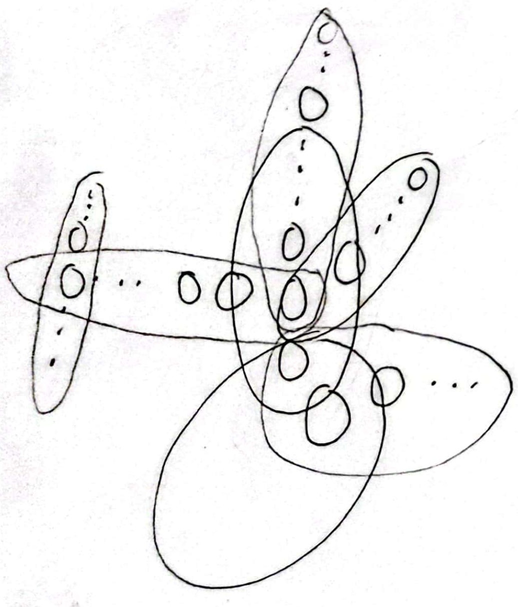
19. Assume ~~A ≈ ... for some natural n, and~~ for the sake of contradiction, that $\langle A, < \rangle \ncong \langle A, \prec \rangle$. Then without loss of generality, we can say that $\langle A, < \rangle \cong \langle \operatorname{seg} a, \prec \rangle$ for some $a \in A$. Therefore, $A \approx \operatorname{seg} a$, which is a proper subset of the finite set A. Hence, this contradicts Corollary 6C. Consequently, it must be that $\langle A, < \rangle \cong \langle A, \prec \rangle$ instead. $\square$

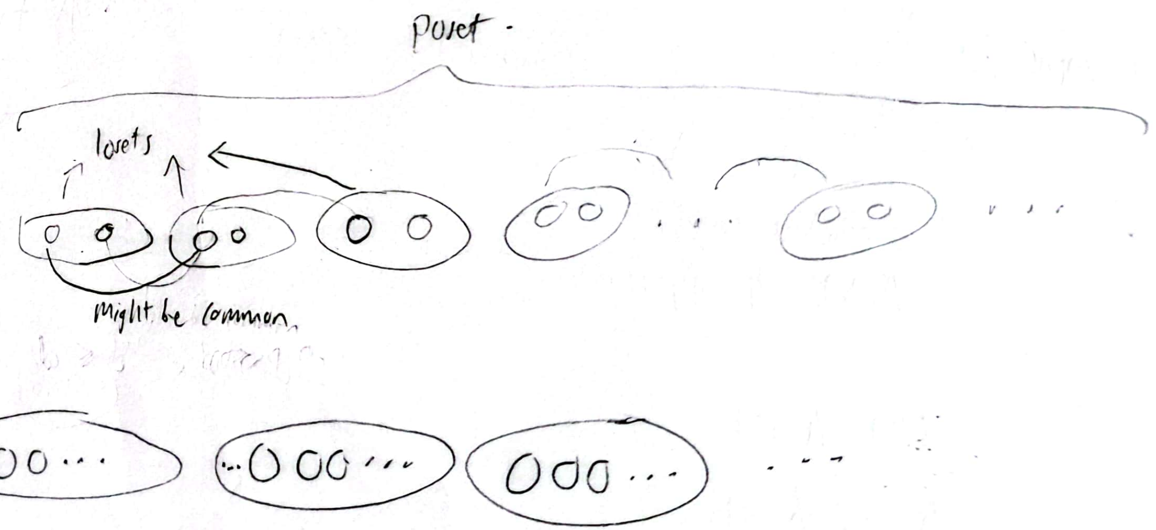
20. Suppose S is infinite (for the sake of contradiction). Then, $\omega \preccurlyeq S$. So either $\langle \omega, \in_\omega \rangle \overset{f}{\cong} \langle S, R \rangle$ or $\langle \omega, \in_\omega \rangle \overset{g}{\cong} \langle \operatorname{seg} s, R \rangle$. In the former, simply let u be the least member of S with respect to R^{-1}, thus u is also the greatest element of S w.r.t R. Consequently, for any $s \in S$, sRu or $s = u$. Which means $f(s) R f(u)$. Since f is a bijection, $f(u)$ is the largest member of ω w.r.t. to membership. Yet $f(u) \in_\omega f(u)^+$, a contradiction. As for the second case, we simply repeat this except with S replaced by $\operatorname{seg} s$ to create a similar contradiction. Wherefore, it must be that S is actually finite. $\square$

21. Set P of all losets of A (by subset axiom on A). $P' := \{ L \in P \mid (\forall L' \in P)[L' \subseteq L \text{ or } (L' \nsubseteq L \,\&\, L \nsubseteq$

~~Losets B and C, $b \in B \cap C \Rightarrow \ell_B = b \,\&\, \ell_C = b$ or $\text{large}_B = b \,\&\, \text{large}_C = b$~~ Let U_L be the set of all upper bounds of $L \in P'$

~~where $B \nsubseteq C$~~
~~$\&\ C \nsubseteq B$~~





17. We evaluate this casewise, based on the results of Theorem 7K:

Case I $\langle A, < \rangle \cong \langle B, <^{\circ} \rangle$, then $\alpha = \beta$ is immediate.

Case II $\langle A, < \rangle \cong \langle \operatorname{seg} b, <^{\circ\circ} \rangle$. Since $b \in A$, this contradicts exercise 15(a) from the same section.

Case III $\langle \operatorname{seg} a, <^{\circ\circ} \rangle \cong \langle B, <^{\circ} \rangle$. We see that $E_A(a) = \beta$. Consequently, $\beta \in \alpha$.

Wherefore, indeed $\beta \underline{\in} \alpha$ holds true. $\square$

15. (a) Let the ordinal numbers of $\langle A, <\rangle$ and $\langle \text{seg}\, t, <^{\circ}\rangle$ be α and β respectively. Notice that $\beta = \text{ran}(E_A \restriction \text{seg}\, t) = E_A(t)$. Thus, $\beta \in \alpha$ and thus $\beta \neq \alpha$ by irreflexivity. By theorem 7I, the above structures mustn't be isomorphic.

(b) This is clear from Theorem 7M (d). $\square$

16. Notice that $\alpha \in \beta^+$ and $\gamma \in \alpha$ implies $\gamma \in \beta^+$ by transitivity. Hence, $\alpha^+ \subseteq \beta^+$ where $\alpha^+ \neq \beta^+$ lest $\bigcup \alpha^+ = \bigcup \beta^+$. i.e. $\alpha = \beta$ which contradicts the assumption that $\alpha \in \beta$. Therefore, it follows from $\alpha^+ \subset \beta^+$ that $\alpha^+ \in \beta^+$. Also, it is clear that whenever $\alpha \neq \beta$, $\alpha^+ \neq \beta^+$. $\square$

17. $\gamma(f, y) \iff y = \iota(A - \text{ran}\, f)$ if $A - \text{ran}\, f \neq \varnothing$, if $A - \text{ran}\, f = \varnothing$ or $y = \varnothing$

$\langle A, <\rangle \not\cong \langle \text{seg}_< b, <^{\circ}\rangle = \langle \text{seg}_{<^{\circ\circ}} b, <^{\circ\circ}\rangle$

$\alpha \ni \delta \quad \gamma$

$\alpha \in \beta$

$F: A \to A \cup \{\varnothing\}$

$\langle A, <\rangle \cong \langle \text{seg}\, b, <^{\circ}\rangle$

$\alpha = \gamma \in \beta$

$F(t) = \iota(A - F[\![\text{seg}\, t]\!])$ $\quad F[\![\text{seg}\, x]\!]$ for $x < t$

$\beta \in \alpha \in A \to \beta \in \bigcup A$

$\alpha \subseteq \bigcup A$

$A \subseteq F[\![\text{seg}\, t]\!]$ $\quad F[\![\text{seg}\, t]\!]$

least l so $A \subseteq F[\![\text{seg}\, l]\!]$ $\quad (t < l \Rightarrow A \not\subseteq F[\![\text{seg}\, t]\!])$

$l \neq l_B$

$x \in \bigcup E[\![B]\!]$ iff exists $y \in E[\![B]\!]$ so $x \in y$

B

$\gamma \in B \Rightarrow \gamma \in E(b) \in E[\![B]\!]$

$\Rightarrow \gamma \in (\bigcup E[\![B]\!])^+$

$\langle A, <\rangle \quad \langle B, <^{\circ}\rangle$

$\cong E \quad \cong E'$

$\langle \alpha, \in_\alpha\rangle \quad \langle B, \in_B\rangle \cong \langle E[\![B]\!], \in_B\rangle$

$\text{ran}\, E' \subseteq \bigcup E[\![B]\!]$

$E'(b) \subseteq E(b)$

$\Longrightarrow \text{ran}\, E' \subseteq \bigcup E[\![B]\!] \subseteq \bigcup \alpha$

$\text{ran}\, E' \subseteq \bigcup E[\![B]\!] \subseteq \alpha \Rightarrow \beta \in \alpha$

Ass. true for any $t < b$, $E'(t) \in E(t) \Longrightarrow E'(t) \in E(b)$

$\Rightarrow E'[\![\text{seg}\, b]\!] \subseteq E(b)$

$t = l_B$ $\quad E'[\![\text{seg}\, b]\!] \subseteq E(b)$

$E'(b) \subseteq E(b)$

$E'(t) \in E(b)$

$E'(b) \subseteq E(b)$

The Burali-Forti Theorem

Assume that A is a set of ordinal numbers. Suppose, for the sake of contradiction, that $(\bigcup A)^+ \in A$. Then, $(\bigcup A)^+ \subseteq \bigcup A$ which means $\bigcup A \in \bigcup A$. However, $\bigcup A$ is an ordinal number by Corollary 7N(d). So, this result contradicts the irreflexivity of the ordinal number $\bigcup A$ (Theorem 7M(c)). Consequently, it must hold true that $(\bigcup A)^+$ is an ordinal number not included in A. Wherefore, since we have proven that for each set of ordinal numbers there exists some ordinal number not included in it, there is no set to which every ordinal number belongs indeed. $\square$

$\alpha \in A \to \alpha \subseteq \bigcup A$

$\alpha \subseteq \bigcup A$

$(\bigcup A)^+ \in A$

Proof of Corollary 7N

(a) Let T be such a transitive set. We see that $\in_T := \bigcup_{\alpha \in T} \in_T$ is certainly a linear order since $\in_\alpha \subseteq \in_\beta$ when $\alpha \in \beta$. Further notice that $\in_T$ is a well-ordering by Theorem 7M(d). Finally, Theorem 7L tells us that T is the ordinal number of $\langle T, \in_T \rangle$.

(b) 0 is the ordinal number of $\langle 0, \in_0 \rangle = \langle \emptyset, \emptyset \rangle$.

(c) By the transitivity of the ordinal number α, α^+ is too, a transitive set of ordinal numbers. By (a) it is clear that α^+ is an ordinal number.

(d) When $\gamma \in \beta \in \bigcup A$, there exists some ordinal number α with $\gamma \in \beta \in \alpha \subseteq \bigcup A$, so that $\gamma \in \bigcup A$. In other words, $\bigcup A$ is a transitive set of ordinals. Part (a) tells us that $\bigcup A$ is certainly an ordinal number.

$\square$

Self Proof of Theorem 7M

(a) By Corollary 7H, $\langle \alpha, \in_\alpha \rangle$ is a well-ordered structure. From Theorem 7J, so is $\langle \beta, \in_\alpha^\circ \rangle$. The transitivity of α tells us that if $\eta \in \gamma \in \beta$, then $\eta \in \gamma \in \alpha$ so $\eta \in \alpha$. Thus, β is also a transitive set. Theorem 7L now entails β is an ordinal number (of $\langle \beta, \in_\alpha^\circ \rangle$).

(b) This is just Theorem 7D(d).

(c) Assume that S is the set of all $\beta \in \alpha$ such that $\beta \notin \beta$, and $\operatorname{seg} \beta \subseteq S$. In other words, for any $\gamma \in \beta$, $\gamma \notin \gamma$. Suppose, for the sake of contradiction, that $\beta \in \beta$. But then $\beta \in \operatorname{seg} \beta$ now, which means $\beta \notin \beta$. Therefore, it must be that $\beta \notin \beta$. So, $\beta \in S$ and $S = \alpha$ by transfinite induction. Consequently, $\alpha \notin \alpha$ lest $\alpha \in \alpha$, implying $\alpha \notin \alpha$.

(d) Let α and β be the $\in$-images of $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$ respectively. Consider the following unique cases brought forth by Theorem 7K:

Case I $\langle A, <_A \rangle \cong \langle B, <_B \rangle$, then $\alpha = \beta$ is trivial using Theorem 7I.

Case II $\langle \operatorname{seg} a, <_A^\circ \rangle \cong \langle B, <_B \rangle$. It follows that $E_A(a) = \operatorname{ran}(E_A \restriction \operatorname{seg} a) = \beta$ (again utilising Theorem 7I). Hence, $\beta \in \alpha$.

Therefore, at least one of $\alpha = \beta$, $\alpha \in \beta$, and $\beta \in \alpha$ holds true. By part (b) and (c), exactly one of them is true. Thence, trichotomy holds.

(e) There exists some ordinal number $\alpha \in S$ since S is nonempty. When α is not already the least element of S, $\alpha \cap S \neq \emptyset$. Let l_S be the least element of $\alpha \cap S$, which must exist because α is well-ordered by Corollary 7H. For any $\beta \in S$, either $\beta \in \alpha$ or $\alpha \subseteq \beta$ by (d). Correspondingly, $l_S \subseteq \beta \in \alpha$ or $l_S \in \alpha \subseteq \beta$. i.e. $l_S \subseteq \beta$ for all $\beta \in S$. Wherefore, there indeed exists a least element of S. (which is l_S if α is not already least).

Remarks: Oh bruh for (c) we can just use Theorem 7D(a) which makes it trivial.

Self Proof of Theorem 7L

ran $\alpha = \{E(\beta) \mid \beta \in \alpha\}$

~~$\emptyset \in \alpha$ if $\alpha \neq \emptyset$, lest $\gamma \in \beta \in \alpha$~~

~~$\bigcup \beta \subseteq \alpha$~~

~~$\bigcup \bigcup \cdots \bigcup \beta \subseteq \alpha$~~

~~Claim: $\emptyset \in \alpha$ is certain~~

$\Rightarrow$ Just use transitivity + well-ordering $\in_\alpha$

$\operatorname{seg} \beta \subseteq S$ of $\beta \in \alpha$ with ~~$E(\beta) \in \alpha$~~ $E(\beta) = \beta$

$E(\beta) = \{E(\gamma) \mid \gamma \in_\alpha \beta\}$ $E(\gamma) = \gamma$

$= \beta$

$\Rightarrow S = \alpha$

$\operatorname{ran} E = \{E(\beta) \mid \beta \in \alpha\}$

$= \{\beta \mid \beta \in \alpha\}$

$= \alpha$

Ideas

$\{\mathbb{R}, \{\mathbb{R}\}, \{\mathbb{R}, \{\mathbb{R}\}\}\}$ cannot be α as $1 \in \mathbb{R} \in \{\mathbb{R}\}$ but $1 \notin \{\mathbb{R}\}$

Show that E bijects α onto α

let $\beta \in \alpha$

Let S be the set of all $\beta \in \alpha$ with $E(\beta) = \beta$. Assume $\operatorname{seg} \beta \subseteq S$. Clearly, $E(\beta) = \beta$:

$$E(\beta) = \{E(\gamma) \mid \gamma \in_\alpha \beta\}$$

$$= \{\gamma \mid \gamma \in \beta\} \text{ by the transitivity of } \alpha,\ \gamma \in \alpha$$

$$= \beta.$$

Therefore, $\beta \in S$ and $S = \alpha$ by transfinite induction. Now,

$$\operatorname{ran} E = \{E(\beta) \mid \beta \in \alpha\}$$

$$= \{\beta \mid \beta \in \alpha\}$$

$$= \alpha.$$

Indeed, α is the ordinal number of the well-ordered structure $\langle \alpha, \in_\alpha \rangle$.

Now consider the case where $\operatorname{dom} F \subset A$. Similarly, pick the least member ℓ_Δ of $A - \operatorname{dom} F$. By the same reasoning as before, $\operatorname{seg} \ell_\Delta = \operatorname{dom} F$. Assume, for the sake of contradiction, that $B - \operatorname{ran} F \neq \varnothing$. Again, choose the least element $\ell_\square$ of $B - \operatorname{ran} F$. Then, $F \cup \{\langle \ell_\Delta, \ell_\square \rangle\} \in F$. This entails that $\ell_\square \in \operatorname{ran} F$, contradicting our assumption that $\ell_\square \in B - \operatorname{ran} F$. Therefore, $\operatorname{ran} F = B$ is certain. So, $\langle \operatorname{seg} \ell_\Delta, <_A^\circ \rangle \cong \langle B, <$

Wherefore, it is indeed true that one of the alternatives

$$\langle A, <_A \rangle \cong \langle B, <_B \rangle,$$
$$\langle A, <_A \rangle \cong \langle \operatorname{seg} b, <_B^\circ \rangle \text{ for some } b \in B,$$
$$\langle \operatorname{seg} a, <_A^\circ \rangle \cong \langle B, <_B \rangle \text{ for some } a \in A;$$

$\square$

holds true.

Se 10 Proof of Theorem 7J

Proof

Let f be a ('two way') order preserving bijection from $\{a\} \cup \operatorname{seg} a$ into $\{b\} \cup \operatorname{seg} b$, and A_f be the set of all such f. Suppose there exists some other bijection f' with the same order preserving property as f, but which maps $\{a'\} \cup \operatorname{seg} a'$ into $b' \cup \operatorname{seg} b'$ where $a \le_A a'$. Further presume, for the sake of contradiction, that $f'(t) \neq f(t)$ for some $t \le_A a$, and S is the set of all such t. By virtue of $<_A$ being a well-ordering, there is a least member l_S of S so that $f'(l_S) \neq f(l_S)$.

~~It is clearly impossible for l_S to be the least element l_A of A, lest $f'(l_S) \neq l_B$ or $f(l_S) \neq l_B$. In the first case, we simultaneously have that $f'(t) = l_B$ for some $t >_A l_S = l_A$. This condition on t implies that $f'(t) >_B f'(l_S)$. While, by the leastness of l_B in B, $f'(l_S) >_B f'(t)$. But this clearly violates trichotomy of the well-order $<_B$. The latter is a similar case as well. Hence, $l_S \neq l_A$~~

Now presume, without loss of generality, that $f'(l_S) < f(l_S)$. The surjectivity of f tells us that there is some $t \le a$ with $f'(l_S) = f(t) <_B f(l_S)$. Accordingly, $t < l_S$. Which means $f'(t) < f'(l_S)$. Therefore, contradicting with the trichotomy of the well-ordering $<_B$.

Consequently, it must be that $f'(t) = f(t)$ for all $t \le a$. In other words, $f \subseteq f'$. As such, the following $F := \bigcup A_f$ must be a func
In the case that $\operatorname{dom} F = A$, if $\operatorname{ran} F = B$ then $\langle A, <_A \rangle \cong \langle B, <_B \rangle$ is immediate. When $\operatorname{ran} F \neq B$, take the least elemen
$l_\square$ of $B - \operatorname{ran} f$. We see that $\operatorname{seg} l_\square \subseteq \operatorname{ran} F$. In fact, $\operatorname{seg} l_\square = \operatorname{ran} F$, lest there is some $b >_B l_\square$ in $\operatorname{ran} F$. But this
mean $l_\square \in \operatorname{seg} b \subseteq \operatorname{ran} F$, contradicting our construction of $l_\square$. Accordingly, $\langle A, <_A \rangle \cong \langle \operatorname{seg} l_\square, <_B^\circ \rangle$.

Let f be a 'tw' opb from seg a $(\alpha_A > \alpha_B)$ into seg b, A_f be the set of all such f

g : seg b $(\alpha_B > \alpha_A)$ into seg a, B_f

Claim: $F := \bigcup A_f$ is a function

If dom $F = A$, result is immediate.

When dom $F = \overline{\text{seg}}\, a \subset A$, ran $F = B$

Claim: dom $G = B$ where $G := \bigcup B_f$

i.e. there is a g : seg $b \to$ seg a for all $b \in B$.

lest there exists $\ell_\square \in B - \text{ran } F$

$F \cup \{\langle a^+, \ell_\square \rangle\}$

$f' : \overline{\text{seg}\, a'} \longrightarrow \overline{\text{seg}\, b'}$

wlog suppose $a < a'$

$\overbrace{f' \upharpoonright \text{seg } a}^{F} = f$

least ℓ_s so $F(\ell_s) \neq f(\ell_s)$ $(t < a)$.

$\ell_s \neq \ell_A$ lest $F(\ell_s) \neq \ell_B$ or $f(\ell_s) \neq \ell_B$

& $F(t) = \ell_B$ $(t > \ell_s = \ell_A)$

$\Rightarrow F(t) > F(\ell_s)$

and $F(t) < F(\ell_s)$

simultaenously

Assume wlog $F(\ell_s) < f(\ell_s)$.

$f(\ell_s^-) = F(\ell_s^-) < F(\ell_s) < f(\ell_s)$

exists some t with $f(t) < f(\ell_s)$

$t < \ell_s$

$F(t) = F(\ell_s)$

Cont.

$\diamond$

ran $F := \{F(x) \mid x \in A\}$

$= \{\text{seg } b$

ran $F \neq B \Rightarrow$ exists $b \in B$ with $b \notin$ ran F.

Take least member of $B -$ ran F, $\ell_\square$

Theorem 7J Ideas

By A.C., either $A \preccurlyeq B$ or $B \preccurlyeq A$. $\underbrace{\text{Assume } A \prec B \text{ wlog}}_{\text{impossible for } \langle A, <_A \rangle \cong \langle B, <_B \rangle}$. With the injection being provided by some function $f : A \to B$

$\operatorname{ran} f \xrightarrow{I} B$

$B \succ \operatorname{ran} f$ lest $A \approx B$

Claim: $\operatorname{ran} f \approx \operatorname{seg} b$ for some $b \in B$

$\{\langle l_A, l_B \rangle\}$

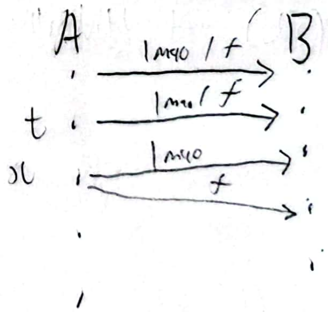
$\{\langle l_A, l_B \rangle, \langle l_A^+, l_B^+ \rangle\}$

Both exists { let f be a 'two way' order preserving injection from seg a into B ($\alpha_A \supset \alpha_B$) ($\alpha_A > \alpha_B$) with $f(l_A) = l_B$, set of all such f be .
g 'tw' op injection seg b into A

f and f' both twopi's from seg a and seg a', respectively, into B
Wlog, $a <_A a'$.

$$\overbrace{f' \restriction \operatorname{seg} a}^{\text{Imao}} = f$$

Let S_f be the set of all $t \in \operatorname{seg} a$ for which $\text{Imao}(t) \neq f(a)$.
Some least l_s exists. Wlog, assume $\text{Imao}(l_s) < f(l_s)$. $l_s \neq l_A$ by defn



Theorem 7C
Idea

Self Proof of Theorem 7J

Define the identity function $I : C \to A$, with $I(c) = c$. Notice that $c <^{\circ} c'$ iff $c < c'$, which in turn is just $I(c) < I(c')$. By Lemma 7F, the result of this theorem is now clear. $\square$

Self Proof of Theorem 7K

If isomorphic, result is immediate.

When $\langle A, <_A \rangle \not\cong \langle B, <_B \rangle$, let S be the set of all (two-way) order preserving injections ~~bijections~~ $f : A \to B$, for some $b \in B$. $f' : B \to A$

$\alpha_A = \alpha_B$, $\alpha_A \subset \alpha_B$, or $\alpha_B \subset \alpha_A$

$\alpha_A \neq \alpha_B$

$\Rightarrow$ For some $E_A(x) \in \alpha_A$, $E_A(x) \notin \alpha_B$

$E_B(y) \in \alpha_B$, $E_B(y) \notin \alpha_A$

$E(x) := E[\operatorname{ran} x]$

$= \{E(t) \mid t < x\}$

$E_A(x)$ $E_B(x)$

α_A	α_B
0	0
1	1
2	2
$\vdots$	$\vdots$

$\cancel{(\forall x \in A)}(E_A(x) \in \operatorname{ran} E_B)$ or

If $E_A(t) \in \operatorname{ran} E_B$ for all $t <_A x$,

$E_A(x) = \{E_A(t) \mid t <_A x\}$ $\quad \operatorname{ran} E_B = \{E_B(y) \mid y \in B\}$

$= \{E_B(y) \mid \quad \}$ exists some $y \in B$ so $E_A(t) = E_B($

$(\exists t <_A x)[E_B(y) = E_A(t)]$

$\subseteq \operatorname{ran} E_B$

Ideas

$\Rightarrow$ need to show

$(\exists t <_A x)[\cdots]$ iff $y < b$ for some $b \in B$

+ What fundamental condition needed for $\operatorname{ran} A \subseteq \operatorname{ran} E_B$? if any

Self Proof of Theorem 7I

(⇐) Assume $\alpha_{A_1} = \alpha_{A_2}$ / ran E_1 = ran E_2

$E_1 \circ E_2^{-1}$ is a bijection from A_1 into A_2 — oops should be $E_2^{-1} \circ E_1$

Injectivity

$$E_1(E_2^{-1}(x)) = E_1(E_2^{-1}(x'))$$
$$E_2^{-1}(x) = E_2^{-1}(x') \quad \text{inj of } E_1$$
$$x = x' \quad \text{inj of } E_2^{-1}$$

Surjectivity

Presume $y \in A_2$. Since E_2^{-1} bijects ran E_2 into A_2, there exists some $x' \in A_2$ so that $E_2^{-1}(x') = y$.

Similarly, $x' = E_1(x)$ for some $x \in A_1$. Therefore,

$E_2^{-1}(E_1(x)) = y$.

Indeed, bijection $\Rightarrow \langle A_1, <_1\rangle \cong \langle A_2, <_2\rangle$

Oh wait we can just say

$\langle A_1, <_1\rangle \cong \langle \text{ran } E_1, \in_{\text{ran } E_1}\rangle = \langle \text{ran } E_2, \in_{\text{ran } E_2}\rangle \cong \langle A_2, <_2\rangle$

Hence by Thm 7E, $\langle A_1, <_1\rangle \cong \langle A_2, <_2\rangle$.

Actually I think I got it and I'll not bother to rewrite this more formally to save on time. Checkbox below after checking with author's proof:

Pt 1 Pt 2

~~Proof on next page~~

Yeah lmao the author uses tfr when I use well-ordered property and vice versa. Shld be fine still :)

Ideas

(⇒) Suppose $\langle A_1, <_1\rangle \cong \langle A_2, <_2\rangle$ and that this bijection from A_1 into A_2 is provided by the function F.

Let the set of $x \in A_1$ s.t.

$$E_1(x) \neq E_2(F(x))$$

be S.

By the well-ordered nature of $\langle A_1, <_1\rangle$, there is some least $l_S \in S$. (as long as $S \neq \emptyset$)

$l_S \neq l_{A_1}$ since $F(l_{A_1}) = l_{A_2}$. So, $E_1(l_{A_1}) = E(F(l_{A_1})) = \emptyset$.

When l_S not least in A_1;

$E_1(x) = E_2(F(x))$ for any $x < l$

$$E_1(l) = \{E_1(x) \mid x <_1 l\}$$
$$= \{E_2(F(x)) \mid x <_1 l\} = \{E_2(x') \mid x' <_2 F(l)\}$$
$$= E_2(F(l))$$

$F(x) <_2 F(l)$

since F preserves order (more precisely, $x <_1 l$ iff $F(x) <_2 F(l)$)

$\Rightarrow S = \emptyset$

$\Rightarrow$ For all $x \in A_1$, $E_1(x) = E_2(F(x))$.

M1) ran $E_1 = \{E_1(x) \mid x \in A_1\}$
$= \{E_2(F(x)) \mid x \in A_1\}$
$= \{E_2(x') \mid x' \in A_2\}$ surjectivity of F
$=$ ran E_2.

M2)
$E_1(x) \in$ ran E_2
ran $E_1 \subseteq$ ran E_2
since argument done wlog, ran $E_2 \subseteq$ ran E_1 as well.
Thence, ran E_1 = ran E_2

17.

Ideas $S \subseteq \mathcal{P}(A)$

Injectivity Suppose $F(a) = F(a')$

$F(a) - seg(a) = F(a') - seg(a)$

$a < a'$

$\Downarrow$

$F(a) - seg(a') = \varnothing$

Cont.

Preserves order $a < a'$

$F(a) \subset F(a')$ is clear because for $x \in F(a)$, $x \leq a < a'$. Thus, $x < a'$ by transitivity of the partial ordering $<$ and $x \in F(a')$.

Proof

Bijectivity

Suppose $F(a) = F(a')$, and for the sake of contradiction, that $a < a'$. Then, $F(a) - seg\ a' = F(a') - seg\ a'$. But now $\varnothing = \{a'\}$ which is clearly impossible. So, it must hold that $a = a'$. As such, injectivity is certain. By definition, $S := \operatorname{ran} F$, guarenteeing surjectivity. Entailing that F bijects A into S.

Preserves Order

Assume $a < a'$. It is clear that $F(a) \subset F(a')$ because for $x \in F(a)$, $x \leq a < a'$. Thus, $x < a'$ by the transitivity of the partial ordering $<$, and so $x \in F(a')$. Consequently, F preserves order.

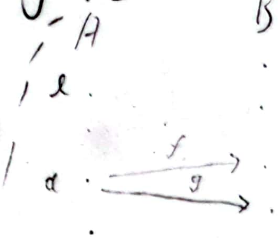
Wherefore, F is an isomorphism from $\langle A, < \rangle$ onto $\langle S, \subset_S \rangle$.

In a sense, S is the $\subset$-image of A.

Exercises

12. See Self Proof of Lemma 6 F

13. Assume that $\langle A, <_A \rangle \cong \langle B, <_B \rangle$ where there are two isomorphisms f and g. Without loss of generality, we can suppose that they map A into B. Also let S be the set of $a \in A$ so that $f(a) = g(a)$. Presume $\text{seg } t \subseteq S$. In other words, $f(a) = g(a)$ for all $a <_A t$. So, notice that $g(a) <_B f(t)$ —— (1). For the sake of contradiction, let's say that $f(t) <_B g(t)$. By the surjectivity of g, $g(a) = f(t)$ for some a, which must be strictly less than t because g preserves order. If t is the least element of A, there is an immediate contradiction as no such $a < t$ can possibly exist. Now consider when t is not least in A. From observation (1), we now have $g(a) < g(a)$ (for that particular $a < t$ aforementioned). But then $a < a$ follows from the order preserving nature of g, which contradicts the trichotomy of $<_A$ since $a = a$ already. To surmise, we arrive at a contradiction as long as $f(t) \neq g(t)$. Consequently, it surely holds that $t \in S$. Wherefore, by transfinite induction, $S = A$, i.e. $f(a) = g(a)$ for any a in their common domain of A. Which tells us that $f = g$. Indeed, there is a unique isomorphism between the well-ordered structures. $\square$



$\text{seg } f(a^+)$

$T = \{a \in A \mid f(a) \neq g(a)\}$

l_T exists

$f(l) = g(l)$ lest order is not preserved.

Suppose $\text{seg } t \subseteq S$. i.e. $f(a) = g(a)$ for all $a < t$

$f(a) <_B f(t$

$g(a) = f(a) <_B \underbrace{f(t)}_{g(\tau) \text{ for some } \tau < t} <_B g(t)$

$\Rightarrow$ $g(\tau) <_B g(\tau)$

Contradiction

Ideas / Brain loading moment

(b) By Theorem 7A (a), it is now sufficient to show that at least one of $x=y$, $x<y$ and $y<x$ holds for $x,y \in A$.
As $<_B$ is a linear order, there are the following two unique cases to consider:

1. $f(x) <_B f(y)$, then clearly $x <_A y$.
2. $f(x) = f(y)$, thus $x=y$ by the injectivity of f.

Therefore, together with the afforementioned Theorem 7A(a), we now know that $<_A$ satisfies trichotomy on A. Combined with the result of (a), we now see that $<_A$ is a linear order on A.

(c) As before, it now is sufficient to simply show that every nonempty subset S of A has a least element. Notice that $f[\![S]\!]$ must have a least element $\ell_{f[\![S]\!]}$ (w.r.t. to $<_B$) by $<_B$ being a well-ordering on B. So, there is some $s \in S$ with $f(s) = \ell_{f[\![S]\!]}$. Which we claim is the least element of S. Suppose otherwise, that there is some $t \in S$ with $t >_A s$. Then $f(t) >_B f(s)$. Which would contradict $f(s)$ being the least member of $f[\![S]\!]$ (w.r.t. to $<_B$). Thence, s is the least element of S (w.r.t. to $<_A$). By virtue of part (b) previously proven, $<_A$ is certainly a well-ordering on A.

Self Proof of Theorem 7G

This comes as a trivial result by using Lemma 7F.

Proof that ω is not isomorphic to $\omega \times \omega$

$\langle m,n\rangle <_L \langle 1,0\rangle$

More specifically, ω under its usual $\in$ ordering and $\omega\times\omega$ under the lexicographic ordering

Assume, for the sake of argument, that they are isomorphic. That is, there exists some bijection $f: \omega\times\omega \to \omega$ that preserves order. So,

$\underbrace{\{\langle 0,n\rangle \in \omega\times\omega\}}_{A} \approx \underbrace{\{n\in\omega \mid n \in f(1,0)\}}_{B}$ as $f\restriction A$ provides this bijection. Injectivity of $f \restriction A$ is clear. To prove surjectivity, suppose $k \in B$.

If $f(0,n) \neq k$ for any n, then $f(m,n) = k$ for some $m \geq 1$ and some other natural n. But then $(m,n) \geq_L (1,0)$ which tells us that $f(m,n) \supseteq f(1,0)$ by the order preserving nature of f. Trichotomy implies $k = f(m,n) \notin f(1,0)$, thus violating the construction of k—asserting that $k \in f(1,0)$. Therefore, as there must be some n so $f(0,n) = k$ surjectivity of $f\restriction A$ is guaranteed. We are now sure that $A \approx B$. However, $A \approx \omega$ and B is finite ($\approx f(1,0)$). Together, they inform us that, in reality, $A \not\approx B$. Hence contradicting the previous result (established from the assumption that an isomorphism f exists). Consequently, $\langle \omega, \in_\omega\rangle$ must not be isomorphic to $\langle \omega\times\omega, <_L\rangle$.

$\square$

Theorem 7E

Relatively simple result, just applying definitions. The first isomorphism is simply provided by I_A, the second by an inversion, and the third by composition. $\square$

✓ Yeah this is essentially what the author wrote too

Self Proof of Lemma 7F

a)

Irreflexivity:

Assume $x <_A x$ for some $x \in A$. But then $f(x) <_B f(x)$, contradicting the irreflexivity of the partial ordering $<_B$.

Transitivity:

Suppose $x <_A y$ and $y <_A z$. Then, $f(x) <_B f(y)$ & $f(y) <_B f(z)$. Thus, $f(x) <_B f(z)$. Hence, $x <_A z$.

By definition, $<_A$ is a partial ordering on A.

10. (a) Let T be the set of all natural numbers n with the $\in$-image of $\langle n, \in_n \rangle$ being n. Suppose $m \in T$ for all $m \in n$. If $n = 0$, then $E[\![0]\!] = 0$, so that $0 \in T$. When $n \neq 0$, $E[\![n]\!] = \{E(m) \mid m \in n\} = \{m \mid m \in n\} = n$ by our supposition. Thus, $n \in T$. By strong induction, $T = \omega$. $\square$

(b)

$\operatorname{ran} E = \{E(n) \mid n \in \omega\} = \{n \mid n \in \omega\} = \omega$ by (a).

11. (a) Define the relation $\lessdot$ on $\mathbb{Z}$ by $m \lessdot n$ if $0 \leq m < n$, $n < m < 0$, or $n < 0 \leq m$. Where $<$ is the regular ordering on $\mathbb{Z}$.

Trichotomy

We simply evaluate casewise (where the following cases are non-intersecting).

1. At least one of m and n are negative. Presume wlog that $n < 0$.

(i) When $m < n < 0$, $n \lessdot m$.

(ii) When $n < m$, either $n < 0 \leq m$ or $n < m < 0$. In both cases, $m \lessdot n$.

(iii) $m = n$.

2. Both m and n nonnegative. If $m \neq n$, assume $m < n$ wlog. So, $m \lessdot n$.

Looking carefully notice that exactly one of $m \lessdot n$, $n \lessdot m$, and $n = m$ can possibly be true. $\square$

Transitivity

Assume $m \lessdot n$ and $n \lessdot k$.

⋮

Seems to be just a tedious casewise proof as above for trans and well-ordering..

11-(b)

$E(3) = 3\omega$

$$E(-1) = E[\![\text{seg } -1]\!] = E[\![\mathbb{N}]\!] = \omega$$

$$E(-2) = E[\![\mathbb{N} \cup \{-1\}]\!] = \omega \cup \{\omega\}$$

$\cancel{\varnothing}$

$\cancel{\{\varnothing\}}$

$\cancel{\{\varnothing, \{\varnothing\}\}}$

$\cancel{\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}}$

~~ran E = $\omega \cup \{\omega\} \cup \{\omega, \{\omega\}\} \cup \{\omega, \{\omega\}, \{\omega \cup \{\omega\}\}\} \cup \cdots$~~

~~Let Ω be the 'ω-inductive' set, the set with $\omega \in \Omega$ and $x \in \Omega \Rightarrow$~~

ran E is essentially ω unioned with a similar Ω whose elements are natural numbers but with ω replacing $\varnothing$, in a sort of 'continuation' of the natural numbers.

Or more precisely, we could describe Ω as the subset of all 'ω-inductive' sets, i.e. ~~having~~ the sets S having the property

$\omega \in S$ and for any $t \in S$, $t^+ \in S$.

(c)

Suppose $s<t$. Then by definition, $E(s) \in E[\![\text{seg } t]\!] = E(t)$. Call this result (i). Conversely suppose $E(s) \in E(t)$. By trichotomy, exactly one of $s<t$, $s=t$ and $t<s$ is true. $t<s$ is impossible by result (i) while $s=t$ is not possible by (a). This leaves only $s<t$ which is indeed possible. As desired, $s<t$ iff $E(s) \in E(t)$. $\square$

(d)

Assume $\gamma \in \beta \in \alpha$. So, $E(s) \in E(t) \in \alpha$ (where $s<t$ must hold by (c)). Then, it is clear that $\gamma = E(s) \in \alpha$ $\square$

Self Proof of Theorem 7D

(a)

If $E(t) \in E(t)$ for some $t \in A$

$\cancel{E(t) = E(x) \text{ for some } x < t}$

Let S be the set of all $t \in A$ so that $E(t) \notin E(t)$.
Suppose $\operatorname{seg} t \subseteq S$, i.e. $E(x) \notin E(x)$ for any $x < t$.[1]
Then $E(t) \neq E(x)$ for each $x < t$, lest $E(x) \in E(x)$.
Hence, $t \in A$. By transfinite induction, $E(t) \notin E(t)$ given any $t \in A$. $\square$

[1] well, if t is least then its just trivial as $\varnothing \notin \varnothing$.

(b)

Since $\alpha := \operatorname{ran} E$, E is immediately surjective (onto α). So, it suffices to prove injectivity. Let S be the subset of A containing all the t's with $E(t)$ being unique in $\operatorname{ran} E$ (i.e. $E(t) = E(t')$ implies $t = t'$). Now assume $\operatorname{seg} t \subseteq S$, (*). In other words, all $E(x)$ is unique so long as $x < t$. Further suppose $E(t) = E(t')$. Then, for all $x < t$, there exists $x' < t'$ so $E(x) = E(x')$. By our assumption (*), $x = x'$. Notice that $\operatorname{seg} t \subseteq \operatorname{seg} t'$. Simply repeat this procedure with t and t' swapped around and we have that $\operatorname{seg} t' \subseteq \operatorname{seg} t$. Thus, $\operatorname{seg} t = \operatorname{seg} t'$. Consequently,

$$E[\![\operatorname{seg} t]\!] = E[\![\operatorname{seg} t']\!]$$
$$E(t) = E(t').$$

Transfinite induction tells us that $S = A$. Wherefore, the function $E : A \to \alpha$ is bijective. $\square$

for all $x < t$, there exists $x' < t'$ so $E(x) = E(x')$

$\operatorname{seg} t \subseteq S$

$E(x)$ unique

Suppose $E(t) = E(t')$.

... $E(x) = E(x')$

$x = x'$

$\operatorname{seg} t \subseteq \operatorname{seg} t'$

$\operatorname{seg} t' \subseteq \operatorname{seg} t$

Oh lol for (b) we can just apply trichotomy and (a) without TI.

Assume that $<$ is a well-ordering on a set A and that for any f there exists a unique so that $\gamma(f, y)$. Also, let S be the subset of A containing all $\tau \in A$ so that there is a unique function f_τ of domain $\operatorname{seg} \tau$ so that

$$\gamma(f_\tau \restriction \operatorname{seg} t,\ f_\tau(t))$$

as long as $t < \tau$. Suppose $\operatorname{seg} \tau \subseteq S$. By assumption, there is some unique y_t nominated by each unique f_t $(t < \tau)$. And hence by a replacement axiom, a set Y containing all such y_t. Now define the function $f_\tau \subseteq (\operatorname{seg} \tau) \times Y$ by $f_\tau(t) = y_t$. To check for uniqueness, presume there is some g_τ also satisfying $\gamma(g_\tau \restriction \operatorname{seg} t,\ g_\tau(t))$. Then, by our supposition that $\operatorname{seg} \tau \subseteq S$, $f_\tau \restriction \operatorname{seg} t = g_\tau \restriction \operatorname{seg} t$. Following from our initial assumption is thus the fact that $f_\tau(t) = g_\tau(t)$ for every $t < \tau$. As such, $f_\tau = g_\tau$ and uniqueness holds true as desired. By transfinite induction, $S = A$. When A has no greatest element, $F := \bigcup_{\tau \in A} f_\tau$[1] is the unique function with domain A that satisfies $\gamma(F \restriction \operatorname{seg} t, F(t))$. Notice that $f_t \subseteq f_\tau$ if $t \leq \tau$, since $f_t = f_\tau \restriction \operatorname{seg} t$ by the uniqueness of such functions f_t proven earlier. It holds that F is a function. Also, $F \restriction \operatorname{seg} t$ is simply f_t, which nominates a unique y_t. So, F satisfies γ. Lastly, presume again that G is a function of domain A that satisfies γ. Again, we similarly have that $F \restriction \operatorname{seg} t = G \restriction \operatorname{seg} t$ by uniqueness (of such functions of domain $\operatorname{seg} t$), from which it again follow (from assumption) that $F(t) = G(t)$. Hence, $F = G$. And uniqueness is also true. Consequently, in the case that A has a largest member simply define $F' = F \cup \{\langle u, y_u \rangle\}$ where u is the largest member of A and $\gamma(F, y_u)$. This must be unique. (verification fo a similar procedure as the above for F). Wherefore, there indeed exists a unique function (F or F') which $\underbrace{\text{satisfies } \gamma}_{\gamma\text{-constructed}}$. $\square$

Phrasing could be better but eh im lazy to rewrite it lmao
Probably kinda confusing to read in lol but I got the key ideas

[1] Oops to assert the existence of $\{f_\tau \mid \tau \in A\}$, we need to apply a replacement axiom.

Exercises

8. For each formula φ not containing B, let $\gamma(x,y)$ be the formula $x=y$ & $\varphi(y,A)$. Suppose $\gamma(x,y_1)$ and $\gamma(x,y_2)$. It follows that $x=y_1$ and $x=y_2$, so $y_1=y_2$. With the hypothesis to the axiom (schema) satisfied, now by a replacement axi there exists some set B with the property that for every y, $y\in B$ iff $y\in A$ & $\varphi(y,A)$.

9. Assume that the formula $\varphi(n,y)$ not involving B represent the formula $(n=0$ & $y=u)$ or $(n\neq 0$ & $y=v)$. Suppose that n is a natural number, $\varphi(n,y_1)$ and $\varphi(n,y_2)$. Then if $n=0$, then $y_1=y_2=u$. Otherwise, $y_1=y_2=v$. Now, again by a replacement axiom, we have that there is some set B so that for each y, $y\in B$ iff $y=u$ or $y=v$.

$\Rightarrow$ Not possible

$\Rightarrow y_1=y_2=v$

$$[(n=0 \,\&\, y_1=u) \text{ or } (n\neq 0 \,\&\, y_1=v)] \,\&\, [(n=0 \,\&\, y_2=u) \text{ or } (n\neq 0 \,\&\, y_2=v)]$$

$\Downarrow$ $y_1=y_2=u$

$\Downarrow$ Not possible

Self Proof of The Transfinite Recursion Theorem Schema

Ideas

Existence:

Assume that $<$ is a well ordering on a set A and that for any f there exists a unique y so that $\gamma(f, y)$.

Then given any f_t of domain $\operatorname{seg} t$ $(t \in A)$, there must also be a unique y_{f_t} nominated by f_t. By a replacement axiom, there exists a set C containing all such unique y.

$\gamma(f \restriction \operatorname{seg} l, y_l)$ $\quad f \restriction \operatorname{seg} l = F \restriction \emptyset = \emptyset \sim y_l$ $\quad$ y_l is unique since $f_l = \emptyset$.

$\gamma(f \restriction \operatorname{seg} l, (f \restriction \operatorname{seg} l^+)(l))$ $\quad f \restriction \operatorname{seg} l^+ = F \restriction \{l\} = \{\langle l, y_l \rangle\} \sim y_{l^+}$

$f \restriction \operatorname{seg} l^{++} = F \restriction \{l, l^+\} = \{\langle l, y_l \rangle, \langle l^+, y_{l^+} \rangle\} \sim y_{l^{++}}$

$\gamma(f \restriction \operatorname{seg} t, (f \restriction \operatorname{seg} \tau)(t))$

For every $\tau \in A$, there exists a unique function f_τ defined by

$$\gamma(f_t, f_\tau(t)) \qquad \begin{cases} f_\tau(a) = f_t(a) & \text{for any } t < \tau \\ \gamma(f_t, f_\tau(t)) \end{cases}$$

Suppose $\operatorname{seg} \tau \subseteq S$ (set of all $\tau \in A$ so that there is the unique function - i.e. there exists a unique f_t so that ... as long as $t < \tau$.

By assumption, there is some unique y_t nominated by each unique f_t and the set Y containing all such y_t exists by a replacement axiom.

Now define the function $f_\tau \subseteq (\operatorname{seg} \tau) \times Y$ by $f_\tau(t) = y_t$

Check $f_t \subseteq f_\tau$ for each $t < \tau$

Take $F := \bigcup_{\tau \in A} f_\tau$. When A has max element u, take $F \cup \{\langle u, y_u \rangle\}$ where $\gamma(F, y_u)$

7. (a)

$$F(0) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg} 0)$$
$$= C \cup \bigcup\bigcup \operatorname{ran} \varnothing$$
$$= C$$

$$F(1) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg} 1)$$
$$= C \cup \bigcup\bigcup \operatorname{ran}(\langle 0, C\rangle)$$
$$= C \cup \bigcup\bigcup\{C\} = C \cup \bigcup C$$

$$F(2) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg} 2)$$
$$= C \cup \bigcup\bigcup \operatorname{ran}(\langle 0, C\rangle, \langle 1, C \cup \bigcup C\rangle)$$
$$= C \cup \bigcup\bigcup \{C, C \cup \bigcup C\}$$
$$= C \cup \bigcup(C \cup C \cup \bigcup C)$$
$$= C \cup \bigcup(C \cup \bigcup C)$$
$$= C \cup \bigcup C \cup \bigcup\bigcup C$$

Making an inference, we can conjecture that $F(n) = C \cup \bigcup C \cup \bigcup\bigcup C \cup \dots \cup \overbrace{\bigcup\bigcup \cdots \bigcup}^{n \text{ times}} C$.

(b) Presume that $a \in F(n)$. Notice that $F(n) \in \operatorname{ran}(F \restriction \operatorname{seg} n^+)$, then $F(n) \subseteq \bigcup \operatorname{ran}(F \restriction \operatorname{seg} n^+)$. So, $a \subseteq \bigcup F(n) \subseteq \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg} n^+)$. Consequently, $a \subseteq F(n^+) = C \cup \bigcup\bigcup \operatorname{ran}(F \restriction \operatorname{seg} n^+)$. $\square$

(c) When $c' \in c \in \overline{C}$, $c' \in c \in F(n)$ for some $n \in \omega$. By part (b)'s result, $c' \in c \subseteq F(n^+)$. Hence, $c' \in \overline{C}$. i.e. $\overline{C}$ is a transitive set.

Since $C = F(0) \in \operatorname{ran} F$, $C \subseteq \bigcup \operatorname{ran} F = \overline{C}$. $\square$

Lemma 7.6.A

Given any set A well-ordered by $<$, and $a \in A$ that is not the greatest member of A, there exists a smallest $a^+ \in A$ so that $a^+ > a$. i.e. $\alpha \geq a^+$ for each $\alpha > a$.

Proof a^+ is simply the least element of $\{\alpha \in A \mid \alpha > a\}$, which is nonempty as long as a is not largest. $\Diamond$

Now, by the above lemma we can now form the function $H : A \to \mathcal{P}\mathbb{Q}$ with

$$H(a) = \begin{cases} \{r \in \mathbb{Q} \mid a < r < a^+\} & \text{if } a \text{ is not largest,} \\ \{r \in \mathbb{Q} \mid r > a\} & \text{if } a \text{ is largest.} \end{cases}$$

Notice that $H(a)$ is pairwise disjoint. Applying AC, there is hence an injection $F : A \to \mathbb{Q}$ which has the property that $F(a) \in H$

As $\mathbb{Q}$ is countable, A must be as well.

5. Let B be the subset of A containing all the x with $x \le f(x)$. Suppose $\text{seg } x \subseteq B$. Assume, for the sake of contradiction, that $f(x) < x$. Immediately, $f(f(x)) < f(x)$ since f preserves order. Simultaenously, as $f(x) < x$ tells us $f(x) \in \text{seg } B$, $f(x) \le f(f(x))$. Clearly, this contradicts the trichotomy of the well-order $<$. Therefore, we conclude that it must be that $x \le f(x)$, i.e. $x \in B$.

Wherefore, by transfinite induction, $B = A$.

$f(\ell) > f(\ell^+)$

least element ℓ and the next bigger element ℓ^+ $(\ell < \ell^+)$

We also know $\ell \le f(\ell) < f(\ell^+)$

Assume $f(\ell^+) < \ell^+$, then $f(\ell^+) = \ell$.
But $f(\ell) < f(\ell^+)$. (contradicts trichotomy

$t < x$

$t \le f(t) < f(x)$ ℓ

$\Rightarrow f(f(x)) < x$

$t < f(x) \in A$

$t < x \Rightarrow t < f(x)$

$\Rightarrow f(t) < f(f(x)) = f(t)$

for all $t \in \text{seg } x$

$f(f(x))$ $f(x)$

seg x

$\Rightarrow$ Since $f(x) \in \text{seg } x$,
$f(x) \le f(f(x))$

Contradict with $f(f(x)) < f($
(follows from $f(x) < x$

Exercises

4.

Trichotomy of R:

By the trichotomy of $<$, exactly one of $f(m) < f(n)$, $f(n) < f(m)$, or $f(n) = f(m)$ is true. In the first two cases, only mRn and nRm are true respectively. (Notice that $n = m$ is not possible because that would mean $f(m) \neq f(n)$. In other words, trichotomy of R holds in these cases.) For the last possibility of $f(m) = f(n)$, then we use the trichotomy of $<$ again, but on n and m. When $m < n$, mRn only. And similarly for $n < m$. Lastly, if $m = n$, then $m \nless n$ and $n \nless m$. As a result, neither mRn nor nRm are possible. By virtue of these facts, we can conclude that R satisfies trichotomy.

Transitivity of R:

Suppose mRn and nRk. There are a couple of unique cases worth considering:

$f(m) < f(n)$ and $f(n) < f(k)$ | Trivially, $f(m) < f(k)$ and mRk.

$f(m) < f(n)$, $f(n) = f(k)$, and $n < k$ | Since $f(m) < f(n) = f(k)$, transitivity is immediate here as well.

$f(m) = f(n) = f(k)$ and $m < n < k$ | Clearly, mRk again as desired.

Thus R is transitive.

This tells us that R is a linear order, a criteria for being a well-order.

R is a well-ordering:

Assume that S is a subset of P. Since we know $<$ is a well-ordering of P, there is some least element $\ell_<^f$ of $f[\![S]\!]$ with respect to $<$. Now, there must exist the other least element $\ell_<$ of $f^{-1}[\![\{\ell_<^f\}]\!]$ (the set of all positive integers k so that $f(k) = \ell_<^f$) w.r.t. $<$. So, given any positive integer $k \in S$ either $f(\ell_<) < f(k)$ or $f(\ell_<) = f(k)$ (notice $f(k) < f(\ell_<)$ is impossible as $\ell_<^f = f(\ell_<)$ is the least member of $f[\![S]\!]$). In the former, $\ell_< R k$ or $\ell_< =$ Similarly for the latter, only $\ell_< < k$ or $\ell_< = k$ are possible. Again, it holds that $\ell_< R k$ or $\ell_< = k$. Consequently, R is a well-ordering on A.

Wherefore, notice that $\langle P, R \rangle$ resembles Fig 45 (d). ordered by number of prime factors $\Rightarrow$ size of number

$$\underbrace{1}_{f(n)=0} < \underbrace{2 < 3 < 5 < 7 < \cdots}_{f(n)=1} < \underbrace{4 < 6 < 9 < \cdots}_{f(n)=2} < \underbrace{8 < 12 < 18 < \cdots}_{f(n)=3} < \cdots$$

Example on page 178

Let S be the subset of A containing only the members $t \in A$ so that (i) holds. Now assume that $\operatorname{seg} t \subseteq S$. If t is the least element of A, then $(F \upharpoonright \operatorname{seg} t) = F \upharpoonright \varnothing = \varnothing$ is indeed a function from $\operatorname{seg} t$ into B, i.e. $(F \upharpoonright \operatorname{seg} t) \in {}^{<A}B$. And when t is not the least element of A, there are two possibilities.

In the case that there is no greatest element in $\operatorname{seg} t$, $(F \upharpoonright \operatorname{seg} t) \in {}^{<A}B$ because for any $\tau_1 \in \operatorname{seg} t$ $F(\tau_1) = (F \upharpoonright \operatorname{seg} \tau_2)(\tau_1) \in B$ for some $t > \tau_2 > \tau_1$. Hence, $F \upharpoonright \operatorname{seg} t$ is a function from $\operatorname{seg} t$ into B.

Alternatively, there is a greatest member τ_{max} in $\operatorname{seg} t$. In such a situation, $F(\tau_{max}) = G(F \upharpoonright \operatorname{seg} \tau_{max})$ is clearly in B. therefore, $(F \upharpoonright \operatorname{seg} t) = (F \upharpoonright \operatorname{seg} \tau_{max}) \cup \{\langle F \upharpoonright \operatorname{seg} \tau_{max}, F(\tau_{max}) \rangle\}$ is a function from $\operatorname{seg} t$ into B, i.e. it is in ${}^{<A}B$

Consequently, it is certain that $(F \upharpoonright \operatorname{seg} t) \in {}^{<A}B$ always holds true. Meaning $t \in S$.

Wherefore, by transfinite induction, $S = A$. $\square$

Self Proof of The Transfinite Induction Principle

$a \in A - B \neq \emptyset$

$\text{seg } a \not\subseteq B$ lest $a \in B$

$\in \text{seg } a - B \neq \emptyset$ $\quad - a' < a$

$\text{seg } a' \not\subseteq B$

$H(0) = a \in A - B$

$H(n^+) = a' \in \text{seg } a - B$

$H(n^{++}) = a'' \in \text{seg}(a') - B$

Assume that $<$ is a well-ordering on A and B is a $<$-inductive subset of A, however $B \subset A$. Let I be the set of nonempty $\text{seg}(a) - B$ for any $a \in A$. By AC on the identity function of I, there is some function $g : I \to A$ with $g(i) \in I(i)$. There now exists the function $f : \omega \to A$ so that

$$f(0) = a;$$

$$f(n^+) = g(\text{seg}(f(n)) - B).$$

Future me: uhh idt we have shown that. First off we dk if $\text{seg}(f(0)) - B = \emptyset$ right?

Where a is any element of $A - B \neq \emptyset$ and the nonemptiness of $\text{seg}(f(n)) - B$ is guaranteed because $f(n) \in B$ otherwise (which is not possible by our construction o-

Consequently, $f(n^+) < f(n)$ since $f(n^+) \in \text{seg}(f(n))$. this contradicts Theorem 7B. Wherefore, $A = B$ is certain.

$\square$

Self Proof of Theorem 7B

Suppose $<$ is a well-ordering on A. Then for any function $f : \omega \to A$, we have some least element l of $f[\![\omega]\!]$, so that $l = f(n)$ for some n. By definition, $f(n^+) \geq f(n)$. Thus, it follows from trichotomy that $f(n^+) \not< f(n)$. Now presume $<$ is a linear ordering on A with the property that there does not exist any function $f : \omega \to A$ with $f(n^+) < f(n)$ for every $n \in \omega$, and let B be a nonempty subset of A. There must exist a least element l of B, lest there exists a $\beta_b < b$ for all $b \in B$. Which would imply the existance of the function $g : \omega \to A$ with

$$g(0) = b_0,$$
$$g(n^+) = \beta_{g(n)}$$

X Ahh right oops missed the fact that β_b might not be unique!!!
We need to utillise exercise 20 of Chapter 6!

Where the existance of such a b_0 is guareenteed by the nonemptiness of B. Hence, contradicting our initial presumption. Consequently, B must have least element, meaning $<$ is a well-ordering. Wherefore, given a linear ordering $<$ on A, it is a well-ordering iff there does not exist any function $f : \omega \to A$ with $f(n^+) < f(n)$ for each $n \in \omega$.

Proof

Assume that $<$ is a linear ordering on A so that the only $<$-inductive subset of A is A itself, but $<$ is not a well-ordering on A. Thus, there exists some nonempty subset B of A with no least element. Let the set C contain (only) all the $a \in A$ such that for all $b \in B$, $a < b$. Suppose that $\text{seg } a \subseteq C$ now. It must be that $a < b$ for each $b \in B$, which tells us that $a \in C$. ~~since A is <-inductive~~

Otherwise, $a \geq b$ for some $b \in B$. And as B has no least element, there exists some $b' \in B$ with $a \geq b > b'$. Entailing that $b' \in \text{seg } a - C$, because B and C are disjoint sets, lest trichotomy is violated. Consequently, $\text{seg } a \nsubseteq C$. But this is not possible because it goes against our supposition. In other words, $a \in C$ indeed is guaranteed. Therefore, C is a $<$-inductive set. Yet $C \subseteq A - B \subset A$, thence contradicting our earlier assumption that A is the only $<$-inductive subset of A. Wherefore, given that $<$ is a linear ordering on A so that the only $<$-inductive subset of A is A itself, $<$ is certainly a well-ordering. $\square$

Self Proof of Theorem 7C

Take $B \subseteq A$ with no least element

If $\in A \Rightarrow A' := \{a \in A \mid a > \alpha\}$ for some fixed $\alpha \in A$. $\Rightarrow$ counter e.g.?

If $\notin A \rightarrow$ counter eg!

What if A has a least element?
∟ You can't make 'the jump' onto the next smallest element

Assume that $<$ is a linear ordering on A so that the only $<$-inductive subset of A is itself, and B is a nonempty subset of A with no least element.

Then, fix some $\beta \in b$ and define B' to be the set of $b \in B$ with $\beta \notin B$.

If A no least element,

$\therefore$ exists $a \in A$ with $a < b$ for all $b \in B$

$\Downarrow$

Always, $\text{seg } t \not\subseteq b$

$\therefore$ No such $a \in A$ exists:

$\text{seg } b \not\subseteq B$

$\beta < b$ & $\beta \notin B$

$Z' := Z \cup \{l\}$

$Z' - \{0\}$

$\text{seg } l \subseteq Z' - \{0\}$

$l \in Z - \{0\}$

For any $a \neq l$ $\quad \text{seg } a \not\subseteq Z' - \{0\}$?

Why?

$\alpha < a$ & $\alpha \notin Z' - \{0\}$

seg 1

$(\text{seg } \alpha) - \{l\} \neq \emptyset$

$\Downarrow$

Knowing $l \in Z' - \{0\}$ does not tell us another element is in it.

Let C be a $<$-inductive subset of A and (which must exists since A itself is such a set)

$C' := \{c \in C \mid (\forall b \in b)(c < b)\}$. Clearly, $C' \cap B = \emptyset$, lest trichotomy is violated.

For any $b \in B$,

$\text{seg } b - C' \neq \emptyset$.

$\text{seg } c' \subseteq C'$

$\Rightarrow \text{seg } c' \subseteq C$

$\Rightarrow c' \in C'$ if $c' <$ for all b, otherwise $c' \geq b$ for some $b \in B$

$\Rightarrow \text{seg } c' - C' \neq \emptyset$

$\Rightarrow \text{seg } c' \not\subseteq C'$

N.A.

Thus, C' is $<$-inductive.

Contradiction with A being the only $<$-inductive subset of A

Ideas

Self Proof of Lemma 6R

Ideas

~~Lemma 6R~~

Lemma 6RA

If every element of $\mathcal{A}$ has at least cardinality λ, and all elements A of $\mathcal{A}$ are (pairwise) disjoint, then

$$(\text{card } \mathcal{A}) \cdot \lambda \leq \text{card } \bigcup \mathcal{A} .$$

Proof

For any $A \in \mathcal{A}$, there exists an injection $g_A : \lambda \longrightarrow A$. Now define the function $F : \mathcal{A} \times \lambda \longrightarrow \bigcup \mathcal{A}$ by

$$F(A, \mu) = g_A(\mu)$$

for some specific injection g_A specified by AC. To verify injectivity, suppose $F(A, \mu) = F(A', \mu')$. Then, $g_A(\ell) = g_{A'}(\ell')$ so $A = A'$ by disjointness. Hence, $g_A = g_{A'}$ by specificity imbued by AC. Consequently, $\mu = \mu'$ follows from the injectivity of $g_A = g_{A'}$. Therefore

$$(\text{card } \mathcal{A}) \cdot \lambda \leq \text{card } \bigcup \mathcal{A}$$

indeed.

~~Utilising exercise 26 of this Chapter,~~ $\cancel{K \times K = \bigcup_{i \in K} \{\langle i, j\rangle \in K \times K \mid j \in K\} \preceq K \times K}$

Oh bruh what was I doing lol Brain loading moment.

Let K be an infinite set.

By the well-ordering theorem, there is some well-order $<$ on K.

Define the well-order $\lessdot$ on $K \times K$ by

$$\langle k_1, k_2 \rangle \lessdot \langle k_3, k_4 \rangle \text{ iff}$$

Call a function $f: A \to B$ injective up to $c \in B$ iff $f(b_1) = f(b_2) \Rightarrow b_1 = b_2$ for each $b_1, b_2 \leq c$.

Suppose ftsoc that $K \not\approx K \times K$, i.e. no injective map from $K \times K$ into K exists.

Every $f: K \times K \to K$ is not injective, i.e. $f(\tau_1) = f(\tau_2)$ for some $\tau_1 \neq \tau_2$ that are both in $K \times K$

Show $(\text{card } \mathcal{A}) \cdot \lambda \leq \text{card } \bigcup \mathcal{A}$ if $\bigcap \mathcal{A} = \emptyset$ / all elements of $\mathcal{A}$ are disjoint

where every member of $\mathcal{A}$ has at least cardinality λ.

i.e. for every $A \in \mathcal{A}$ there exists an injection $g_A: L \to A$.

Define the function

$$F: \mathcal{A} \times L \to \bigcup \mathcal{A}$$

by

$$F(A, \ell) = g_A(\ell)$$

for the specific injection g_A given by AC.

By this specificity and the fact that all elements in $\mathcal{A}$ are disjoint, F must be injective..

$$F(A, \ell) = F(A', \ell')$$

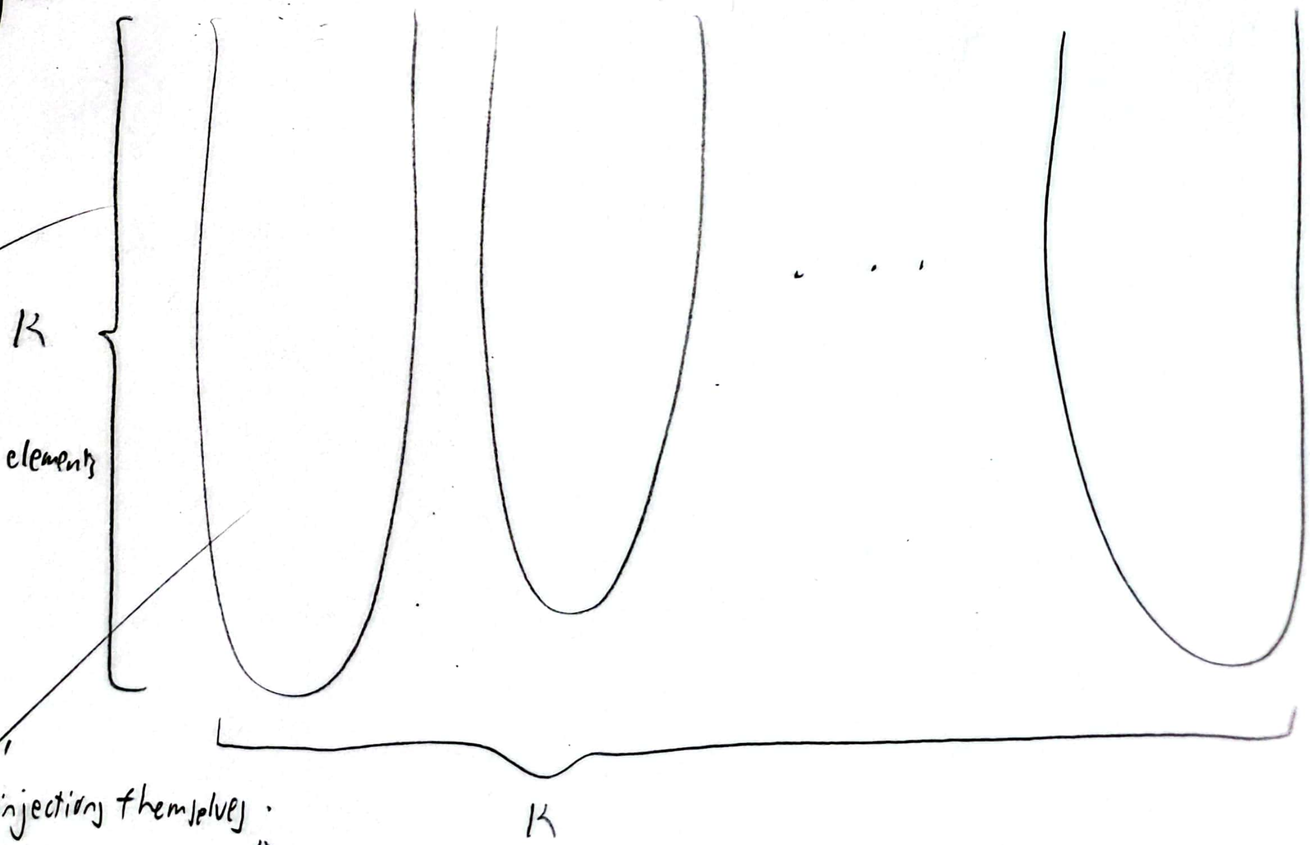
$$g_A(\ell) = g_{A'}(\ell')$$

$\Rightarrow A = A'$ by disjointness $\Rightarrow g_A = g_{A'}$

$\Rightarrow \ell = \ell'$ as $g_A = g_{A'}$ are injections themselves.

$\Rightarrow (\text{card } \mathcal{A}) \cdot \lambda \leq \text{card } \bigcup \mathcal{A}$

$\diamond$



Self Proof of Lemma 6R: For any infinite cardinal κ, $\kappa \cdot \kappa = \kappa$.

Infinite $\kappa \implies n \prec \omega \preccurlyeq \kappa$

$\kappa \prec \kappa \cdot \kappa$ is trivial.

Given any $n \in \omega$, there exists an injection $f_n : n \longrightarrow \kappa$ that is never surjective.

$$\kappa \cdot \kappa \leq \kappa$$

$$\pi_a := \{ \langle a, b \rangle \in \kappa \times \kappa \} \approx \kappa$$

$$\Pi_\kappa := \underbrace{\{ \pi_a \in \mathcal{P}(\kappa \times \kappa) \mid a \in \kappa \}} \approx \kappa$$

$$\kappa \times \kappa = \bigcup \Pi_\kappa \preccurlyeq \kappa \times \kappa .$$ Lol

$$\kappa \times \kappa \preccurlyeq \kappa$$

$$\begin{tikzcd}
\omega \arrow[r] & \omega \times \omega \arrow[r] & \kappa \arrow[r] & \kappa \times \kappa \arrow[lll, bend right] \arrow[l, bend left, "?"]
\end{tikzcd}$$

$$\kappa = \bigcup_{k \in \kappa} m.$$

$$\kappa \preccurlyeq \kappa \times \omega$$

24.

Clearly, $A \cap [1, \infty)$ must be finite, lest the sum of its finite subsets be unbounded above. So, we need only focus on $A \cap (0,1)$. Notice that for any $A \cap (\frac{1}{n}, 1)$ and a finite subset F of it, $\sum\{a \in \mathbb{R} \mid a \in F\} = n \sum F \leq nb$. Hence, for this to hold $A \cap (\frac{1}{n}, 1)$ must also be finite as every $an > 1$ given $a \in A \cap (\frac{1}{n}, 1)$. We also see that when $a \in A \cap [1, \infty)$, then $a \in (\frac{1}{n}, 1)$ for some natural $n > \frac{1}{a}$ whose existance is ensured by the Archimedean property. Consequently, $A = A \cap [1, \infty) \cup \bigcup_{n \in \omega} A \cap (\frac{1}{n}, 1)$, a countable union of countable (more specifically, finite) sets. Wherefore, A is countable

$\square$

30. $Sq(A) = \bigcup_{n \in \omega} {}^{n}A$

$\emptyset \longrightarrow \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 2,0\rangle, \ldots\}$

0 sequence of length n $\longrightarrow \{\langle 0, n^{+}\rangle, \langle 1,0\rangle, \langle 2,0\rangle \ldots\}$

Any other sequence of length n $\longrightarrow \{\langle 0,0\rangle, \langle 1,0\rangle, \ldots, \langle \underbrace{n}_{\frac{1}{2}n(n-1)}, 0\rangle, \langle \underbrace{n}_{\frac{1}{2}n(n-1)}+m, f(m)\rangle, \ldots\}$

$\Sigma 4$. bound b

Finite $\mathcal{F} \subseteq A$ $\quad J_{\mathcal{F}} : \mathcal{F} \longrightarrow \omega$

exists $n_{\mathcal{F}} \ni J_{\mathcal{F}}(a)$ for all $a \in \mathcal{F}$

for every finite $\mathcal{F}, \mathcal{F}' \subseteq A$, there is $\overline{J}_{\mathcal{F},\mathcal{F}'} : \mathcal{F}' \longrightarrow \omega$

with $\overline{J}_{\mathcal{F},\mathcal{F}'}(a) = J_{\mathcal{F}'}(a) + n_{\mathcal{F}}$

$b > \Sigma \mathcal{F}$

If A is not countable, then $\omega \overset{\Pi}{<} A$ by AC.

$\cancel{\Sigma \Pi[n] < b}$

$\Sigma \Pi[\![n]\!] < b$

$\omega < A - \Pi[\![\omega]\!] \neq \emptyset$

for any $b \in \mathbb{R}$, show that there is some finite subset A' of A s.t. $\Sigma A' > b$.

$i : A' \longrightarrow \omega \qquad A' < \omega < A$

$\inf(A) = i$

At least finite number of x with $x > 1$, $i = 0$

$\{ S^{\omega}_{A} \in \mathcal{P}(\omega \times A) \mid \text{dom } S^{\omega}_{A} = \omega \text{ \& ran } S \subseteq A \}$

& converges to 0

$\bigcup_{i \in I} S_i[\![\omega]\!] \simeq \omega$ $\qquad \Sigma S_i(n) \leq b$

$I > \omega$

28. $\{(-\infty, \pi), (\pi, \infty)\}$, proof is trivial and left as an exercise to the reader $\square$

27. (b) No. We can construct an injection f from $\mathbb{R}^+$ into the set C_p of all circles in the plane, no two of which intersect, so that

$$f(r) = \{\langle x, y\rangle \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}.$$

Injectivity and non-intersection is left as an exercise to the reader. Consequently, $\omega \prec \mathbb{R}^+ \preccurlyeq C_p$. Wherefore, $C_p \succ \omega$ is not countable. $\square$

$\Downarrow$

Assume $r_1 \neq r_2$, then $r_1^2 \neq r_2^2$ and $r_1^2 - x^2 \neq r_2^2 - x^2$ for each $x \in \mathbb{R}$. Thus, this suffices to show that the circles centered at $(0,0)$ of radii r_1 and r_2 never intersect. Clearly, they must be nonempty since $(0, r_1)$ and $(0, r_2)$ are elements of each of them, respectively. Which tells us that $f(r_1) \neq f(r_2)$.

(a) For every disc $D \in A$, there exists the nonempty set Q_D of the points $\langle p, q\rangle \in \mathbb{Q} \times \mathbb{Q}$. Since the discs are non-intersecting, these Q_D are pairwise disjoint. By AC, there is the injection $f : A \longrightarrow \mathbb{Q} \times \mathbb{Q}$ defined by $f(D) \in Q_D$. As we know that $\mathbb{Q} \approx \omega$ and $\omega \times \omega \approx \omega$, thus it follows that $A \preccurlyeq \omega$. Consequently, A is countable. $\square$

(c) Yes, C needs to be countable. This follows from a similar procedure as in (a).

Self of Theorem 7X

(a) Assume ftsoc that $A \in A$ for some set A. But then $A \in A \cap \{A\}$, contradicting the regularity axiom. $\square$

(b) Suppose, for the sake of contradiction, that such sets a and b exist. But from Theorem 7V and 7W, rank $a \in$ rank b and rank $b \in$ rank a simultaneously, thus contradiction the irreflexivity of ordinals wrt $\in$. $\square$

(c) Presume that there is some function f of domain ω with $f(n^+) \in f(n)$ for all $n \in \omega$. By regularity, there exists some natural n with $f(n) \cap f[\![\omega]\!] = \varnothing$. But $f(n^+) \in f(n) \cap f[\![\omega]\!]$, a contradiction. $\square$

, more like this than is insufficient to complete the q...

26. By Theorem 7W we can immediately conclude α is grounded, but let's avoid 'cheating'. Anyways, we'll have to complete the qns with the following technique, to show rank $\alpha = \alpha$:

Assume that for all $\beta \in \alpha$, $\beta \in V_\beta$ and rank $\beta = \beta$. Then α is grounded by theorem 7V(b) so that rank $\alpha = \bigcup\{(\text{rank } \beta)^+ \mid \beta \in \alpha\}$
$= \bigcup\{\beta^+ \mid \beta \in \alpha\} = \alpha$. So, by exercise 25 of this chapter, this is true for all ordinals α. $\square$

31. (a) Ideas

$B \in A \Rightarrow$ rank $B \in$ rank A $\qquad A \approx B$ $\qquad \{B \in V_{(\text{rank } A)^+} \mid A \approx B \text{ and for any } C \text{ with rank } C \in \text{rank } B,\ C \not\approx A\}$

rank $C \in$ rank $B \Rightarrow C \not\approx B$
$C \not\approx A$

Applying a subset axiom on $V_{(\text{rank } A)^+}$, we obtain a set kard A of all $B \in V_{(\text{rank } A)^+}$ such that $A \approx B$ and for any set C with rank $C \in$ rank B, $B \not\approx C$. Condition (i) and (ii) hold by the above construction.

(b) Let ord be the set of ordinals $\alpha \subseteq$ rank A such that there exists $B \approx A$ with rank $B = \alpha$. Since $A \approx A$, rank $A \in$ ord. There must be some least ordinal l_{ord} of ord with some corresponding set $B \approx A$ of rank l_{ord}, so that $B \in$ kard A.

(c) If $A \approx B$, kard $A \approx$ kard B is quite straightforward because $l_{ord_A} = l_{ord_B}$. Similarly, when kard $A =$ kard B, then for any either one, $C \approx A \approx B$.

32. A similar argument can be used as in 31. $\square$

33. Idea

$b \in a \in D \Rightarrow b \in D$

$b \in a \subseteq B \Rightarrow b \in B$ & $a \in B$

D

must be ∈ so its 'irreducible' in a sense

$\boxed{b_0, b_1, b_2, \ldots, a, \ldots}$

$\underbrace{b_0, b_1, b_2}_{\text{If in } B, \text{ then } a \in B}$

$D \subseteq V_{\text{rank } D}$

$b \subseteq V_{\text{rank } b} \subset V_{\text{rank } a} \subset V_{\text{rank } D}$

$a \subseteq$

Claim: for every V_α, $V_\alpha \cap D \subseteq B$.

Suppose that for all $\beta \in \alpha$, $V_\beta \cap D \subseteq B$. Let $a \in V_\alpha \cap D$. If $a \in V_\beta$ for some $\beta \in \alpha$, $a \in B$ is immediate. So, consider $a \notin V_\beta$ for each $\beta \in \alpha$. That is, $(\text{rank } a)^+ = \alpha$. Given any element b of a, we know $\text{rank } b \in \text{rank } a$. Accordingly, $b \in V_{(\text{rank } b)^+}$ where $(\text{rank } b)^+ \in \text{rank } a \in \alpha$. Thus, $b \in B$ as long as $b \in a$. In other words, $a \subseteq B$ and therefore $a \in B$ by its construction. Consequently, $V_\alpha \cap D \subseteq B$ (we proved the above for arbitrary $a \in V_\alpha \cap$ ...). Wherefore, by transfinite induction over ordinals (exercise 25), $V_{(\text{rank } D)^+} \cap D = D \subseteq B$. $\square$

34. Either $x = u$ and $\{x, y\} = \{u, v\}$ or $x = \{u, v\}$ and $\{x, y\} = u$. The latter contradicts Theorem 7X (b) since $x \in u$ ..., hence the former must hold instead. Indeed, $x = u$ tells us that $y = v$ (because $\{x, y\} = \{u, v\} = \{x, v\}$). $\square$

35. Assume for the sake of contradiction that $a \neq b$ yet $a \cup \{a\} = b \cup \{b\}$. Now, $a \in b \in a$ which again contradicts Theorem 7X (b).
So, if $a^+ = b^+$, then it must instead hold that $a = b$. $\square$

is a transitive set of transitive ...
... the transfinite induction schema in exercise 25 and regularity, we conclude every transitive set a

36. $TC\,S := \bigcup \operatorname{ran} F$ for some unique function F of domain ω so that $F(n) = S \cup \bigcup\bigcup F[\![\operatorname{seg} n]\!]$. So, let S be the set of $n \in \omega$ with $F(n) = V_{\operatorname{rank} S}$. Notice that $F(0) = S$, thus $0 \in S$. Now assume $n \in S$:

$$\begin{aligned} F(n^+) &= S \cup \bigcup\bigcup F[\![\operatorname{seg} n^+]\!] \\ &= S \cup \bigcup\bigcup (F[\![\operatorname{seg} n]\!] \cup \{F(n)\}) \\ &= S \cup \bigcup\bigcup F[\![\operatorname{seg} n]\!] \cup \bigcup F(n) \\ &= F(n) \cup \bigcup F(n). \end{aligned}$$

By our assumption, $F(n) = V_{\operatorname{rank} S}$, therefore $\bigcup F(n) = V_{\operatorname{rank} S}$. (Since for $x \in y \in F(n)$, $\operatorname{rank} x \in \operatorname{rank} y \in \operatorname{rank} F(n) = \operatorname{rank} S$)
In other words, $n^+ \in S$ and hence $S = \omega$ by induction. As such, $\bigcup \operatorname{ran} F$ has rank S because, again, for $x \in F(n)$, $\operatorname{rank} x \in \operatorname{rank} F(n) = \operatorname{rank} S$.
(which means $x \in V_{\operatorname{rank} x} \subseteq V_{\operatorname{rank} F(n)} = V_{\operatorname{rank} S}$. i.e. $\bigcup \operatorname{ran} F \subseteq V_{\operatorname{rank} S}$, if it still was not clear to any reader) $\square$

37. By Theorem 7M we already know any ordinal α is a transitive set well-ordered by $\in$. The converse holds true by Theorem 7L.
(The linear order $\in_\alpha$ must be a well-order lest there is an infinitely descending elementship chain) $\square$

38. Ideas
$\kappa \in \mu \in \lambda \Rightarrow \kappa \in \lambda$
$\bigcup\lambda \subseteq \lambda$

Show $\lambda \subseteq \bigcup\lambda$
$\mu \in \lambda \Rightarrow \mu \in \mu^+ \in \lambda$
$\mu \in \mu^+ \in \lambda$ since λ is a limit ordinal

Proof
$\bigcup\lambda \subseteq \lambda$ holds trivially by the transitivity of ordinal numbers. Assume $\mu \in \lambda$, then $\mu \in \mu^+ \in \lambda$ since λ is a limit ordinal. So, $\mu \in \lambda$ and $\lambda \subseteq \bigcup\lambda$. Consequently, when λ is a limit ordinal, $\lambda = \bigcup\lambda$.

39. We already know an ordinal number is a transitive set of transitive sets. So, we only need to be concerned with the converse. To that end, suppose that for all $\mu \in \lambda$, any transitive set of rank μ containing only transitive sets are ordinals. Further assume α of rank λ is a transitive set of transitive sets. For any $\beta \in \alpha$, $\operatorname{rank} \beta \in \operatorname{rank} \alpha = \lambda$. So, α is a transitive set of ordinals. Hence, it must be an ordinal number itself. By the transfinite induction schema in exercise 25 and regularity, we conclude every transitive set of transitive sets is an ordinal.

- Proof of the Transfinite Recursion Schema on the ordinals

Assume that for every f there is a unique set y such that $\gamma(f, y)$. By 'regular' transfinite recursion given in the previous chapter, there is the unique function f_α for any ordinal α so that $\gamma(f_\alpha \upharpoonright \text{seg } \beta, f_\alpha(\beta))$ for each $\beta \in \alpha$. Now, we can define the formula φ by $\varphi(\alpha, y)$ iff $\gamma(f_\alpha, y)$ for some unique set y. Now suppose g_α is a function whose domain is an ordinal number α such that $\varphi(\beta, g_\alpha(\beta))$ for all $\beta \in \alpha$. By our construction of φ, it must hold that $\gamma(f_\beta, g_\alpha(\beta))$. From uniqueness of such f, $f_\alpha \upharpoonright \text{seg } \beta = f_\beta$. Again utilising uniqueness, $f_\alpha(\beta) = g_\alpha(\beta)$ as long as $\beta \in \alpha$. [Since we know $\gamma(f_\alpha \upharpoonright \text{seg } \beta, g_\alpha(\beta))$]. Hence, $f_\alpha = g_\alpha$ so

$$\varphi(\alpha, y) \text{ iff } \gamma(g_\alpha, y)$$

by definition.

Self Proof of Theorem 8A

(a) Assume that for all $\alpha \in \beta$, and each ordinal number γ, $\aleph_\gamma < \aleph_\alpha$ iff $\gamma \in \alpha$. If $\alpha \in \beta$, then $\aleph_\alpha < \aleph_\beta$ lest $\aleph_\beta \leq \aleph_\alpha$ but this means $\beta \in \alpha$ from our assumption, a contradiction

To prove the converse, suppose $\beta \in \alpha$. By the same argument as above, $\aleph_\beta \leq \aleph_\alpha$.

So it's def wrong since I didn't use the defn of f. Yeah 1 billion %.

yhh this feels scuffed af.

for all $\alpha \in \beta$, and each ordinal number γ, $\text{Bruh}_\gamma < \text{Bruh}_\alpha$ iff $\gamma \in \alpha$.

$$(\forall \alpha \in \beta)\ (\forall \text{ ordinals } \gamma)\ (X_\gamma < X_\alpha \Leftrightarrow \gamma \in \alpha\ (\in \beta))$$

Ah right can't believe I made such a mistake :(. By assuming for all ordinals γ, ..., it means we're assuming

$$(\forall \alpha \in \beta)(\aleph_\beta < \aleph_\alpha \text{ iff } \beta \in \alpha):$$

Well, clearly this is a false (actually fucking absurd) statement and you can prove any bs with a false assum

Well, we learn and get better :O

Ok well to compensate for this mistake, let's make sure we 100 000 ... 000 % understand the author's proof:

$\aleph_\alpha$ is the least (infinite) cardinal so $\aleph_\alpha \neq \aleph_\gamma$ for all $\gamma \in \alpha$.

Similarly for $\aleph_\beta$, it is ...

Also, $\aleph_\alpha \neq \aleph_\beta$ assuming $\alpha \in \beta$ because $\aleph_\beta$ must be different from $\aleph_\alpha$ by construction.

By leastness, $\aleph_\alpha < \aleph_\beta$. Otherwise, it would contradict $\aleph_\alpha$ being the least infinite cardinal so...

(b) Ideas

Assume there exists some infinite cardinal κ with $\kappa \neq \aleph_\alpha$ for every α. There must exist some least such infinite cardinal $\ell \in K$. Every infinite cardinal smaller than ℓ must now be $\aleph_\beta$ for some ordinal number β. Let α be the set of all such ordinal numbers β, because by (a), α must be an ordinal number itself. Then, $\aleph_\alpha$ being the least infinite cardinal that is (from (a)) bigger than every infinite cardinal smaller than ℓ it must itself be ℓ. This contradicts the assumption that $\ell \neq \aleph_\alpha$ for any α. Consequently, every infinite cardinal is of the form $\aleph_\alpha$ for some α.

Self Proof of Lemma 8B

Let κ be a set of cardinal numbers. ~~If there exists a largest cardinal number in κ, $\bigcup\kappa$ is simply that cardinal number. So consider when there is no largest cardinal number in κ~~ For $\mu \in \bigcup\kappa$, we know $\mu \in \lambda \in \kappa$ for some λ. As λ must be a cardinal number and $\lambda \subseteq \bigcup\kappa$, we see that $\mu < \lambda \leq \bigcup\kappa$ follows from cardinal comparability. Hence, $\mu \not\approx \bigcup\kappa$ as long as μ is smaller than $\bigcup\kappa$ (as ordinal numbers). Consequently, $\bigcup\kappa$ is the least ordinal that is equinumerous to itself. Wherefore, it is again a cardinal number.

$\square$

Ideas

If there exists a largest cardinal number $\lambda \in \kappa$, $\bigcup\kappa = \lambda$.

When there is no largest cardinal number in κ, let $\mu \in \bigcup\kappa$, i.e. $\mu \in \lambda \in \kappa$ for some λ.
Since λ is a cardinal number, $\mu \in \lambda$ means $\mu < \lambda \leq \bigcup\kappa$ (as $\lambda \subseteq \bigcup\kappa$) (with AC)

$\Rightarrow \mu < \bigcup\kappa$

$\Rightarrow \mu \not\approx \bigcup\kappa$

Trying to define beth numbers myself

Define the formula γ by

$$\gamma(f, y) \quad \text{iff}$$

Hmm I don't see a way to combine the three eqns describing beth numbers together.
I mean I could split them apart in the definition of $\vee$ γ, but that feels not elegant.
Huh yeah this is what the author does.

Exercises

1. Let us define the formula γ by

$$\gamma(f,y) \iff \begin{cases} \text{dom } f = 0 \ \& \ y = S, \\ \text{dom } f \text{ is some successor ordinal } \alpha^+ \ \& \ y = f(\alpha)^+, \\ \text{dom } f \text{ is some limit ordinal } \lambda \ \& \ y = \bigcup_{\alpha \in \lambda} f(\alpha), \text{ or} \\ \text{dom } f \text{ is something else} \ \& \ y = \varnothing. \end{cases}$$

By transfinite recursion, we define t_α for each ordinal α to be the unique y so that $\varphi(\alpha, y)$, where φ is the corresponding 'function-class' over all the ordinals, as specified in the transfinite recursion schema on the ordinals. In other words, t_α is such that

$$t_0 = S;$$
$$t_{\alpha^+} = t_\alpha^+,$$
$$t_\lambda = \bigcup_{\alpha \in \lambda} t_\lambda.$$

2. Let S be the set of $n \in \omega$ with $t_n = S + n$. Clearly, $t_0 = S + 0$. Thus assume $n \in S$ and notice $t_{n^+} = t_n^+ = (S+n)^+ = S + (n+1)$. That is, $n^+ \in S$. Therefore, $S = \omega$ by induction. Now suppose $\omega \subseteq \alpha$, and for any β such that $\omega \in \beta \in \alpha$, $t_\beta = \beta$.

When $\alpha = \omega$, we see that $t_\omega = \bigcup_{n \in \omega} S + n = \omega$ from the previous part. Consider $\omega \in \alpha$. If α is a successor ordinal, i.e. $\alpha = \beta^+$ for ordinal β, $t_{\beta^+} = t_\beta^+ = \beta^+ = \alpha$ by our supposition. Lastly, if α is a limit ordinal, we again utilize our supposition to see that $t_\alpha = \bigcup_{\mu \in \alpha} t_\mu = \bigcup_{\mu \in \alpha} \mu$. Which must be α itself as α is a limit ordinal. Regardless, $t_\alpha = \alpha$ given $\omega \subseteq \alpha$. In fact, this now holds true for all ordinal numbers α by transfinite induction.

Self Proof of Theorem Schema 8C

Ideas

Assume $\alpha \in \beta$. If β is a limit ordinal, $t_\alpha \in t_{\alpha^+} \in \sup\{t_\gamma \mid \gamma \in \beta\} = t_\beta$ by continuity. When β is a successor ordinal, $\beta = \gamma^+$ for some ordinal number γ so that $\alpha \underline{\in} \gamma$. Using hypothesis of TI, $t_\alpha \underline{\in} t_\gamma \in t_{\gamma^+} = t_\beta$ ...

Proof

Assume t_β is locally monotonous — that is, $t_\gamma \in t_{\gamma^+}$ for every ordinal γ — and $\alpha \in \beta$. Further suppose, for the sake of using transfinite induction, that $t_\alpha \in t_\gamma$ for all ordinals α and γ such that $\alpha \in \gamma \in \beta$. Now, there are three cases to consider given any $\alpha \in \beta$:

(i) β is a limit ordinal. Then $t_\alpha \in t_{\alpha^+} \in \sup\{t_\gamma \mid \gamma \in \beta\} = t_\beta$ by continuity and local monotonicity.

(ii) β is a successor ordinal. It follows from our supposition that $t_\alpha \underline{\in} t_\gamma$. And local monotonicity tells us $t_\alpha \underline{\in} t_\gamma \in t_{\gamma^+} = t_\beta$.
(for the ordinal γ with $\beta = \gamma^+$)

(iii) $\beta = 0$. $t_\gamma \in t_\beta$ is vacuously true for each $\gamma \in \beta = 0$.

Regardless, we now see that when $\alpha \in \beta$ for any ordinals α and β, $t_\alpha \in t_\beta$ must hold true by transfinite induction. □

Self Proof of Theorem Schema 8E

When S has some largest member L, $t_{\sup S} = t_L = \bigcup_{\alpha \in S} t_\alpha = \sup\{t_\alpha \mid \alpha \in S\}$. So, consider S not having any largest member L now. Then for all $\alpha \in S$, there exists β and γ in S with $\alpha \in \beta \in \gamma$ such that $\alpha^+ \in \beta \in \gamma$. In other words, $\alpha^+ \in S$ and which means $\sup S$ is a limit ordinal. By continuity, $t_{\sup S} = \sup\{t_\alpha \mid \alpha \in \sup S\}$. Since given $\alpha \in \beta$, we know $t_\alpha \in t_\beta$ from monotonicity, thus $t_\alpha \subset t_\beta$. Accordingly, $t_{\sup S}$ is just $\sup\{t_\beta \mid \beta \in S\}$.

In any case, we can conclude $t_{\sup S} = \sup\{t_\alpha \mid \alpha \in S\}$. $\square$

Self Proof of Theorem Schema 8D

Ideas

Claim: $\beta \underline{\in} t_\beta$ always

Suppose true for any $x \in \beta \cdot (\alpha \underline{\in} t_\beta)$

Then for all $\alpha \in \beta$, $\alpha \in t_\beta$

because $\alpha \in t_\alpha \in t_\beta$ by monotonicity.

$\Rightarrow \beta \subseteq t_\beta$

$\beta \underline{\in} t_\beta$.

By monotonicity,

$\alpha \in \beta \Rightarrow t_\alpha \in t_\beta$

$t_\gamma \underline{\in} \beta \Rightarrow t_\gamma \subseteq \beta$

While $\beta \in \alpha \Rightarrow t_\beta \in t_\alpha$
$\Rightarrow \beta \in t_\alpha$

For $t_\gamma \in \beta$, it's impossible to have $\beta \in \gamma$

So, for any ordinal number γ with $t_\gamma \underline{\in} \beta$, $\gamma \underline{\in} \beta$.

Apply leastness on $\beta^+ - \{\gamma \mid t_\gamma \underline{\in} \beta\} = \beta^+ - \{\gamma \underline{\in} \beta \mid t_\gamma \underline{\in} \beta\}$ (or take $\bigcup\{\gamma \in \beta^+ \cdots$

$\ell \neq 0$ since $t_0 \underline{\in} \beta$ by defn of β. i.e. $0 \in \ell$ / $1 \underline{\in} \ell$

when ℓ is a limit ordinal,

$t_\ell = \bigcup_{\mu \in \ell} t_\mu$ by continuity

Claim: ℓ cannot be a limit ordinal.

$\ell \underline{\in} t_\ell$

$\ell \in t_\ell$
$\ell = t_\mu$ for some $\mu \in \ell$

$\ell = t_\ell$
$\gamma \in \ell \Rightarrow \gamma = t_\mu$ for some $\mu \in \ell$
&
$\gamma \underline{\in} t_\gamma$

Proof

Lemma 8D(i)

For any ordinal number β, $\beta \underline{\in} t_\beta$.

Proof

Assume this is true for each $\alpha \in \beta$. Then, for all such α, $\alpha \underline{\in} t_\alpha \in t_\beta$ by monotonicity. In other words, $\beta \subseteq t_\beta$ so that $\beta \underline{\in} t_\beta$.

Notice that for $t_\gamma \underline{\in} \beta$, it is impossible to have $\beta \in \gamma$ because that would imply $t_\beta \in t_\gamma$ by monotonicity after which $\beta \in t_\gamma$ follows from the above. So, it must be that $\gamma \underline{\in} \beta$ in order for $t_\gamma \underline{\in} \beta$. Now take the least element $\ell := \beta^+ - \{\gamma \mid t_\gamma \underline{\in} \beta\} = \beta^+ - \{\gamma \underline{\in} \beta \mid t_\gamma \underline{\in} \beta\}$. We see that $\ell \neq 0$ because $t_0 \underline{\in} \beta$ by definition. Furthermore we claim: ℓ can never be a limit ordinal either. To prove this, first suppose for the sake of contradiction that ℓ is a limit ordinal. Then for any ordinal $\alpha \in t_\ell$, $\alpha = t_\gamma \underline{\in} \beta$ (follows from continuity for some $\gamma \in \ell$). In fact, since ℓ is a limit ordinal, $\gamma^+ \in \ell$ such $\alpha = t_\gamma \in t_{\gamma^+} \underline{\in} \beta$. Therefore, $t_\ell \subseteq \beta$, telling us $t_\ell \underline{\in} \beta$. This contradicts the construction of ℓ which says $t_\ell \not\underline{\in} \beta$. Indeed we have verif that ℓ isn't a limit ordinal. Consequently, it must be a successor ordinal. Wherefore, the γ_{max} with $\gamma_{max}^+ = \ell$ is the greatest ordinal such that $t_\gamma \underline{\in} \beta$ we are looking for.

ℓ is a limit ordinal

$\ell = t_\gamma$ for some $\gamma \in \ell$

$\Rightarrow \gamma \in t_\gamma$

Show $t_\ell \in \beta$

$t_\ell \subseteq \beta$

$\alpha \in t_\ell \Rightarrow \alpha = t_\gamma$ for some $\gamma \in \ell$

$\Rightarrow \alpha \in \beta$

Indeed, $t_\ell \subseteq \beta$.

Continue of ideas / sketch of TS 8D

Self - Proof of Veblen Fixed - Point Theorem Schema (1907)

Examples: $\aleph_\alpha = \alpha$

$\aleph_\omega \neq \omega$

$\aleph_{\aleph_n} \neq \aleph_n$ because $n \in \aleph_n$ so that $\aleph_n \in \aleph_{\aleph_n}$

$\aleph_\alpha = \aleph_{\aleph_\alpha}$ iff $\aleph_\alpha = \alpha$

Define

$\gamma(f,y)$ iff $\operatorname{dom} f = \alpha$ for some ordinal α & $y = \aleph_{\bigcup \operatorname{im} f}$,

$\operatorname{dom} f \neq \alpha$ for any ordinal α & $y = \emptyset$.

By TR, we let $\eta_\alpha = \aleph_{(\bigcup_{\beta\in\alpha} \eta_\beta)} = \aleph_{\sup\{\eta_\beta \mid \beta\in\alpha\}}$.

Let λ be a limit ordinal

$\eta_\lambda^+ = \aleph_{\eta_\lambda} \overset{?}{=} \eta_\lambda$ $\quad \eta_\lambda \overset{?}{=} \sup\{\eta_\beta \mid \beta\in\lambda\}$

$\aleph_{\sup\{\eta_\beta \mid \beta\in\lambda\}} = \sup\{\eta_\beta \mid \beta\in\lambda\}$

If $\alpha \in \sup\{\eta_\beta \mid \beta\in\lambda\}$, there exist β with

$\alpha \in \eta_\beta \subseteq \eta_\lambda$

$\alpha \in \eta_\beta \leq \eta_\lambda$

$\alpha \in \eta_\lambda$

$\Rightarrow \sup\{\eta_\beta \mid \beta\in\lambda\} \leq \eta_\lambda$

L1 tells us $\aleph_{\sup\{\eta_\beta \mid \beta\in\gamma\}} \subseteq \aleph_{\sup\{\eta_\beta \mid \beta\in\lambda\}}$

L1

Assume $\bigcup_{\mu\in\gamma} \eta_\mu \subseteq \bigcup_{\mu\in\beta} \eta_\mu$ for all $\gamma\in\beta\in\alpha$.

If $\gamma\in\alpha$,

$\bigcup_{\mu\in\gamma} \eta_\mu \subseteq \bigcup_{\mu\in\alpha} \eta_\mu$

$\bigcup_{\mu\in\gamma} \eta_\mu \subseteq \bigcup_{\mu\in\alpha} \eta_\mu$

$\Rightarrow$ True for α.

(Let's call this weak monotonicity)

$\eta_n = \aleph_{\sup\{\eta_{n-1}\ldots}$

Lω

$\sup\{\eta_m \mid m\in 1\} = \eta_0$

Assume $\sup\{\eta_m \mid m\in n\} = \eta_{n^-}$

$\sup\{\eta_m \mid m\in n^+\} = \left(\bigcup_{m\in n} \eta_n\right) \cup \eta_n$

$= \eta_{n^-} \cup \eta_n$

$= \eta_n$ $\quad$ by L1

Suppose α st. such that $\alpha \in \aleph_{\sup\{\eta_\beta \mid \beta\in\lambda\}}$ but for all $\gamma\in\lambda$, $\alpha \notin \aleph_{\sup\{\eta_\beta \mid \beta\in\ldots}$

α finite $\Rightarrow \alpha \in \aleph_0 = \eta_0$

Consider α infinite. $\alpha = \aleph_\mu$ for some μ

(i) $\mu = \lambda$

Self-Proof of Veblen Fixed-Point Theorem Schema (1907)

Ideas

Note: $T_0 = t_0$

Let γ be the formula $\gamma(f,y)$ iff dom $f = \alpha$ for some ordinal α & $y = t_{\bigcup \operatorname{ran} f}$, or
dom $f \neq \alpha$ for all ordinals α & $y = \emptyset$.

By TR, we can define $T_\alpha := t_{\sup\{T_\beta \mid \beta \in \alpha\}}$

Monotonicity: Suppose $T_\gamma \subseteq T_\beta$ for all $\gamma \in \beta \in \alpha$. For any $\beta \in \alpha$,

$$\{T_\gamma \mid \gamma \in \beta\} \subseteq \{T_\gamma \mid \gamma \in \alpha\}$$
$$\sup\{T_\gamma \mid \gamma \in \beta\} \subseteq \sup\{T_\gamma \mid \gamma \in \alpha\}$$
$$\sup\{T_\gamma \mid \gamma \in \beta\} \in \sup\{T_\gamma \mid \gamma \in \alpha\}$$
$$T_\beta := t_{\sup\{T_\gamma \mid \gamma \in \beta\}} \in t_{\sup\{T_\gamma \mid \gamma \in \alpha\}} := T_\alpha$$ by the monotonicity of t

$\Rightarrow T_\beta \in T_\alpha$ for all $\beta \in \alpha$.

$\Rightarrow$ For any $\beta \in \alpha$, $T_\beta \in T_\alpha$ L1.1
$T_\beta \subseteq T_\alpha$ L1.2

$$t_{\sup A} = \sup\{t_\alpha \mid \alpha \in A\}$$
$$t_{\sup A} \ni t_\alpha$$ by monotonicity

If A has a largest element, sup A is just that.
Then $t_{\sup A} = \sup\{t_\alpha \mid \alpha \in A\}$ since $\sup A \in A$.

If A no largest element, sup A either 0 or lim ord.

Want to show $t_{T_\lambda} = T_\lambda$
$$T_{\lambda^+} = T_\lambda$$
$$\left(\bigcup_{\mu\in\lambda} T_\mu\right) \cup T_\lambda = T_\lambda$$
$$T_\lambda = T_\lambda$$

1. $T_{\lambda^+} = t_{T_\lambda}$ by L1.2
2. $$\bigcup_{\mu\in\lambda^+} T_\mu = \left(\bigcup_{\mu\in\lambda} T_\mu\right) \cup T_\lambda$$
$$= T_\lambda$$ by L1.2
$$\stackrel{?}{=} \bigcup_{\mu\in\lambda} T_\mu$$

$\bigcup_{\mu\in\lambda} T_\mu \subseteq T_\lambda$ trivial by L1.2

So we need to see if $T_\lambda \subseteq \bigcup_{\mu\in\lambda} T_\mu$.
Assume $\alpha \in T_\lambda$, i.e. $\alpha \in t_{\bigcup_{\mu\in\lambda} T_\mu}$. (Prove $\alpha \in \mu$ s.t. $\alpha \in T_\mu$ for some $\mu \in \lambda$)
Factually, either (i) $T_\lambda \in u$
(ii) $u \in T_\lambda$

(ii) There exists $\alpha \in T_\lambda$ s.t. $\alpha \notin T_\mu$ for every $\mu \in \lambda$

$$t_{\sup\{T_\mu \mid \mu\in\lambda\}} = \sup\{t_{T_\mu} \mid \mu \in \lambda\}$$
$$= \sup\{T_{\mu^+} \mid \mu \in \lambda\}$$
$$= \sup\{T_\mu \mid \mu \in \lambda\}$$

Assume $T_\beta = T_{\aleph_0}$ for all $\beta \in \alpha$, if $\aleph_0 \leq \beta$
$$T_\alpha := t_{\sup\{T_\beta \mid \beta\in\alpha\}}$$
$$= t_{\sup\{T_{\aleph_0} \mid \beta\in\alpha\}}$$
$$= t_{\aleph_0}$$

Self Proof of Veblen's Fixed Point Theorem Schema

Let γ be the formula

$$\gamma(f,y) \quad \text{iff} \quad \begin{array}{llll} \operatorname{dom} f = \alpha & \text{for some ordinal } \alpha & \& & y = t_{\sup f[\![\alpha]\!]}\text{, or} \\ \operatorname{dom} f \neq \alpha & \text{for all ordinals } \alpha & \& & y = \varnothing . \end{array}$$

We shall prove two lemmas first before proceeding further:

Lemma VFP 1

For any normal operation t and nonempty set A of ordinals,

Bruh we also proved this in Thm 8IE Brain loading moment...

$$t_{\sup A} = \sup\{t_\alpha \mid \alpha \in A\}.$$

Proof

When A has a largest element, $\sup A$ is just that. Then, $t_{\sup A} \subseteq \sup\{t_\alpha \mid \alpha \in A\}$ since $\sup A \in A$. The other direction follows by monotonicity. Hence, $t_{\sup A} = \sup\{t_\alpha \mid \alpha \in A\}$.
Now consider A having no largest element. By the nonemptiness of A, $\sup A$ must be a limit ordinal. Thus, the claim holds by continuity. ◁

Lemma VFP 2

T is 'weakly monotonous', i.e. if $\beta \in \alpha$ then $T_\beta \subseteq T_\alpha$.

Proof

Suppose $\beta \in \alpha$, thus $\{t_\gamma \mid \gamma \in \beta\} \subseteq \{t_\gamma \mid \gamma \in \alpha\}$. As such, $\sup\{t_\gamma \mid \gamma \in \beta\} \subseteq \sup\{t_\gamma \mid \gamma \in \alpha\}$. From the monotonicity of t, $T_\beta := t_{\sup\{t_\gamma \mid \gamma \in \beta\}} \subseteq t_{\sup\{t_\gamma \mid \gamma \in \alpha\}} := T_\alpha$.

We now continue proving Veblen's:

Assume λ is a limit ordinal, which must be nonempty. Notice that

$$\begin{aligned} T_\lambda &:= t_{\sup\{T_\mu \mid \mu \in \lambda\}} \\ &= \sup\{t_{T_\mu} \mid \mu \in \lambda\} && \text{by Lemma VFP 1} \\ &= \sup\{T_{\mu^+} \mid \mu \in \lambda\} && \text{as } T_{\mu^+} = t_{T_\mu} \text{ by virtue of lemma VFP 2} \\ &= \sup\{T_\mu \mid \mu \in \lambda\} && \text{by Lemma VFP 2 and } \lambda \text{ being a limit ordinal} \end{aligned}$$

Consequently, it follows that $T_\lambda := t_{\sup\{T_\mu \mid \mu \in \lambda\}} = t_{T_\lambda}$. Since the class of all infinite cardinals, which are limit ordinals, is unbounded, so must the class of all limit cardinals. In other words, for every ordinal number β we can find another limit ordinal $\lambda \cancel{\supseteq \beta}$ with $t_{T_\lambda} = T_\lambda$. To ensure $\beta \in T_\lambda$ (a property currently missing), we can simply adjust the definition of γ to have $\gamma(0, t_\beta)$ instead of $\gamma(0, t_0)$. Wherefore, $\beta \subseteq t_\beta = T_0 \in T_\lambda$ by Lemma 8D(i) and Lemma VFP2.

7. Assume $t_\gamma = \gamma$ where $\beta \in \gamma$. Let S be the set of natural numbers n with $t_\beta^n \subseteq \gamma$. We see that $t_\beta^0 = t_\beta \leq t_\gamma = \gamma$ by monotonicity such that $0 \in S$. Suppose that $n \in S$; $t_\beta^{n^+} := t_{t_\beta^n} \subseteq t_\gamma = \gamma$ then follows, again from monotonicity. Hence $S = \omega$. Consequently, $\lambda := \sup\{t_\beta^n \mid n \in \omega\} \leq \sup\{\gamma\} = \gamma$. $\square$

Wherefore, $\lambda \subseteq \gamma$, proving the leastness of λ.

[Exercise]

4. Suppose λ is a limit ordinal and $\gamma \in t_\lambda$. Thus, $\gamma \in t_\alpha$ for some $\alpha \in \lambda$, so that $\gamma^+ \overset{\text{by continuity}}{\subseteq} t_\alpha \in t_{\alpha^+} \in t_\lambda$ follows from normality. Hence, t_λ is not a successor ordinal. Since λ is a limit ordinal, $\lambda \neq 0$, i.e. $0 \in \lambda$. Thus, $t_0 \overset{\text{from monotonicity}}{\in} t_\lambda$. So, $t_\lambda \neq 0$ lest there is some element of 0. Consequently, t_λ must be a limit ordinal.

$\square$

5. Proven in self-proof of Theorem Schema 8D, see Lemma 8D(i).

$\square$

6. Let β be any ordinal number. By Theorem Schema 8D, there exists a greatest ordinal γ such that $t_\gamma \in \beta$; In other words, $\beta \in t_{\gamma^+}$, which is, informally, a member of the mentioned class. Since for every ordinal number there is a larger t_{γ^+} 'in the class', it is unbounded. And it must be closed by definition (given any element t_α of the class, it must be an ordinal. Accordingly, t_{t_α} is also in that class).

$\square$

8. Rigorously, we can define γ to be the formula

$\gamma(f, y)$ iff y is the least fixed point t_γ not in $\operatorname{ran} f$.[1]

Then by transfinite recursion, we can define t'_α to be the unique y so that $\varphi(\alpha, y)$ where φ is the formula specified in transfinite recur-
In other words, t'_α is the unique least fixed point t_γ different from t'_β for each $\beta \in \alpha$ (these fixed points must exist by Veblen's Fixed Point Theorem Sche-
Time to prove normality:

1. Monotonicity: Suppose $\alpha \in \beta$. Clearly, $t'_\beta \neq t'_\alpha$ by definition. Thus, either $t'_\beta \in t'_\alpha$ or $t'_\alpha \in t'_\beta$. For the former, it contradicts the fact that t'_α is the least fixed point $t_\gamma \neq t'_\beta$ (for every $\beta \in \alpha$) because there is indeed a smaller fixed point t'_β. Hence, the latter must be true, implying monotonicity holds true.

2. Continuity: Assume λ is a limit ordinal. By monotonicity, we know t'_λ is the least ordinal larger than t'_α for $\alpha \in \lambda$. Furthermore, we also know $\sup\{t'_\alpha \mid \alpha \in \lambda\}$ is the ordinal least upper bounding every t'_α, which must in fact be larger than each t'_α because $t'_\alpha \in t'_{\alpha^+} \in \{t'_\alpha \mid \alpha \in \lambda\}$ by monotonicity and λ being a limit ordinal (informing us that $\alpha \in \alpha^+ \in \lambda$). Therefore, continuity holds as well.

Indeed, t' is again a normal operation.

1 Did this before self-proving Veblen's and then reading the author's proof where he defined t'

TXT IF NEW,

Lemma 8G

This is just tedious so I just mentally checked it.

$$\begin{aligned}
\bar{1} + \bar{3} &= it\langle 1, \in_1\rangle + it\langle 3, \in_3\rangle \\
&= it\langle \{3\}, \varnothing\rangle + it\langle 3, \in_3\rangle \\
&= it\langle \{3\} \cup 3, \varnothing \cup \in_3 \cup (\{3\}\times 3)\rangle \quad \text{by definition} \\
&= it\langle \{0,1,2,3\}, \in_3 \cup \{\langle 3,n\rangle \mid n \in 2\}\rangle \\
&= it\langle 4, \in_4\rangle
\end{aligned}$$

$\langle 3,0\rangle, \langle 3,1\rangle, \langle 3,2\rangle$

because we can define a bijection $f : 4 \to 4$ with

$$f(n) = \begin{cases} n^+ & \text{if } n \in 2, \\ 0 & \text{if } n = 3. \end{cases}$$

$f(0) = 1$
$f(1) = 2$
$f(2) = 3$
$f(3) = 0$

which is an isomorphism.

Ideas

$\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle$

$\{\langle 3,0\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$

$\in_3 \quad \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle\}$

$\in_4 = \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 0,3\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$

$$\begin{aligned}
\bar{1} + \bar{\omega} &= it\langle 1, \in_1\rangle + it\langle \omega, \in_\omega\rangle \\
&= it\langle \{\omega\}, \varnothing\rangle + it\langle \omega, \in_\omega\rangle \\
&= it\langle \omega^+, \varnothing \cup \in_\omega \cup (\{\omega\}\times\omega)\rangle \quad \text{by definition} \\
&= it\langle \omega^+, \in_\omega \cup \{\langle \omega, n\rangle \mid n \in \omega\}\rangle \\
&= it\langle \omega, \in_\omega\rangle
\end{aligned}$$

because we can again define an isomorphism $g : \omega^+ \to \omega$ by

$$g(x) = \begin{cases} x^+ & \text{if } x \in \omega, \\ 0 & \text{if } x = \omega. \end{cases}$$

$\in_\omega = \{\langle n,m\rangle \in \omega\times\omega \mid n \in m\}$

under g, $\in_\omega \cup \{\langle \omega, n\rangle \mid n \in \omega\}$

becomes $\{\langle n^+, m^+\rangle \in \omega\times\omega \mid n \in m\} \cup \{\langle 0, n\rangle \mid n \in \omega\}$

$= \in_\omega$

Exercises

10. (a) By exercise 26 of the previous chapter, rank $0 = 0$, rank $1 = 1$ and rank $2 = 2$. Also, we know that

$V_0 = \varnothing = 0,$

$V_1 = \{\varnothing\} = 1,$

$V_2 = \{\varnothing, \{\varnothing\}\} = 2.$

Hence, the only subset of V_0 that is equinumerous to 0 is 0 itself. That is, kard $0 = \{0\}$. Clearly, no subset of V_0 is equinumerous to 1. But $1 \in V_1$ and 1 is the only subset of V_1 which is equinumerous to 1. Accordingly, kard $1 = 1$. Similarly, kard $2 = 2$. $\square$

(b) Everything in kard 3 must have the same rank, or at most rank $3 = 3$ by exercise 26 of the previous chapter. Again, notice that

$$V_0 = \varnothing,$$
$$V_1 = \{\varnothing\},$$
$$V_2 = \{\varnothing, \{\varnothing\}\},$$
$$V_3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = 3.$$

We see that nothing of rank 2 or below, i.e. any subset of V_2, is equinumerous to 3. Therefore, 3 is the only set equinumerous to itself such that nothing of a smaller rank than it is equinumerous to 3. In other words, kard $3 = 3$.

1. Assume there exists the set IT of every structure isomorphic to $\langle A, R\rangle$. By a replacement axiom, there is a set S of all sets B so that there exists a binary relation R with $\langle A, R\rangle \cong \langle B, F$ Now let B be any set equinumerous to A and f be a bijection from A into B. Define a new binary relation $R_B := \{\langle f(a_1), f(a_2)\rangle \mid a_1 R a_2\}$. Clearly, f is now an isomorphism from $\langle A, R\rangle$ into $\langle B, R_B\rangle$. Which means that the afforementioned set S must contain all sets B that are equinumerous to A. However, this contradicts exercise 6 of chapter 6. As such, it must be that, instead, no set contains every structure isomorphic to $\langle A, R\rangle$, given $A \neq \varnothing$.

$$\bar{\omega} + \bar{1} = it\langle \omega, \in_\omega \rangle + it\langle 1, \in_1 \rangle$$
$$= it\langle \omega, \in_\omega \rangle + it\langle \{\omega\}, \varnothing \rangle$$
$$= it\langle \omega^+, \in_\omega \cup \varnothing \cup (\omega \times \{\omega\}) \rangle$$
$$= it\langle \omega^+, \in_\omega \cup \{\langle n, \omega \rangle \mid n \in \omega\} \rangle$$
$$= it\langle \omega^+, \in_{\omega^+} \rangle$$

Define the function $f : 2 \times \omega \to \omega$ by

$$f(n,m) = \begin{cases} 2m & \text{if } n = 0, \\ 2m+1 & \text{otherwise.} \end{cases}$$

Assume $\langle n, m \rangle = \langle n', m' \rangle$.

Case 1 $n = n' = 0$: $f(n,m) = 2m = 2m' = f(n', m')$

Case 2 $n = n' \neq 0$: $f(n,m) = 2m+1 = 2m'+1 = f(n',m')$

Regardless, $f(n,m) = f(n',m')$. Hence, f is injective

Suppose $k \in \omega$:

Case I $k = 2m$ for some natural m: $f(0,m) = 2m = k$

Case II $k = 2m+1$ for some natural m: $f(1,m) = 2m+1 = k$.

Thus, f is also surjective.

Presume $\langle n, m \rangle <_H \langle n', m' \rangle$ $\xrightarrow{\text{continuation}}$

When $m \in m'$

n	n'	$f(n,m)$ vs $f(n',m')$
0	0	$f(0,m) = 2m \in 2m' = f(0,m')$
0	1	$f(0,m) = 2m \in 2m'+1 = f(1,m')$
1	0	$f(1,m) = 2m+1 \in 2m'$ (as $m' - m$ is at least 1)
1	1	$f(1,m) = 2m+1 \in 2m'+1 = f(1,m')$

When $m = m'$ & $n \in n'$ (i.e. $n = 0 \in 1 = n'$)
$f(n,m) = 2m \in 2m'+1 = f(n', m')$

Consequently, we see that f does indeed preserve order. Therefore, f is an isomorphism from $\langle 2 \times \omega, <_H \rangle$ to $\langle \omega, \in_\omega \rangle$. Indeed, we can verify that $\bar{2} \cdot \bar{\omega} = \bar{\omega}$.

Self - Proof of Theorem 82

(a)

$$\begin{aligned}(\rho+\sigma)+\tau &= (\mathrm{it}\langle A,R\rangle + \mathrm{it}\langle B,S\rangle) + \mathrm{it}\langle C,T\rangle \text{ , where } A\wedge B\wedge C \neq \emptyset \\ &= \mathrm{it}\langle A\cup B, R\oplus S\rangle + \mathrm{it}\langle C,T\rangle \\ &= \mathrm{it}\langle A\cup B\cup C, (R\oplus S)\oplus T\rangle \\ &= \mathrm{it}\langle A\cup B\cup C, R\cup S\cup(A\times B)\cup T\cup[(A\cup B)\times C]\rangle \\ &= \mathrm{it}\langle A\cup B\cup C, R\cup S\cup T\cup(A\times B)\cup(A\times C)\cup(B\times C)\rangle \\ &= \mathrm{it}\langle A\cup B\cup C, R\cup S\cup T\cup(B\times C)\cup[A\times(B\cup C)]\rangle \\ &= \mathrm{it}\langle A\cup B\cup C, R\oplus(S\oplus T)\rangle \\ &= \mathrm{it}\langle A,R\rangle + (\mathrm{it}\langle B,S\rangle + \mathrm{it}\langle C,T\rangle) \\ &= \rho + (\sigma+\tau)\end{aligned}$$

$$\begin{aligned}R\oplus(S\oplus T) &= R\oplus[S\cup T\cup(B\times C)] \\ &= R\cup S\cup T\cup(B\times C)\cup[A\times(B\cup C)]\end{aligned}$$

$$\begin{aligned}(\rho\cdot\sigma)\cdot\tau &= \mathrm{it}\langle A\times B, R*S\rangle\cdot\mathrm{it}\langle C,T\rangle \\ &= \mathrm{it}\langle (A\times B)\times C, (R*S)*T\rangle \\ &= \mathrm{it}\langle A\times(B\times C), R*(S*T)\rangle \\ &= \rho\cdot(\sigma\cdot\tau)\end{aligned}$$

as they are isomorphic. Doing it rigorously would be an exercise in tedium but the intuitive idea is that

1. $(A\times B)\times C$ is very similar to $A\times(B\times C)$
2. $(R*S)*T$ looks at the T-[illegible] first, then the S and R coords. And so does $R*(S*T)$.

$\mathrm{it}\langle A\cup B, R*S\rangle + \mathrm{it}\langle A\cup C; R*T\rangle$

(b)

$$\begin{aligned}\rho\cdot(\sigma+\tau) &= \mathrm{it}\langle A,R\rangle\cdot(\mathrm{it}\langle B,S\rangle + \mathrm{it}\langle C,T\rangle) \\ &= \mathrm{it}\langle A,R\rangle\cdot\mathrm{it}\langle B\cup C, S\oplus T\rangle \\ &= \mathrm{it}\langle A\times(B\cup C), R*(S\oplus T)\rangle \\ &= \mathrm{it}\langle (A\times B)\cup(A\times C), (R*S)\oplus(R*T)\rangle \text{ because when } \ldots \\ &= \ldots \\ &= (\rho\cdot\sigma)+(\rho\cdot\tau)\end{aligned}$$

$R*(S\cup T\cup(B\times C))$

$(R*S)\cup(R*T)\cup[(A\cup B)\times(A\cup C)]$

(c)

$$\begin{aligned}\rho+\bar{0} &= \mathrm{it}\langle A,R\rangle + \mathrm{it}\langle\emptyset,\emptyset\rangle \\ &= \mathrm{it}\langle A\cup\emptyset, R\cup\emptyset\cup(A\times\emptyset)\rangle \\ &= \mathrm{it}\langle A,R\rangle \\ &= \rho\end{aligned}$$

$$\begin{aligned}\bar{0}+\rho &= \ldots \\ &= \ldots \\ &= \rho\end{aligned}$$

(same thing as to the left)

$$\begin{aligned}\rho\cdot\bar{1} &= \mathrm{it}\langle A,R\rangle\cdot\mathrm{it}\langle 1,\emptyset\rangle \\ &= \mathrm{it}\langle A\times 1, R*\emptyset\rangle \\ &= \mathrm{it}\langle A\times\{0\}, R\rangle \text{ as } 0 \text{ is the only member of } 1 \text{ and } 0=0 \\ &= \mathrm{it}\langle A,R\rangle \\ &= \rho\end{aligned}$$

$\vdots$

IS. ^Self-proof.

Exercises

11. Assume the structures $\langle A,R\rangle$ and $\langle B,S\rangle$ have order types ρ and σ respectively (they must exist as ρ and σ are isomorphism types of some structures by definition). Then, the structure $\langle B\times\{A\}, S'\rangle$ is of order type σ where $\langle b_1, A\rangle S' b_2$ iff $b_1 S b_2$. Furthermore, $(B\times\{A\})\cap A=\varnothing$ is guaranteed, lest if $\langle b,A\rangle\in A$ for some $b\in B$. But then $A\in\{b,A\}\in\langle b,A\rangle\in A$, hence forming an infinitely descending membership chain, contradicting regularity.

12. We see that

$$\text{it}\langle A,R\rangle+\text{it}\langle B,S\rangle=\text{it}\langle\{0\}\times A, R'\rangle+\text{it}\langle\{1\}\times B, S'\rangle \text{ where } \langle 0,a_1\rangle R'\langle 0,a_2\rangle \text{ iff } a_1Ra_2 \text{ and } \langle 1,b_1\rangle S'\langle 1,b_2\rangle \text{ iff } b_1Sb_2$$
$$=\text{it}\langle(\{0\}\times A)\cup(\{1\}\times B), R'\oplus S'\rangle$$

We claim that $R'\oplus S'$ is just $<_L$. Assume $\langle n,x\rangle(R'\oplus S')\langle m,y\rangle$. There are three cases to consider:

I. $\langle 0,a_1\rangle(R'\oplus S')\langle 0,a_2\rangle$, then $n=m$ and a_1Ra_2 by definition.

II. $\langle 1,b_1\rangle(R'\oplus S')\langle 1,b_2\rangle$; again $n=m$ and b_1Sb_2 by definition.

(or if you prefer, $a_1(R\cup S)a_2$ and $b_1(R\cup S)b_2$)

III. $\langle 0,a\rangle(R'\oplus S')\langle 1,b\rangle$, so $0\in 1$.

The converse is clear cut as well. Hence, $R'\oplus S'$ is indeed the lexicographic ordering $<_L$ on $(\{0\}\times A)\cup(\{1\}\times B)$ with $\in$ and $R\cup S$ 'dictating' the first and second coordinates, in a sense. Therefore, $\text{it}\langle A,R\rangle+\text{it}\langle B,S\rangle=\text{it}\langle(\{0\}\times A)\cup(\{1\}\times B), <_L\rangle$ indeed. $\square$

14. Assume, instead, that $\rho\neq\bar{0}$ and $\sigma\neq\bar{0}$. Let $\langle A,R\rangle$ and $\langle B,S\rangle$ be of types ρ and σ respectively, where A and B (by extension, R and S too) are nonempty. Then, we see that

$$\rho\cdot\sigma=\text{it}\langle A,R\rangle\cdot\text{it}\langle B,S\rangle$$
$$=\text{it}\langle A\times B, R*S\rangle$$

where once again $A\times B\neq\varnothing$. Hence, $\langle A\times B, R*S\rangle\not\cong\bar{0}$ because there does not even exist a bijection between $A\times B$ and 0 in the first place, much less an isomorphism. Consequently, $\rho\cdot\sigma=\text{it}\langle A\times B, R*S\rangle\neq\bar{0}$. Wherefore, taking the contrapositive, we indeed have that if $\rho\cdot\sigma=\bar{0}$, either $\rho=\bar{0}$ or $\sigma=\bar{0}$.

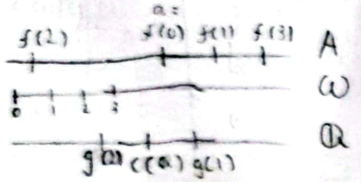
15. see Self-proof of Theorem 8I.

Bijection $f : \omega \to A$ Choice function C for $\mathbb{Q}$

$\hat{h_L} := h_L(n) :=$ the $m \in n$ with the largest $f(m)$ s.t. $f(m) \, R \, f(n)$

$\hat{h_u} := h_u(n) :=$ the $m \in n$ with the least $f(m)$ s.t. $f(n) \, R \, f(m)$

g



$$g(n) = \begin{cases} C(\mathbb{Q}) & \text{if } n = 0, \\ C((g(\hat{h_L}), g(\hat{h_u}))) & \text{if } \hat{h_L} \text{ and } \hat{h_u} \text{ exist,} \\ C((g(\hat{h_L}), \infty)) & \text{if } \hat{h_u} \text{ does not exist but } \hat{h_L} \text{ does,} \\ C((-\infty, g(\hat{h_u}))) & \text{if } \hat{h_L} \text{ does not exist but } \hat{h_u} \text{ does,} \\ e & \text{otherwise.} \end{cases}$$

iso $:= g \circ f^{-1}$

Let S be the set of $n \in \omega$ with $g(n) \in \mathbb{Q}$.

Assume seg $n \subseteq S$ and $0 \in n$ because for $n = 0$, $g(n) \in \mathbb{Q}$ by defn.

So, $f(0) \, R \, f(n)$ or $f(n) \, R \, f(0)$. Thus, there must at least exist one of $\hat{h_L}$ or $\hat{h_u}$. Therefore, it indeed holds that $g(n) \in \mathbb{Q}$. Now, $S = \omega$. Hence, $g : \omega \to \mathbb{Q}$.

Suppose T is the set of $m \in \omega$ such that for all $n \in m$, if $f(n) \, R \, f(m)$ then $g(n) <_{\mathbb{Q}} g(m)$, and if $f(m) \, R \, f(n)$ then $g(m) <_{\mathbb{Q}} g$

Presume seg $k \in T$.

Case I $f(m) \, R \, f(k)$ (where we know $m \in k$) so $\underset{\in k}{h_L^k}$ exists with $f(m) \subseteq f(h_L^k)$. By our presumption, $g(m) \leq_{\mathbb{Q}} g(h_L^k)$. Hence, since $g(m)$ is 'below' (ordered by $<_{\mathbb{Q}}$) $(g(h_L^k), g(\hat{h_u}))$ and $(g(h_L^k), \infty)$, while $g(k)$ is in one of them, it certainly holds that $g(m) <_{\mathbb{Q}} g(k)$.

Case II similar.

Injectivity

This is immediately true of $g \circ f^{-1}$ since it is order-preserving.

Wherefore, $\langle A, R \rangle \overset{g\circ f^{-1}}{\cong} \langle g[\![\omega]\!], <_{\mathring{\alpha}} \rangle$ (in the case where $A \neq \varnothing$). When $A = \varnothing$, $\langle \varnothing, \varnothing \rangle \cong \langle \varnothing, \varnothing \rangle$ trivially. $\square$

Oh oops AC is actually unnecessary here but no harm in using it.

17. Proof

By countability and density, we know there exists some bijection $f : \omega \to A$ and AC tells us there is a choice function C for $\mathbb{Q}$.

Now, we can define the function $g : \omega \to \mathbb{Q}$ by transfinite recursion:

$$
g(n) = \begin{cases} C(\mathbb{Q}) & \text{if } n = 0, \\ C((g(l_n), g(u_n))) & \text{if } l_n \text{ and } u_n \text{ exist}, \\ C((g(l_n), \infty)) & \text{if, only } l_n \text{ exists}, \\ C((-\infty, g(u_n))) & \text{if, only } u_n \text{ exists}, \\ e & \text{otherwise.} \end{cases}
$$

Where e is some extraneous element not in $\mathbb{Q}$, l_n is the $m \in n$ with the largest $f(m)$ such that $f(m) R f(n)$, and similarly, u_n is the $m \in n$ with the least $f(m)$ such that $f(n) R f(m)$. We claim $g \circ f^{-1}$ gives the desired isomorphism:

<u>$g : \omega \to \mathbb{Q}$</u>:

Let S be the set of all $n \in \omega$ with $g(n) \in \mathbb{Q}$. Assume seg $n \subseteq S$ and $0 \in n$, because for $n = 0$, $g(n) \in \mathbb{Q}$ holds trivially by defn. Thus, $f(0) R f(n)$ or $f(n) R f(0)$ must hold. Accordingly, at least one of l_n and u_n will exist. Therefore, it indeed holds that $g(n) \in \mathbb{Q}$ and $n \in S$. By transfinite induction / strong induction, $S = \omega$.

<u>Order-preserving</u>

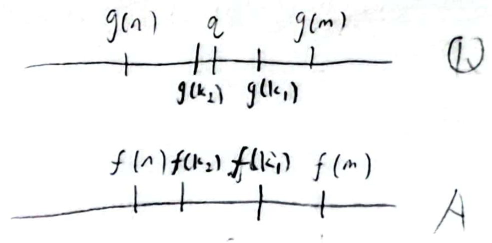
Similarly, suppose T is the set of $m \in \omega$ such that for every $n \in m$, if $f(n) R f(m)$ then $g(n) <_{\mathbb{Q}} g(m)$ and if $f(m) R f(n)$ then $g(m) <_{\mathbb{Q}} g(n)$. Presume seg $m \subseteq T$ and, without loss of generality, that $f(n) R f(m)$ where $n \in m$. So, l_m exists with $f(n) \leq f(l_m)$. By our presumption, $g(n) \leq_{\mathbb{Q}} g(l_m)$. And $g(l_m) <_{\mathbb{Q}} g(m)$ from the construction of g. Thence, $g(n) <_{\mathbb{Q}} g(m)$ and $m \in T$. Consequently, $T = \omega$ follows once more from transfinite induction / strong induction. Now we see that $g \circ f^{-1}$ is order-preserving.

10 Ideas

$\operatorname{ran} C = \mathbb{Q}$ since $C(\{q\}) = q$ for all $q \in \mathbb{Q}$

Let $q \in \mathbb{Q}$, wlog we can suppose $g(n) <_a q <_a g(m)$ for some $n, m \in \omega$

$f(n) \; R \; f(m)$



Let the set of all $k \in \omega$ such that $f(k) \in (f(n), f($ be S; having same least element l.

Self Proof of Theorem 8K

This is just a special case of Theorem 8I translated to ordinal arithmetic.

Self Proof of Theorem 8L

oh huh actually I thought up to $(\{0\}\times\alpha)\cup(\{1\}\times\lambda) = (\{0\}\times\alpha)\cup\bigcup\{\{1\}\times\beta \mid \beta\in\lambda\}$ in one of my ideas. Perhaps I should have continued in that direction instead.

$\gamma\in\Sigma$ iff $\gamma\in\alpha+\beta$ for some β

it $\langle\Sigma, \in_\Sigma\rangle$

Results (A1), (M1) are from Theorem 8K immediately, thus no further work is required.

(A2): $\alpha+\beta^+ = \alpha+(\beta+1)$ by previous example,

$= (\alpha+\beta)+1$ by associativity given from Thm 8K,

$= (\alpha+\beta)^+$ again by the afforementioned example.

(A3):

Ideas

ok well this should work but no No I won't write this idea out properly, it'll be pain.

I bet 100% that the intended solution is way more elegant.

order-preserving ✓ $\Rightarrow$ Injective ✓

$\Uparrow$

For $\langle 0,a\rangle <_L \langle 1,\gamma\rangle$, just repeat a similar proof.

$\overline{\alpha+\lambda} = \overline{\alpha+\sup\{\beta\mid\beta\in\lambda\}}$

it $\langle(\{0\}\times\alpha)\cup(\{1\}\times\lambda), <_L\rangle$

Assume $\gamma\in\Sigma$, i.e. $\gamma\in\alpha+\beta$ for some $\beta\in\lambda$

Either $\gamma\in\alpha$ or $\alpha\subseteq\gamma$.

1. $f(0,\gamma)=\gamma$ ✓
2. When $\alpha=\gamma$, $f(1,0)=\alpha+0=\gamma$.
   When $\alpha\in\gamma$, $\gamma\setminus\alpha\neq\varnothing$.

$\langle\gamma\setminus\alpha, \in_\gamma\cap(\gamma-\alpha)\rangle \cong \langle\gamma', \in_{\gamma'}\rangle$

for some $\gamma'(\in\gamma)$

$\Rightarrow f(1,\gamma') = \alpha+\gamma'$

$\vdots$

$= \gamma$

surjective ✓.

$f: (\{0\}\times\alpha)\cup(\{1\}\times\lambda) \longrightarrow \Sigma$

$$f(\langle n,\gamma\rangle) = \begin{cases}\gamma & \text{if } n=0,\\ \alpha+\gamma & \text{if } n=1.\end{cases}$$

$\langle 0,a_1\rangle <_L \langle 0,a_2\rangle \Longrightarrow f(0,a_1)=a_1\in a_2 = f(0$

$\langle 1,\gamma_1\rangle <_L \langle 1,\gamma_2\rangle$ $(\gamma_1\in\gamma_2)$

$\Rightarrow f(1,\gamma_1) = \alpha+\gamma_1$

$f(1,\gamma_2) = \alpha+\gamma_2$

$\langle(\{0\}\times\alpha)\cup(\{1\}\times\gamma_1), <_{\gamma_1 L}\rangle \cong \langle\operatorname{seg}[\langle 1,\gamma_1\rangle], <^{\circ}_{\gamma_2 L}\rangle$

$\Longrightarrow \langle\alpha+\gamma_1, \in_{\alpha+\gamma_1}\rangle \cong \langle\operatorname{seg}\text{ something}, \in^{\circ}_{\gamma_2}\rangle$

$\Longrightarrow \alpha+\gamma_1 \in \alpha+\gamma_2$.

Assume for $\varphi_1\in\varphi_2\in\psi$, and any ordinal α, $\alpha+\varphi_1\in\alpha$-

Then suppose $\varphi\in\psi$:

~~If $\psi=0$, immediate.~~

~~If ψ is a succ ord: 1. $\varphi^+\in\psi \Rightarrow \alpha+\varphi\in(\alpha+\varphi)^+=\alpha+\varphi$~~

~~2. $\varphi\in\varphi'\in(\varphi')^+=\psi$ ...~~

If ψ is a lim ord:

~~Set 7 – Proof of Lemma 8M~~

$\overline{\alpha} \cdot \overline{\lambda} = \text{tt} \langle \alpha \times \lambda, <_H \rangle$

$\alpha \times \lambda = \bigcup \{\alpha \times \beta \mid \beta \in \lambda\}$

$\langle \alpha \times \beta, <_H \rangle$

Lemma 8M applicable ✓

Qns: 1. Why Hebrew lexicographical ordering?

2. Why are order types at all, over TR? ✓

└ Are order types useful! where

Self - Proof of The Subtraction Theorem

Assume that for every $\alpha \in \gamma \in \beta$, there exists a unique ordinal number δ with $\alpha + \delta = \gamma$. Now consider $\alpha \in \beta$:

Case 1 $\beta = 0$, then $0 + \beta = \beta$.

Case 2 β is a successor ordinal, thus $\beta = \gamma^+$ for some $\gamma \in \beta$ with a corresponding δ. Therefore, $\alpha + \delta^+ = (\alpha + \delta)^+ = \gamma^+ = \beta$.

Case 3 β is a limit ordinal, so $\alpha + \sup\{\delta_\gamma \mid \gamma \in \beta\} = \sup\{\alpha + \delta_\gamma \mid \gamma \in \beta\} = \sup\{\gamma \mid \gamma \in \beta\} = \beta$.

Regardless, we see that there exists a ordinal number δ with $\alpha + \delta = \beta$. Which is, in fact, unique by Corollary 8P (b). Consequently, by transfinite induction this result is true for each ordinal number β. $\square$

Self - Proof of The Division Theorem

Ideas By Thm 8D, there exists largest ordinal number β with $\delta \cdot \beta \in \alpha$ (as $\delta \neq 0$, so $\cdot$ is normal). The subtraction thm tells us there is some unique ord γ so $\alpha = \delta \cdot \beta + \gamma$. $\gamma \in \delta$ lest $\gamma = \delta + \varepsilon$ by the same thm. But then $\alpha = \delta \cdot \beta + (\delta + \varepsilon) = (\delta \cdot \beta + \delta) + \varepsilon = \delta \cdot \beta^+ + \varepsilon$, contradicting the largeness of β.

Proof By Theorem 8D, there exists the largest ordinal number β with $\delta \cdot \beta \in \alpha$ since $\delta \neq 0$ which tells us $\cdot$ is normal. The Subtraction Theorem now informs us that there is some unique ordinal number γ so $\alpha = \delta \cdot \beta + \gamma$. In fact, $\gamma \in \delta$ lest $\gamma = \delta + \varepsilon$ for some ordinal number ε, again by the Subtraction Theorem. But then $\alpha = \delta \cdot \beta + (\delta + \varepsilon) = (\delta \cdot \beta + \delta) + \varepsilon = \delta \cdot \beta^+ + \varepsilon$ Hence contradicting the fact that β is supposed to be largest. $\square$

Remarks $\beta' \in \beta$ $\quad \beta'^+ \in \beta$

$\delta \cdot \beta' + \gamma'$ $\quad \delta \cdot \beta + \gamma$

Ah yeah it might have been good to add a section to show the uniqueness of β for any $\beta' \in \beta$ like what the aut did.

Exercises

20. By the Division Theorem, every ordinal number $\alpha = \omega \cdot \beta + n$ for the unique ordinal numbers β and $n \in \omega$. We claim $\omega \cdot \beta$ is the limit ordinal λ we are looking for if $\beta \neq 0$. Consider β being a successor ordinal first, i.e. $\beta = \gamma^+$ for some ordinal γ. Then we see that $\omega \cdot \gamma^+ = \omega \cdot \gamma + \omega = \sup\{\omega \cdot \gamma + m \mid m \in \omega\}$, so that for any $\delta \in \beta$ which we know must be in some $\omega \cdot \gamma + k$, we will have $\delta^+ \in (\omega \cdot \gamma + k)^+ = \omega \cdot \gamma + k^+$. And hence, $\delta^+ \in \omega$, thus confirming $\omega \cdot \beta$ is a limit ordinal. Now when β is a limit ordinal, the result follows similarly. Consequently, every ordinal number is indeed uniquely expressible in the form $\lambda + n$ where either $\lambda = 0$ or λ is a limit ordinal. $\square$

21. By the fact that there is an infinite number of primes and every positive integer can be uniquely prime factorised, thus there is an infinite number of positive integers $n \neq 1$ with m prime factors for each $m \geq 0$. As such,

$\langle P, < \rangle \cong \langle \{1_2\} \cup (\omega \times \omega), \{\langle 1_2, n \rangle \mid n \in \omega\} \cup <_H \rangle$ where the second coordinate (for ordered pairs in $\omega \times \omega$) represents the number of prime factors, say m, and the first coordinate, say k, is the kth positive integer with m prime factors. Now, we can tell that the ordinal number of $\langle P, R \rangle$ should be $1 + \omega \cdot \omega = \omega \cdot \omega$.

(Of course, to do it rigorously we should find the isomorphism but ehh that's just an exercise in tedium).

23. (a) First, notice that ω^2 is a limit ordinal, so

$$\begin{aligned}
\omega + \omega^2 &= \sup\{\omega + \alpha \mid \alpha \in \omega^2\} \\
&= \sup\{\omega + \alpha \mid (\exists n \in \omega)(\alpha \in \omega \cdot n)\} \\
&\in \sup\{\omega + \omega \cdot n \mid n \in \omega\} \\
&= \sup\{\omega \cdot (1+n) \mid n \in \omega\} \\
&= \sup\{\omega \cdot n \mid n \in \omega\} \\
&= \omega \cdot \omega.
\end{aligned}$$

Furthermore, we know that by Theorem 8Q, $\omega \cdot \omega = 0 + \omega \cdot \omega \in \omega + \omega \cdot \omega$. Hence, $\omega + \omega^2 = \omega^2$.

23. (b) The Subtraction Theorem tells us there exists a ordinal number δ with $\omega^2 + \delta = \beta$. As such,

$$\begin{aligned} \omega + \beta &= \omega + (\omega^2 + \delta) \\ &= (\omega + \omega^2) + \delta \\ &= \omega^2 + \delta \qquad \text{by (a)} \\ &= \beta \end{aligned}$$

as desired. $\square$

24. Ideas $\omega^2 \in \alpha^2$

$\omega + \alpha^2 = \alpha^2$

$\alpha + \alpha^2 = \omega + \delta + \alpha^2$

$\omega \in \alpha$

$\omega \in \gamma$

$$\begin{aligned} \alpha + \alpha^2 &= \gamma^+ + (\gamma^+)^2 \\ &= \gamma + 1 + (\gamma+1)^2 \\ &= \gamma + \underbrace{1 + \gamma^2}_{\gamma^2} + \gamma + \gamma + 1 \\ &= \gamma + \gamma^2 + \gamma + \gamma + 1 \\ &= \gamma^2 + \gamma + \gamma + 1 \\ &= (\gamma+1)^2 \\ &= (\gamma^+)^2 \\ &= \alpha^2 \end{aligned}$$

$$\begin{aligned} \gamma^2 &= \omega + \delta \\ 1 + \gamma^2 &= 1 + \omega + \delta \\ &= \omega + \delta \\ &= \gamma^2 \end{aligned}$$

24. Assume that for all $\gamma \in \alpha$, if $\omega \subseteq \gamma$ then $\gamma + \gamma^2 = \gamma^2$. Consider α being a successor ordinal, i.e. $\alpha = \gamma^+$ for some ordinal $\gamma \in \alpha$. Then

$$\begin{aligned}
\alpha + \alpha^2 &= \gamma^+ + (\gamma^+)^2 \\
&= \gamma + 1 + (\gamma + 1)^2 \\
&= \gamma + 1 + \gamma^2 + \gamma + \gamma + 1 \\
&= \gamma + \gamma^2 + \gamma + \gamma + 1 && \text{since } 1 + \gamma^2 = \gamma^2. \\
&= \gamma^2 + \gamma + \gamma + 1 && \text{by assumption} \\
&= (\gamma + 1)^2 \\
&= (\gamma^+)^2 \\
&= \alpha^2
\end{aligned}$$

As for when α is a successor ordinal, the procedure is similar to that of 23.(a). So, this is true for α. Consequently, transfinite induction tells us this is true for all ordi[nals]. Now, by a repeat of what was done in 23.(b), we can conclude that $\alpha + \beta = \beta$. $\square$

25. Ideas

$\cancel{\bar{\alpha} + \bar{\Theta} = \text{it } \langle (\{0\} \times \alpha) \cup (\{1\} \times \Theta), <_L \rangle}$

$\cancel{f(n, \beta) = \begin{cases} \beta & \text{if } n = 0, \\ \alpha + \beta & \text{if } n = 1 \end{cases}}$

$\alpha + \delta$

Consider any $\beta \in \alpha + \Theta$ and wlog that $\alpha \subseteq \beta$

$\beta = \alpha + \delta$

$\alpha + \delta \in \alpha + \Theta$

$\delta \in \Theta$

Proof

Clearly, $\alpha \cup \{\alpha + \delta \mid \delta \in \Theta\} \subseteq \alpha + \Theta$ by monotonicity. Now consider any $\beta \in \alpha + \Theta$, and without loss of generality that $\alpha \subseteq \beta$. From the Subtraction Theorem, $\beta = \alpha + \delta \in \alpha + \Theta$ and Corollary 8P(a) says $\delta \in \Theta$. Hence, we are assured that $\beta \in \alpha \cup \{\alpha + \delta \mid \delta \in \Theta\}$. In other words, $\alpha + \Theta \subseteq \alpha \cup \{\alpha + \delta \mid \delta \in \Theta\}$ too. Therefore we conclude equality holds indeed. $\square$

26. Ideas

$\gamma_m \in \gamma_{m-1}$

$\omega^{\gamma_m} \cdot n_m \in \omega^{\gamma_m} \cdot \omega$
$= \omega^{\gamma_m^+}$
$\in \omega^{\gamma_{m-1}}$
$\in \omega^{\gamma_{m-1}} \cdot n_{m-1}$

$\Rightarrow \omega^{\gamma_m} \cdot n_m \in \omega^{\gamma_{m-1}} \cdot n_{m-1}$

$1 \in S$:

$\omega^{\gamma_{m-1}} \cdot n_{m-1} + \omega^{\gamma_m} \cdot n_m \in \omega^{\gamma_{m-1}} \cdot n_{m-1} + \omega^{\gamma_{m-1}} \cdot n_{m-1}$
$= \omega^{\gamma_{m-1}} \cdot n_{m-1} \cdot 2$
$= \omega^{\gamma_{m-1}} \cdot 2n_{m-1}$
$\in \omega^{\gamma_{m-2}} \cdot n_{m-2}$

$i \in S$: $\cancel{\omega^{\gamma_{m-i}} \cdot n_{m-i}} + \omega^{\gamma_{m-i}} \cdot n_{m-i} + \dots + \omega^{\gamma_m} \cdot n_m \in \cancel{\omega^{\gamma_{m-i}} \cdot n_{m-i}} + \omega^{\gamma_{m-i}} \cdot n_{m-i} + \omega^{\gamma_{m-i}} \cdot n_m$
$\in \cancel{\omega^{\gamma_{m-i}} \cdot n_{m-i}} + \omega^{\gamma_{m-i}} \cdot 2n_{m-i}$
$\in \omega^{\gamma_{m-i}} \cdot \omega$
$\in \omega^{\gamma_{m-i^+}} \cdot n_{m-i^+}$

$i^+ \in S$

Assume there exists least $i \in \omega$ with γ_i or n_i not unique.

If $i = 0$, log thm

$\omega^{\gamma_0} \cdot n_0 + \dots + \omega^{\gamma_j} \cdot n_j + \cancel{\rho} = \alpha$ for some unique $\cancel{\rho}$ lest violate Corollary 8P(a) — just use $\omega^{\dots}, \dots$ instead of ρ

If $i = j^+$,

$j^+ \dots \quad \cancel{\rho} = \omega^{\gamma_{j'}} \cdot n_{j'} + \rho$ by log thm ...

26 . [illegible] for the entirety of this proof for Lemma 26A

$$\omega^{\gamma_{k-m-1}} \cdot n_{k-m-1} + \cdots + \omega^{\gamma_k} \cdot n_k \in \omega^{\gamma_{k-m}} \cdot n_{k-m} \text{ for any natural } m \in k$$

Let S be the set of all $m \in \omega$ so that if $m \in k$, then the above result holds. $0 \in S$ clearly holds because $0 \in \omega^{\gamma_{k-m}} \cdot n_k$.

So assume $m \in S$, and wlog, that $m^+ \in k$. Then,

$$\begin{aligned}
\omega^{\gamma_{k-m}} \cdot n_{k-m} + \omega^{\gamma_{k-m-1}} \cdot n_{k-m-1} + \cdots + \omega^{\gamma_k} \cdot n_k &\in \omega^{\gamma_{k-m}} \cdot n_{k-m} + \omega^{\gamma_{k-m}} \cdot n_{k-m} && \text{by our assumption} \\
&= \omega^{\gamma_{k-m}} \cdot 2n_{k-m} \\
&\in \omega^{\gamma_{k-m}} \cdot \omega && \text{by Corollary 8P(a)} \\
&= \omega^{(\gamma_{k-m}^+)} \\
&\in \omega^{\gamma_{k-m^+}} \cdot n_{k-m^+} .
\end{aligned}$$

Hence, $n^+ \in S$ indeed. As such, $S = \omega$ by induction. ◇

Now suppose T is the set of all $m \in \omega$ with $m \in k$ implying that $\gamma_0 \in \cdots \in \gamma_m$ and $n_0, \ldots, n_m \in \omega$ are unique. Consider $m = 0$. By Lemma 26A, we know that for $\omega^{\gamma_0} \cdot n_0 + (\omega^{\gamma_1} \cdot n_1 + \cdots + \omega^{\gamma_k} \cdot n_k)$, we have $(\omega^{\gamma_1} \cdot n_1 + \cdots + \omega^{\gamma_k} \cdot n_k) \in \omega^{\gamma_0} \cdot n_0$, in addition to $n_0 \in \omega$. By the Logarithm Theorem, γ_0, n_0, and the sum $(\omega^{\gamma_1} \cdot n_1 + \cdots + \omega^{\gamma_k} \cdot n_k)$ must be unique. Accordingly, $0 \in S$. Assume $m \in T$, and wlog, that $m^+ \in k$. Then we know the sum (not necessarily the individual terms (yet)) $\omega^{\gamma_{m^+}} \cdot n_{m^+} + \cdots + \omega^{\gamma_k} \cdot n_k$ must be unique lest Corollary 8P(a) is violated as the first m terms are uniquely determined by our assumption. Again, Lemma 26A tells us that $(\omega^{\gamma_{m^{++}}} \cdot n_{m^{++}} + \cdots + \omega^{\gamma_k} \cdot n_k) \in \omega^{\gamma_{m^+}} \cdot n_{m^+}$ where we also know $n_{m^+} \in \omega$. Therefore, the Logarithm Theorem informs us once more that γ_{m^+} and n_{m^+} must be uniquely determined. As such, $m^+ \in T$. By induction, $T = \omega$.

□

Self - Proof of Theorem 9A, trivial.

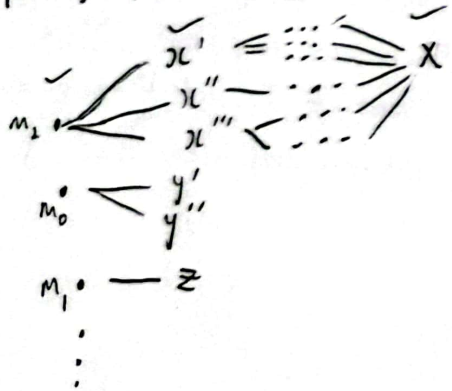
Assume that the relation R is well founded and f is a function from ω into $\mathrm{fld}\, R$. Then there exists a minimal element $f(m)$ for $\mathrm{ran}\, f$ so there is no $n \in \omega$ with $f(n)\, R\, f(m)$. So, $f(m^+)\, R\, f(m)$ is false. Thus, there exists no infinitely descending R-chain f. Conversely, suppose R is not well founded. In other words, for each $y \in \mathrm{fld}\, R$ there exists $x \in \mathrm{fld}\, R$ with $x R y$. Let the set of all such x's be the nonempty set X_y. By AC there exists a function f with $f(x) \in X_y$. By recursion we define $h : \omega \to \mathrm{fld}\, R$ by

$$h(0) = a,$$
$$h(n^+) = f(h(n)),$$

where a is some fixed element of $\mathrm{fld}\, R$ that must exist from the non-well-foundedness of R. Thus, $h(n^+)\, R\, h(n)$ is certain. That is, f is an infinitely descending R-chain f.

$\square$

Tranfinite Induction Principle, intuition



Self - Proof of Theorem 9B

Ideas

$T := \{ S \mid R \subseteq S \subseteq \text{fld}\, R \times \text{fld}\, R$ & S is transitive $\}$

$\text{fld}\, R \times \text{fld}\, R \in T$ clearly.

Let $R^t := \bigcap T$.

Assume $x R^t y$ and $y R^t z$. So, for any $R' \in T$, $x S y$ and $y S z$, thus $x S z$. Therefore, $x R^t z$ and R^t is a transitive relation with $R \subseteq R^t$. Hence satisfying (a). By definition, (b) follows as well.

Uniqueness holds by virtue of (b).

$\square$

Self - Proof of Theorem 9C

Ideas

We know $R^t \subseteq \text{fld}\, R \times \text{fld}\, R$. Hence suppose nonempty $S \subseteq \text{fld}\, R$ and m is its R-minimal element. Consider the case where m is not the R^t-minimal element of S. Let $T \subseteq \text{fld}\, R$ contain all the y's so that there exists $x \in S$ with $x R^t y R^t m$, or $x = y$ and $x R^t m$. $T \neq \emptyset$ by the nonminimality of m in S. Hence, T has some R-minimal element n, which must be the R^t-minimal element of T, lest there exist $y R n$ or $y R_k y' R n$ for some $y, y' \in \text{fld}\, R$ and $k \in \omega$, contradicting the R-minimality of n. Now, n is the R^t-minimal element of S

Self - Proof of Corollary 9D

Ideas

Irreflexivity follows trivially from the fact that R^t is well-founded and R^t is transitive by construction. Hence, R^t is indeed a partial ordering on A (again since $R \subseteq A \times A$, so is R^t).

3. Suppose that S is the set of all x such that $\{z \mid z R^t x\}$ is finite and that $\mathrm{seg}_{R^t}\, y \subseteq S$. Now, notice that

$$\{z \mid z R^t y\} = \{x \mid x R y\} \cup \{z \mid z R^t x R y \text{ for some } x\}$$

$$= \{x \mid x R y\} \cup \bigcup_{xRy} \{z \mid z R^t x\} .$$

By our supposition, $\{z \mid z R^t x\}$ is finite for each x with xRy. From the provided assumption we also know $\{x \mid x R y\}$ is finite so that $\bigcup_{xRy} \{z \mid z R^t x\}$ is a finite set of finite sets, and hence is itself finite. Therefore, $\{z \mid z R^t y\}$ is definitely finite. Transfinite induction then ensures $S = \mathrm{fld}\, R^t$ (because by defn, every nonempty set contains an R^t-minimal element, including every subset of $\mathrm{fld}\, R^t$, meaning R^t is a well-founded relation on $\mathrm{fld}\, R^t$). Lastly, if $y \notin \mathrm{fld}\, R^t$, then $\{z \mid z R^t y\} = \varnothing$ is trivial finite.

$\square$

3. Ideas (Trying to do it without transfinite induction lol)

For each x s.t. xRy, there is the set $T_x := \{t \mid tR^t x\}$. If $T_x \neq \emptyset$, exists the nonempty set of minimal elements M.

By recursion we can define:

$h_x(0) = \{t \mid tRx\}$

$h_x(m^+) = \{t' \mid t'Rt$ for some $t \in h_x(m)\}$

For $tR^t y$, $tR_n y$ where $n=0$ or $n \in 1$. In the latter case, $tR_{n-1} x R y$ for some x.
tRy

$\{t \mid tR^t y\} = \{t \mid tRy\} \cup \{t \mid tR_n xRy$ for some x & $n \in \omega\}$ $\Longrightarrow$ There two sets are disjoint lest irreflexivity of the partial order R^t is violated.

(Well even if they're not we can just slap in the set minus opera

Self-Proof of Theorem 9E

$xM^t a$

Let T be the set of $a \in TC\, S$ with $E(a) = \text{rank } a$ and assume $\text{seg } a \subseteq T$. Then, $E(a) = \{E(x) \mid x \in a\} = \{\text{rank } x \mid x \in a\}$[1]

$xM^t a$

$\subseteq \bigcup\{(\text{rank } x)^+ \mid x \in a\} = \text{rank } a$ by Theorem 7V(b). For the converse, suppose $\alpha \in \text{rank } a$, in other words, $\alpha \subseteq \text{rank } x$ for some $x \in a$.

$yM^t x$ & $x \in a$

If equality holds, $\alpha = \text{rank } x = E(x) \in E(a)$ by assumption. Otherwise, $\alpha \in E(x) = E[\![\text{seg } x]\!]$ so that $\alpha = E(y) \in E(a)$ for some $y \in x \in a$, again

$yM^t a$

by virtue of our assumption (where $y \in a$ holds because $TC\, S$ being a transitive set tells us $y \in x \in TC\, S$, and hence, $yM^t x$). Therefore, $\text{rank } a \subseteq E(a)$ too.

Consequently, $E(a) = \text{rank } a$ and $T = TC\, S$ by transfinite induction. Lastly, $\text{ran } E = \text{rank } S$ can be similarly proven. $\square$

Oh actually if we choose to use $TC\{S\}$ instead, we can immediately say $\text{ran } E = E'(S) = \text{rank } S$.

[1] Oops I realised I had some confusion about M^t! But the general idea is there.

Exercise

See self-proof of Theorem 9C $\square$

Since R^t is certainly transitive regardless of what R is, thus for R^t to be irreflexive, R itself must be such that if $xR_n y$ for some $n \in \omega$, then $yR_m x$ for no $m \in \omega$.

Self-Proof of Lemma 9G

Ideas

$\langle \omega \times 2, <_H \rangle \cong \langle \omega, \sqsubset \rangle$

where $\sqsubset$ is a well-ordering on ω given by $n \sqsubset m$ iff n is ~~odd~~ even and m is ~~even~~ odd, or n, m both odd and $n \in m$, or n, m both even and $n \in m$.

bijection provided by $f : \omega \longrightarrow \omega \times 2$ with

$$f(n) = \begin{cases} \langle m, 0 \rangle & \text{if } n = 2m, \\ \langle m, 1 \rangle & \text{if } n = 2m+1; \end{cases}$$

Injectivity: Assume $f(n) = f(m)$. Clearly, $n = m$ by construction.

Surjectivity: Suppose $\langle n, m \rangle \in \omega \times 2$

when $m = 0$: $f(2m) = \langle m, 0 \rangle$

when $m = 1$: ...

Bijective ✓

Since $\langle \omega \times 2, <_H \rangle$ has the ~~same~~ ordinal number, namely $\omega \cdot 2$, so must $\langle \omega, \sqsubset \rangle$. Since rank $\omega \cdot 2 = \omega \cdot 2$, $\omega \cdot 2 \notin V_{\omega \cdot 2}$ $\square$

Not sure if $\sqsubset \in V_{\omega \cdot 2}$ is an important criteria to satisfy "well-ordered structure in $V_{\omega \cdot 2}$" or is it just the underlying set that needs to be in $V_{\omega \cdot 2}$.

Either ways, $\sqsubset \subseteq \omega \times \omega \subseteq \mathcal{P}\mathcal{P}\omega$ so $\sqsubset \in V_{\omega \cdot 2}$.

$S \subseteq \omega \subseteq V_\omega \Rightarrow S \subseteq V_\omega \Rightarrow S \in \mathcal{P} V_\omega = V_{\omega^+}$

$\Longrightarrow \mathcal{P}\omega \subseteq V_{\omega^+}$

$\vdots$

$\Longrightarrow \mathcal{P}\mathcal{P}\omega \subseteq V_{\omega^{++}}$

oh yeah $\sqsubset \in V_{\omega \cdot 2}$ is probably important in this context for our model arguments.

Self-Proof of Corollary 9H

Assume, for the sake of contradiction, that all the replacement axioms are true in $V_{\omega\cdot 2}$. Combined with $\omega\cdot 2$ being a limit ordinal, ensuring Theorem 9F holds true, we can hence apply transfinite recursion to $\langle \omega, \in \rangle$ (where ω and $\in$ are both in $V_{\omega\cdot 2}$ by the proof of Lemma 9G) which enables us to form the usual function E with domain ω given by $E(n) = E[\![\operatorname{seg} n]\!]$. Now, $\operatorname{ran} E$ supposedly exists in $V_{\omega\cdot 2}$. But $\operatorname{ran} E = \omega\cdot 2 \notin V_{\omega\cdot 2}$ was shown in the proof of Lemma 9G, a contradiction.

Corollary 9I follows trivially from Corollary 9H.

Self-Proof of Lemma 9J

Assume that for any $\beta \in \alpha$, $\operatorname{card} V_{\omega+\beta} = \beth_\beta$. If $\alpha = 0$, $\operatorname{card} V_\omega = \aleph_0 =: \beth_0$ since V_ω is a countable union of countable sets. When α is a successor ordinal, i.e. $\alpha = \beta^+$ for some $\beta \in \alpha$, $\operatorname{card} V_{\omega+\alpha} = \operatorname{card} V_{(\omega+\beta)^+} = \operatorname{card} \mathcal{P} V_\beta = 2^{\beth_\beta} = \beth_\alpha$ by assumption. Lastly, consider the case where α is a limit ordinal. Then, $\operatorname{card} V_{\omega+\alpha} = \operatorname{card} \bigcup\{V_\gamma \mid \gamma \in \omega+\beta \text{ for some } \beta \in \alpha\} = \operatorname{card} \bigcup\{V_{\omega+\beta} \mid \beta \in \alpha\}$ follows from Theorem 7U(a). Thus, exercise 26 of chapter 6 tells us $\operatorname{card} V_{\omega+\alpha} \le (\operatorname{card} \alpha)\cdot \beth_\alpha$; which is just $\beth_\alpha$ by exercise 5 of chapter 8 and the Absorption Law of Cardinal Arithmetic. Conversely, we see that for any $\beth_\beta < \beth_\alpha$ (i.e. any $\beta \in \alpha$), $\beth_\beta < \operatorname{card} V_{\omega+\alpha}$ by assumption. Therefore, given $\gamma \in \beth_\alpha$, $\gamma \in \beth_\beta \in \operatorname{card} V_{\omega+\alpha}$. Hence, $\beth_\alpha \subseteq \operatorname{card} V_{\omega+\alpha}$ and $\beth_\alpha \le \operatorname{card} V_{\omega+\alpha}$ results. Consequently, $\operatorname{card} V_{\omega+\alpha} = \beth_\alpha$. Transfinite induction asserts $\operatorname{card} V_{\omega+\alpha} = \beth_\alpha$ for every ordinal α.

Self-Proof of Lemma 9K

(a) Follows from transfinite induction. The zeroth case being given by part (a) (of the definition of inaccessible cardinals), the successor ordinal case by (b) and lastly the limit ordinal case by (c).

(b) Similarly follows from ~~transfinite induction and~~ the definition of inaccessible cardinals;

Theorem 9L

Since κ is a limit ordinal, theorem 9F immediately informs us all the Zermelo axioms must hold true in V_κ.

To prove this for the replacement axioms, first assume φ is a formula not containing B with the property that $(\forall x \in A)(\forall y_1 \in V_\kappa)(\forall y_2 \in V_\kappa)(\varphi(x, y_1)\ \&\ \varphi(x, y_2) \Rightarrow y_1 = y_2)$. Now, we form a modified formula ψ defined by

$$\psi(x, y) \quad \text{iff} \quad \varphi(x, y) \text{ and } y \in V_\kappa .$$

Therefore, ψ fulfills the property that $(\forall x \in A)\, \forall y_1 \forall y_2 (\psi(x, y_1)\ \&\ \psi(x, y_2) \Rightarrow y_1 = y_2)$. By a replacement axiom, there exists a set B with $y \in B$ iff $(\exists x \in A)\psi(x, y)$. To show $B \in V_\kappa$, first define yet another formula $\gamma(x, y)$ iff $y = (\text{rank } x)^+$. Applying another replacement axiom on B and γ, there is a set C of all $(\text{rank } y)^+$ for $y \in B$. We know $(\text{rank } y)^+ \in \kappa$ and Lemma 9K (b) tells us $\text{card } C \le \text{card } B \le \text{card } A < \kappa$. Thus, C is a set of ordinals less than κ with $\text{card } C < \kappa$. By part (c) of the definition of inaccessible cardinals, $\text{rank } B \overset{\text{Thm 7V(b)}}{=} \sup C \in \kappa$. Therefore, $(\text{rank } B)^+ \in \kappa$ which means $B \in V_\kappa$. $\square$

Exercises

5. Assume $S \in V_\omega$. Then $S \subseteq V_{\text{rank } S}$, where rank $S \in \omega$ so $V_{\text{rank } S}$ is finite. Notice that for any $t \in TC\, S$, $t \in \cdots \in S$. As such, rank $t \in \cdots \in$ rank S. Thus, for any $t \in S$, $t \in V_{\text{rank } S}$' Which also means $TC\, S \subseteq V_{\text{rank } S}$ and must hence be finite. Conversely, suppose $S \notin V_\omega$. ~~That is, $\omega \subseteq$ rank S~~. If S is infinite, the result is immediate, thus further suppose S is finite. That is, $\omega \in$ rank S. We first prove the result below.

Presume that for all $\beta \in \alpha$, given any $x \in V_\beta - V_\omega$, $TC\, x$ is infinite. If $\alpha \in \omega$ this would be vacuously true for α. Therefore, consider ~~wlog that~~ $\omega^+ \in \alpha$ and the two cases that follow:

Case 1 $\alpha = \beta^+$ for some ordinal $\beta \in \alpha$. Accordingly, for any $x \in V_\alpha - V_\omega = \mathcal{P} V_\beta - V_\omega \neq \varnothing$, either x is an infinite subset of V_ω, or there exists $y \in x$ with $y \notin V_\omega$ which tells us $TC\, y \subseteq TC\, x$ is infinite by our presumption.

Case 2 α is a limit ordinal. Then, let $x \in V_\alpha - V_\omega = \left(\bigcup_{\beta \in \alpha} \mathcal{P} V_\beta\right) - V_\omega \neq \varnothing$. In other words, $x \in V_{\beta^+}$ for some ordinal $\beta \in \alpha$. Hence, $TC\, x$ is infinite again by our presumption.

Regardless we see that the result is true of α too. Consequently, this is true for every ordinal α via transfinite induction.

Now, our further supposition says there exists $x \in S - V_\omega \neq \varnothing$, thus $x \in V_{(\text{rank } S)^+} - V_\omega$ and $TC\, x$ is infinite by the above result. Wherefore, $TC\, S$ is infinite regardless. $\square$

Could do some improvements to the phrasing but I don't wanna rewrite this whole paragraph.

... and each of its elements has rank β. Accordingly, every of S's elements are $\in^\circ$-minimal (in S) lest

6. Assume the set $A \in V_\omega$ and φ is a formula with the property that $(\forall x \in A)(\forall y_1 \in V_\omega)(\forall y_2 \in V_\omega)[\varphi(x, y_1) \& \varphi(x, y_2) \rightarrow y_1 = y_2]$. Again, we define a modified formula ψ with $\psi(x,y)$ iff $\varphi(x,y)$ & $y \in V_\omega$. Then by a replacement axiom, there exists a set B with $y \in B$ iff $(\exists x \in A)\psi(x,y)$. Since $A \in V_\omega$, we know that $A \subseteq V_{\text{rank } A}$ must be a finite set as rank $A \in \omega$. Therefore, by definition of our modified formula ψ, B must be a finite set of rank less than ω. Hence, $B \in V_\omega$ is guaranteed. $\square$

The only axiom not true in V_ω is infinity, as shown in the proof of theorem 9F.

7. Ideas

$\langle \omega, E \rangle$ and $\langle V_\omega, \in^\circ \rangle$ are well-founded structures

Let S_n be a subset of ω having cardinality n and T contain all natural numbers n with S_n having an E-minimal element if $n \neq 0$. So, $0 \in T$ immediately. Assume $n \in T$ now. Then taking a fixed member $m \in S_{n^+} \neq \emptyset$, we have $S_{n^+} = (S_{n^+} - \{m\}) \cup \{m\}$ where $S_{n^+} - \{m\} \approx n$. If m is an E-minimal element of S_{n^+}, the result follows. Otherwise, $\overset{S_{n^+} - \{m\}}{n}$ must be nonempty (i.e. $n \neq 0$). Hence, by assumption, there exists an E-minimal element of it, which is also the E-minimal element of S_{n^+}. Regardless, we see that $n^+ \in T$. Accordingly, $T = \omega$ holds from induction. Lastly, consider any S_ω. Th

Let S be a nonempty subset of ω and ℓ be its $\in$-least element. We claim that ℓ must be E-minimal in S, lest there exists $m \in S$ with $m \in g(\ell)$. But then $m \in \ell$, contradicting the $\in$-leastness of S.

Assume that for any $\beta \in \overset{\in\omega}{\alpha}$, any nonempty subset of V_β has a $\in^\circ$-minimal element.

Case 1, $\alpha = 0$, then no nonempty subsets of $V_\alpha = \emptyset$ exist. Accordingly, the above is true of $\alpha = 0$ immediately.

Case 2 $\alpha = \beta^+$ for some $\beta \in \alpha$. Now take any nonempty subset S of V_α. If $S \cap V_\beta \neq \emptyset$ then any of its $\in^\circ$-minimal elements, say I, must be an $\in^\circ$-minimal element of S because for any $X \in S - V_\beta$, X has rank β. So if, contrary to our expectations, $X \in I$ then rank $X = \beta \in$ rank I, but $I \in S \cap V_\beta$ means rank $I \in \beta$, a contradiction. Suppose instead that $S \cap V_\beta = \emptyset$ now. It follows that $S \subseteq V_\alpha - V_\beta$ and each of its elements has rank β. Accordingly, every of S's elements are $\in^\circ$-minimal (in S) lest $\beta \in \beta$.

Case 3: α is a limit ordinal. Suppose wlog that S is a subset of V_α with rank α. Then $S \cap V_\beta \neq \varnothing$ for some $\beta \in \alpha$, any of its $\in^\circ$-minimal members, say I, must again be an $\in^\circ$-minimal member of S: Given $X \in S - V_\beta$, $\beta \subseteq \text{rank } X$. Therefore, if $X \in I$, $\beta \subseteq \text{rank } X \in \text{rank } I \in \beta$, a contradiction.

Regardless, every nonempty subset of V_α has a $\in^\circ$-minimal element. Transfinite induction tells us this is true of all ordinals α, such as ω. As a result, $\langle V_\omega, \in^\circ \rangle$ is a well-founded structure.

(or instead of TR just using a rank-based + replacement argument probably works)

$g(0) = \varnothing$ (lest $n \in 0$)

$g(1) = \{0\}$ (lest $n \in 1$ for some $n \neq 0$)

$g(2) = \{1\}$ or $\{0,1\}$ (i.e. 2)

$g(n)$ is one of ($2 \subseteq n$): $\{1\}$, $\{2\}$, ..., $\{n-1\}$, $\{$some number of natural numbers $m \in n\}$, ...

$\sum_{r=0}^{n} \binom{n}{r} - \overbrace{1}^{\text{card } \varnothing} - (n-2) - \overbrace{1}^{\{0\}}$

$= 2^n - 1 - n + 2 - 1$

$= 2^n - n$

$g(4) = \{2\}$

$0 \to 1 \to 2 \to 4$

$3 \in 8$

5

$V_0 = \varnothing$

$V_1 = \{\varnothing\}$

$V_2 = \{\varnothing, \{\varnothing\}\}$

$V_3 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$
$= \{0, 1, 2, \{1\}\}$

$V_4 = V_3 \cup \{\{2\}, \{\{1\}\}, \{0,1,2\}, \{0,2,\{1\}\}, \{0,1,\{1\}\}, \{1,2,\{1\}\}, \{0,\{1\}\}, \{1,\{1\}\}, \{2,\{1\}\}, \{0,1,2,\{1\}\}, \{0,2\}, \{1,2\}\}$

$\varnothing$, $\{0\}$, $\{0,1\}$, $\{1\}$

12

$-(n-2)$

$V_1 - V_0 = \{0\}$ 0 is global min

$\{1\}$

$\{\{1\}, 2\}$

$\text{card } V_{n^+} = 2^{\text{card } V_n}$

$2^{\text{card } V_n} - \text{card } V_n$

$h(m) \in h[\![g(n)]\!]$ $\{h(m) \mid m \in g(n)\}$

$m \in g(n)$

7. Define the isomorphism $h : \omega \longrightarrow V_\omega$ by $h(n) = h[\![g(n)]\!]$ using transfinite recursion. The codomain of h must be V_ω since $h(0) = \emptyset$ with rank 0 and rank $h(n^+) \in (\text{rank } h[\![n]\!])^+$ (as $g(n) \subseteq n$).

Injectivity: Assume that for all $m' \in m \in n$, $h(m') \neq h(m)$.

Case 1 $n = 0$, then this is vacuously true.

Case 2 $n = k^+$ for some natural $k \in n$. The bijective nature of g tells us that $g(m) \neq g(n)$ for $m \in n$. Hence there exists some $i \in n$ that is in exactly one of $g(m)$ and $g(n)$. By assumption, $h(i)$ must be different from every other $h(j)$ for each $j \in n$. So, $h(i) \in h(m)$ but $h(i) \notin h(n)$ or vice versa because both are subsets of $\{h(j) \mid j \in n\}$.

Regardless, the statement is true for n. By Strong Induction, this is true for each $n \in \omega$.

Two-Way Order-Preserving:

If $m E n$, $m \in g(n)$ so $h(m) \in h(n)$. Conversely, when $h(m) \in h(n)$, it must be that $m \in g(n)$, lest there exists $k \in g(n) \subset n$ with $g(k) = g(m)$. But then $k = m$ by injectivity, a contradiction.

Surjectivity: Suppose that for every $m \in n \in \omega$, any $X \in V_m$ is such that $X \in \text{ran } h$.

Case 1 $n = 0$, vacuous.

Case 2 $n = k^+$ for some natural $k \in n$. Consider $X \in V_n$, thus $X \subseteq V_m$ for some $m \in n$. It follows that $X = h(g^{-1}\{g^{-1}[\![h^{-1}[\![X]\!]]\!]\})$.

Therefore, this is true of n too. Strong induction again tells us this holds for all $n \in \omega$.

Consequently, ran $h = V_\omega$.

Wherefore, $\langle \omega, E \rangle \overset{h}{\cong} \langle V_\omega, \in^\circ \rangle$.

8. See self-proof.

$\square$

9. First, notice that $\alpha+\lambda$ is a limit ordinal, because if $\beta \in \alpha+\lambda$, then $\beta^{+} \subseteq \alpha+\delta \in \alpha+\lambda$ by the normality of ordinal addition (for some $\delta\in\lambda$ with $\beta\in\alpha+\delta$). So,

$$\begin{aligned} V_{\alpha+\lambda} &= \bigcup\{V_\beta \mid \beta\in\alpha+\lambda\} \\ &= \bigcup\{V_{\alpha+\delta} \mid \delta\in\lambda\} \end{aligned}$$

Since for $\beta\in\alpha+\delta\in\alpha+\lambda$, $V_\beta \subseteq V_{\alpha+\delta}$.

$\square$

0. By transfinite recursion we define the function F of domain ω by $F(0)=\omega$ and $F(n^{+}) = \mathcal{P}F(n)$.
Then let $S := \bigcup F[\![\omega]\!]$:

Extensionality

Suppose $T\in S$, so $T\in F(n)$ for some $n\in\omega$. If $n=0$, $m\in T\in\omega$ always implies $n\in\omega\subseteq S$. Thus consider $1\in n$, meaning $T\subseteq F(m)$ where m is the natural number for which $n=m^{+}$. Therefore, we again have that any T' in T must be in $F(m)$, and hence, in S. Accordingly, S is a transitive set that models extensionality as shown in the proof of Theorem 9F.

Empty Set

Well, $0\in\omega\subseteq S$. Immediately, S models the empty set axiom is seen.

Pairing

Presume $u, v\in S$ and wlog that $n\in m$ where $u\in F(n)$ and $v\in F(m)$. We claim that $n\in m$ implies $F(n)\subseteq F(m)$:

Let N be the set of $n\in\omega$ with $F(n)\subseteq F(m)$ for all $n\in m$ and M_n be the set of $m\in\omega$ such that if $n\in m$, then $F(n)\subseteq F(m)$.

Immediately, $0\in M_0$ as $0\notin 0$. So assume $m\in M_0$. First consider $m=0$, then given $k\in\omega$, $k\subseteq\omega$ such that $k\in F(1)$. That is, $1\in M_0$ can be similarly proven. Therefore, suppose $2\in m$. In other words, $k^{+}=m$ for some $1\in k\in m$. Thus, $F(k)\subseteq F(m)$ by assumption. Given any $X\in F(m)$, we know $X\subseteq F(k)$. As $F(k)\subseteq F(m)$, X is also a subset of $F(m)$. Hence, $X\in F(m^{+})$ and $m^{+}\in M_0$. Accordingly, $M_0=\omega$ and $0\in S$.

40. My definition makes that an ass to prove so let us define another function G of domain ω by $G(0) = \omega$ and $G(n^+) = (\bigcup G[\![n]\!]) \cup \mathcal{P}G(n)$. Clearly, $S = G[\![\omega]\!]$ so nothing should be affected.

Presume $u, v \in S$ and wlog that $n \in m$ where $u \in G(n)$ and $v \in G(m)$. By our construction above, $G(n) \subseteq G(m)$ follows from $n \in m$. That is, $u, v \in G(m)$. So, $\{u, v\}$ which exists by the Pairing Axiom is a subset of $G(m)$, and hence in $G(m^+)$.

$\square$

10. Let us define the function G of domain ω using recursion on ω by

$$G(0) = \omega,$$
$$G(n^+) = G(n) \cup \mathcal{P}G(n).$$

Now, we have that $S := G[\![\omega]\!]$ is certainly $\omega \cup \mathcal{P}\omega \cup \mathcal{P}(\omega \cup \mathcal{P}\omega) \cup \cdots$. It remains to verify $S = \omega \cup \mathcal{P}\omega \cup \mathcal{P}\mathcal{P}\omega \cup \cdots$ before proving it models the Zermelo Axioms. We shall do this with Lemma 10A below:

Lemma 10A
If $n \in m \in \omega$, then $G(n) \subseteq G(m)$.

Proof
Suppose T is the set of all $m \in \omega$ with $G(n) \subseteq \mathcal{P}G(m)$ for any natural number $n \in m$. Every natural number is a subset containing natural numbers, and hence a subset of ω, i.e. in $\mathcal{P}G(0)$. Therefore, $0 \in T$. Consider $m \in T$ now and any $x \in G(m^+)$. Since $m \in T$, we can safely assert that $x \subseteq G(m) \subseteq \mathcal{P}G(m) = G(m^+)$. So, $x \subseteq G(m^+)$ and must be in $\mathcal{P}G(m^+)$. This suffices to show $m^+ \in T$. Thus, $T = \omega$ by induction.

As such, we can rewrite the mapping of G equivalently as $G(0) = \omega$, $G(n^+) = \mathcal{P}G(n)$. Which informally translates to $\omega \cup \mathcal{P}\omega \cup \mathcal{P}\mathcal{P}\omega \cup \cdots$ (as desired)

Onto showing S models the Zermelo Axioms!

Extensionality

From the proof of Theorem 9F, it suffices to show S is transitive. Take any $x \in y \in S$. And $x \in y \in G(n)$ for some $n \in \omega$ by construction of G. By the previous lemma, $y \in G(n^+)$ so $x \in y \subseteq G(n)$. As $x \in G(n)$, it follows that $x \in S$. In other words, S is indeed transitive.

Empty Set

Clearly, $\varnothing := 0 \in \omega \subseteq S$.

Pairing

Assume $u, v \in S$ and wlog that $n \in m$ where $u \in G(n)$ and $v \in G(m)$. We note from the previous lemma that $u \in G(m)$ also. Hence, the subset $\{u, v\}$ of $G(m)$, that exists by the pairing axiom / a subset axiom is also in $G(m^+) \subseteq S$.

Union

Let $X \in S$. That is, $X \in G(n)$ for some least $n \in \omega$. Accordingly, consider any $y \in \bigcup X$ whose existance is guarenteed by the Union Axiom, i.e. $y \in x \in X$ for some x. Again with Lemma 10A we see that $X \subseteq G(n)$, telling us $x \in G(n)$. Repeating this, we have that $x \subseteq G(n)$ and then, $y \in G(n)$. In other words, $\bigcup X \subseteq G(n)$ and $\bigcup X \in G(n^+)$.

Power set

Consider any $x \in G(n)$ for some $n \in \omega$. Once more, $x \subseteq G(n)$ and as a result every subset of x is a subset of $G(n)$. Thence is in $G(n^+)$. Now, the powerset $\mathcal{P}x$ which exists by the Powerset Axiom / a subset axiom must be in $G(n^{++})$ (since it is a subset of $G(n^+)$).

Infinity

$\omega \in \mathcal{P}\omega = G(1) \subseteq S$.

Choice

Presume R is any relation in S. By AC there exists a subfunction F of R with $\operatorname{dom} F = \operatorname{dom} R$. As before, we know $R \subseteq G(n)$ for some $n \in \omega$, thus $F \subseteq G(n)$ as well. Then, $F \in G(n^+) \subseteq S$.

Wherefore, S is indeed a model of the Zermelo axioms.

11.

Ideas

κ must be a limit ordinal since it is an infinite cardinal number. This tells us that $\beth_\kappa = \bigcup_{\alpha\in\kappa} \beth_\alpha$.

For any $\alpha \in \kappa$, $\alpha \in \alpha^+ \in \kappa$ & $\alpha \in \alpha^+ \subseteq \beth_{\alpha^+}$. So, $\kappa \subseteq / \subseteq \beth_\kappa$.

Assume that for any $\beta \in \alpha \in \kappa$ we have $\beth_\beta \in \kappa$.

Case 1 $\alpha = 0$, then $\beth_\alpha = \aleph_0 \in \kappa$ by defn

Case 2 $\alpha = \beta^+$ for some ordinal $\beta \in \alpha$. By assumption we know $\beth_\beta \in \kappa$ so $\beth_\alpha = 2^{\beth_\beta} \in \kappa$ by condition 2 of defn

Case 3 α is a limit ordinal. So, $\beta \in \beth_\alpha$ is such that $\beta \in \underset{\in \kappa}{\beth_\gamma}$ for some ordinal $\gamma \in \alpha$. Thus, $\beta \in \kappa$ which tells us $\beth_\alpha \subseteq \kappa$.

In fact, $\beth_\alpha \subset \kappa$ as the set of ordinals less than κ, $\{\beth_\gamma \mid \gamma \in \alpha\}$ has cardinality less than κ (as $\alpha \in \kappa$). It follows that $\beth_\alpha \in \kappa$.

Regardless, we have shown that $\beth_\alpha \in \kappa$. By transfinite induction, this holds for any ordinal $\alpha \in \kappa$.

Now, we see that $\beth_\kappa \subseteq \kappa$ because $\alpha \in \beth_\beta \in \kappa$ implies $\alpha \in \kappa$. Consequently, $\beth_\kappa = \kappa$.

Similarly, notice that $\kappa \subseteq V_\kappa$, since for any $\alpha \in \kappa$, $\alpha \in \alpha^+ \subseteq V_{\alpha^+} \subseteq V_\kappa$. Hence, $V_\kappa \geq \kappa$.

Suppose that for all $\beta \in \alpha \in \kappa$, card $V_\beta < \kappa$. As before, there are 3 cases to evaluate.

Case 1 $\alpha = 0$, it follows that card V_α = card $\varnothing = 0 < \aleph_0 \leq \kappa$.

Case 2 $\alpha = \beta^+$ for some ordinal $\beta \in \alpha$. Then card V_α = card $\mathcal{P} V_\beta = 2^{\text{card } V_\beta} < \kappa$ since card $V_\beta < \kappa$ by assumption and using part (b) of the definition of inaccessible cardinals.

Case 3 α is a limit ordinal. Since $\alpha \in \kappa$, card $\alpha < \kappa$. Thus, card $V_\alpha \leq$ (card α) $\cdot\, \kappa = \kappa$ from exercise 26 of Chapter 6 and the Absorption Law of Cardinal Arithmetic. And further, equality mustn't hold, lest part (c) of κ's definition is violated.

In any case, card $V_\alpha < \kappa$ is guaranteed. Hence by transfinite induction, this is true of all ordinals α. Now, card $V_\kappa \leq$ (card κ) $\cdot\, \kappa = \kappa$. Combined with the fact we established earlier, that card $V_\kappa \geq \kappa$, we can be certain that card $V_\kappa = \kappa$.

Wherefore, as long as κ is an inaccessible cardinal, $\beth_\kappa = \kappa$ and card $V_\kappa = \kappa$ indeed. $\square$

Self - Proof of Theorem 9M

Assume, for the sake of contradiction, that $\text{cf}\,\aleph_{\alpha+1} < \aleph_{\alpha+1}$ for some ordinal number α. In other words, there exists a subset S of $\aleph_{\alpha+1}$ with $\sup S = \aleph_{\alpha+1}$ but $\text{card}\, S \leq \aleph_\alpha$. Hence, suppose $f : S \to \aleph_\alpha$ provides this injection. By the definition of $\aleph_{\alpha+1}$, every $\beta \in S$ must have cardinality strictly less than $\aleph_{\alpha+1}$, that is, $\text{card}\,\beta \leq \aleph_\alpha$ too. Therefore, $\text{card} \bigcup S \leq (\text{card}\, S) \cdot \aleph_\alpha \leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ by exercise 26 of chapter 6 and the Absorption Law of Cardinal Arithmetic. However, $\text{card} \bigcup S = \aleph_{\alpha+1} > \aleph_\alpha$ by assumption, a contradiction. $\square$

Self - Proof of Theorem 9N

$\text{card}\, S = \text{cf}\,\lambda$ & $\sup S = \lambda$

Replacement axiom : Let $\alpha \in S$.

There exists $A_\alpha := \{\aleph_\beta \mid \beta \in \alpha\}$ $\quad$ $\sup A_\alpha = \aleph_\alpha$ if $\alpha \neq 0$

There exists $T := \{\sup A_\alpha \mid \alpha \in S\}$

$\sup T = \aleph_\lambda$? $\quad$ Suppose $\gamma \in \aleph_\lambda = \sup\{\aleph_\beta \mid \beta \in \lambda\}$

$\leq$ trivial $\quad$ i.e. $\gamma = \aleph_\beta$ for some $\beta \in \lambda$

$\beta \in \alpha \in S$

$\aleph_\beta \in A_\alpha$

$\aleph_\beta \in \sup T$

$\Rightarrow \sup T = \aleph_\lambda$

Assume that $S \subseteq \lambda$ with $\sup S = \lambda$ and $\text{card}\, S = \text{cf}\,\lambda$. By two replacement axioms we can construct, firstly, $A_\alpha := \{\aleph_\beta \mid \beta \in \alpha\}$ for each $\alpha \in S$ and also $T := \{A_\alpha \mid \alpha \in S\}$. Clearly, $\sup T \subseteq \aleph_\lambda$ because for $\aleph_\beta \in A_\alpha$ for some $\beta \in \alpha \in S$, $\beta \in \lambda$ since $S \subseteq \lambda$ by definition, hence $\aleph_\beta \in \aleph_\lambda$ by the normality of the aleph operation. For the converse, suppose $\gamma \in \aleph_\lambda$. So, $\gamma = \aleph_\beta$ for some $\beta \in \lambda (= \sup S)$ by continuity. That is, $\beta \in \alpha \in S$ for some α. Thus, $\gamma = \aleph_\beta \in A_\alpha$ and therefore $\gamma = \aleph_\beta \in \sup T$. We have shown $\aleph_\lambda \subseteq \sup T$ too. Now, $\aleph_\lambda = \sup T$ where $\text{card}\, T = \text{card}\, S = \text{cf}\,\lambda$. Consequently, $\text{cf}\,\aleph_\lambda \leq \text{cf}\,\lambda$. Repeating a similar procedure on the set $S' \subseteq \aleph_\lambda$ with $\sup S' = \aleph_\lambda$ and $\text{card}\, S' = \text{cf}\,\aleph_\lambda$, we can arrive at the conclusion that $\text{cf}\,\lambda \leq \text{cf}\,\aleph_\lambda$ as well. Wherefore, $\text{cf}\,\aleph_\lambda = \text{cf}\,\lambda$. $\square$

Self-Proof of Lemma 9P

By transfinite recursion define the α-sequence h with $h(\delta)$ being $f(\gamma)$ where $\gamma \in \alpha$ is the least ordinal so $h(\eta) \in f(\gamma)$ for all $\eta \in \delta$, and if such an ordinal does not exist, $h(\delta) :=$ some extraneous set e. Let β be the least $\delta \in \alpha$ with $h(\delta) = e$ or α if $e \notin \operatorname{ran} f$. Presume that for each $\eta \in \delta \in \beta$, $f(\eta) \subseteq h(\eta)$. If $\delta = 0$, it is easy to see that $f(\delta) = h(\delta)$. And when δ is a successor or limit ordinal, $f(\delta) \subseteq h(\delta)$ lest $h(\delta) \in f(\delta)$, meaning $h(\delta) = f(\eta)$ for some $\eta \in \delta$. But then $f(\eta) \subseteq h(\eta) \in f(\eta)$ by our presumption, a contradiction. Thus, it is certain that $f(\delta) \subseteq h(\delta)$.

By transfinite induction, $f(\delta) \subseteq h(\delta)$ for every $\delta \in \beta$. Hence, $\sup \operatorname{ran} f = \sup \operatorname{ran} h$ because if $\zeta \in f(\delta)$ for some $\delta \in \alpha$, then $\zeta \in h(\eta)$ for some $\eta \in \delta$. (The converse is trivial)

Now, the β-sequence $g := h \restriction \beta$ is our desired increasing function. □

Self Proof of Theorem 9Q

Let S be the subset of λ with $\sup S = \lambda$ and $\operatorname{card} S = \mathrm{cf}\,\lambda$, with the $(\mathrm{cf}\,\lambda)$-sequence $f : \mathrm{cf}\,\lambda \to S$ providing this bijection. From Lemma 8P, there exists an increasing β-sequence g for some $\beta \subseteq \mathrm{cf}\,\lambda$ so $\sup \operatorname{ran} g = \sup \operatorname{ran} f = \sup S = \lambda$. Now, $\mathrm{cf}\,\lambda = \beta$ is guaranteed because otherwise it contradicts the leastness of $\mathrm{cf}\,\lambda$. □

Self-Proof of Theorem 8S

Ideas

$S \subseteq \lambda$ with $\sup S = \lambda$ & $\operatorname{card} S = \mathrm{cf}\,\lambda$, increasing $f : \mathrm{cf}\,\lambda \to S$ $\sup \operatorname{ran} f = \sup S$

$T \subseteq \lambda$ with $\sup S = \mathrm{cf}\,\lambda$ & $\operatorname{card} T = \mathrm{cf}\,\mathrm{cf}\,\lambda$ increasing $g : \mathrm{cf}\,\mathrm{cf}\,\lambda \to T$ $\sup \operatorname{ran} g = \sup T = \mathrm{cf}\,\lambda$

$\sup f[\![\operatorname{ran} g]\!]$ ($\operatorname{ran}(f \circ g)$) $= \sup \operatorname{ran} f = \lambda$ no equality as $\mathrm{cf}\,\lambda$ is bound to be an infinite cardinal

lest exists $\alpha \in f(\beta)$ for some $\beta \in \mathrm{cf}\,\lambda$ with $f(g(\gamma)) \in f(\beta)$ for each $\gamma \in \operatorname{dom} g$ / $\mathrm{cf}\,\mathrm{cf}\,\lambda$ so that

$g(\gamma) \in \beta$ for all $\gamma \in \mathrm{cf}\,\mathrm{cf}\,\lambda$. But then $\beta \notin \sup \operatorname{ran} g$, and as such, $\sup \operatorname{ran} g \neq \mathrm{cf}\,\lambda$, a contradi[illegible]

$f \circ g : \mathrm{cf}\,\mathrm{cf}\,\lambda \to S$ is s.t. $\sup \operatorname{ran}(f \circ g) = \lambda$

i.e. $\mathrm{cf}\,\lambda = \mathrm{cf}\,\mathrm{cf}\,\lambda$

Self-Proof of Theorem 9S

Proof

Let f and g be an increasing $(\operatorname{cf}\lambda)$-sequence converging to λ and an increasing $(\operatorname{cf}\operatorname{cf}\lambda)$-sequence converging to $\operatorname{cf}\lambda$ respectively. We claim that $\sup \operatorname{ran}(f\circ g) = \sup \operatorname{ran} f = \lambda$, lest there exists some ordinal $\alpha \in f(\beta)$ for some $\beta \in \operatorname{cf}\lambda$ so $f(g(\gamma)) \subseteq \beta$ for each $\gamma \in \operatorname{cf}\operatorname{cf}\lambda$. Furthermore, equality mustn't hold for any such γ because then $\beta \in f(g(\gamma^+))$. Therefore, $g(\gamma) \in \beta$ given $\gamma \in \operatorname{cf}\operatorname{cf}\lambda$. But then $\beta \notin \sup \operatorname{ran} g$. As such, $\sup \operatorname{ran} g \neq \operatorname{cf}\lambda$, a contradiction. Indeed, it must be that $\sup \operatorname{ran}(f\circ g) = \sup \operatorname{ran} f = \lambda$. Now, $\operatorname{ran}(f\circ g) \subsetneq \lambda$ with supremum λ and $\operatorname{cf}\operatorname{cf}\lambda \geq \operatorname{cf}\lambda$. Consequently, $\operatorname{cf}\lambda = \operatorname{cf}\operatorname{cf}\lambda$ must hold. $\square$

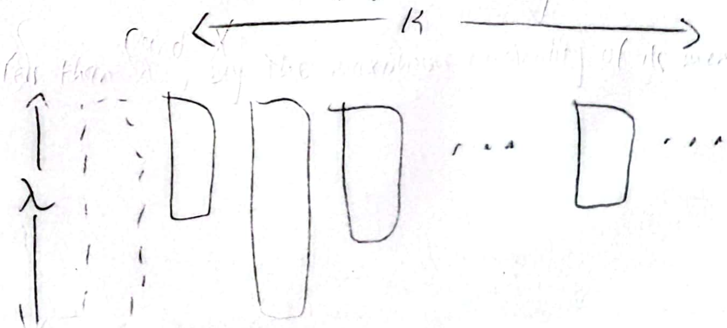
Self-Proof of Theorem 9T

Ideas

Let S be a set of cardinality κ with $\bigcup S = \lambda$ and all its members having cardinality less than λ. Define the function $F: S \to \lambda$ by $F(X) = (\sup X)^+$. Notice that X must be a subset of λ with cardinality less than λ. Accordingly,

$\lambda = \aleph_\mu$ for some limit ordinal μ

$\operatorname{cf}\lambda = \operatorname{cf}\mu$



By the well-ordering theorem, there exists some well-order $<$ on S. For each $X \in S$, let α_X be the ordinal number of $\langle X, <\rangle$. As $\operatorname{card} X < \lambda$, $\alpha_X \in \lambda$. By transfinite recursion, define the function F given by

$F(X) = (\sup F[\![\operatorname{seg} X]\!]) + \alpha_X$

Self Proof of Theorem 6T (=US)

Ideas

Define the lexicographical well-ordering $\trianglelefteq$ on λ by

$\alpha_X \trianglelefteq \beta_Y$ iff $X < Y$ or ($X = Y$ & $\alpha_X \in \beta_Y$)

Assume wlog each $X \in S$ is pairwise disjoint and nonempty

$X \subseteq \lambda$

We know there is the usual function E with domain λ given by $E(\alpha) = E[\![\operatorname{seg}\alpha]\!]$. (or we can also define $E(\alpha)$ as the least ordinal not in $E[\![\operatorname{seg}\alpha]\!]$

~~Suppose for any $\beta \in \alpha$, $E(\beta) \in \lambda$~~

~~Case 1 $\alpha = 0$ $E(\alpha) = 0 \in \lambda$~~

~~Case 2 $\alpha = \beta^+$ for some $\beta \in \alpha$,~~

~~Case 3 α is a limit ordinal.~~ ~~Presume~~ ftsoc that $\lambda = E(\alpha)$.

$E(\alpha) = E[\![\operatorname{seg}\alpha]\!]$

$\alpha \in \lambda$

$\operatorname{card}(\operatorname{seg}\alpha) < \lambda$

$\operatorname{card}(\operatorname{seg}\alpha) = \operatorname{card} E(\alpha) < \lambda$

$E(\alpha) < \lambda$

$E(\alpha) \in \lambda$

$\Longrightarrow E(\alpha) \in \lambda$ for each $\alpha \in \lambda$ (dom E)

$E[\![\lambda]\!] \subseteq \lambda$

$\lambda \preccurlyeq E[\![\lambda]\!]$

$\lambda \subseteq E[\![\lambda]\!]$

$\Longrightarrow \lambda = E[\![\lambda]\!]$

E^{-1}: some ordinal $\operatorname{ran} E \xrightarrow{E[\![\lambda]\!]} \lambda$

If there exists a largest $L \in S$,

$\operatorname{card}\left(E[\![L]\!] \cup \bigcup_{Y<L} E[\![Y]\!]\right)$

$= \operatorname{card} E[\![L]\!] + \mu$ where $\mu < \lambda$ as $\bigcup_{Y<L} E[\![Y]\!] < \lambda$

$E[\![L]\!] \approx L \approx \lambda$

a contradiction.

Define the function $G : S \to \lambda$ by $G(X) = \bigcup_{Y \leq X} E[\![Y]\!]$

If $X \neq X'$, $X < X'$ so there exists some $\alpha \in X'$ with $\alpha \notin X$. Accordingly, $E(\alpha) \in G(X')$ but $E(\alpha) \notin G(X)$. Hence, $G(X) \neq G(X')$

Clearly, sup ran G = ran $E = \lambda$ since for each $E(\alpha)$, $\alpha \in X$ for some $X \in S$.

sup ran G

Proof See PDF

Self-Proof of Konig's Theorem

$\mathrm{cf}\, 2^\kappa \approx S \quad (S \subseteq 2^\kappa \text{ \& } \sup S = 2^\kappa)$

E.g. $\aleph_0 < \mathrm{cf}\, \underbrace{2^{\aleph_0}}_{\text{maybe } \aleph_{\aleph_1}}$

Ideas

$S \subseteq 2^\kappa$, $\sup S = 2^\kappa$, then

If $\mathrm{card}\, 2^\kappa < \kappa$

$\kappa < \mathrm{cf}\, 2^\kappa = \mathrm{cf}\, \mathrm{card}\, \mathcal{P}\kappa$

$= \aleph_\mu \;\; \mathrm{cf}\, \aleph_\alpha$ for some $\alpha \ni \mu$

Show α cannot be a limit ordinal with cofinality less than or equal to $\aleph_\mu$.

i.e. if α is a limit ordinal λ, $\aleph_\mu < \mathrm{cf}\, \lambda$

Assume ftsoc that α is some limit ordinal λ but $\aleph_\mu \geq \mathrm{cf}\, \lambda \;\; (\geq \omega)$ $\quad \mathrm{cf}\, \aleph_\lambda$

Then $\mathrm{cf}\, \lambda \in \aleph_\mu$

14. Ideas

card $\{x \mid xRtR_ny$ for some $t \in \operatorname{fld} R\}$ card $\{t \mid tR_n y\} \underset{=\mu}{<} \kappa$

$\operatorname{cf} \kappa = \kappa$

Proof

Let S be the set of all natural numbers n with card $\{x \mid xR_n y\} < \kappa$. When $n=0$, this follows immediately from our assumption. So suppose $n \in S$. Then card $\{t \mid tR_n y\} = \mu$ for some cardinal $\mu < \kappa$. And again by assumption, card $\{x \mid xRt\} = \lambda_t$ for some cardinal $\lambda_t < \kappa$ and any t. Notice that $\lambda := \sup\{\lambda_t \mid tR_n y\}$ must be less than κ because card $\{\lambda_t \mid tR_n y\} = \mu < \kappa = \operatorname{cf} \kappa$. Therefore, card $\{x \mid xR_{n^+} y\} = $ card $\bigcup_{t \mid tR_n y} \{x \mid xRt\} \leq \mu \cdot \lambda < \kappa$. Consequently, $n^+ \in S$. Now, $S = \omega$ by induction.

Wherefore, card $\{x \mid xR^t y\} = $ card $\bigcup_{n \in \omega} \{x \mid xR_n y\} \leq \aleph_0 \cdot \kappa < \kappa$ if $\kappa > \aleph_0$. The case of $\kappa = \aleph_0$ is trivial since $\{x \mid xRy\}$ is always finite.

15. Ideas

Suppose $\operatorname{cf} \kappa \neq \aleph_\kappa$ for some inaccessible cardinal κ.

$= \kappa$

$= \aleph_\lambda$ for some $\lambda \leq \kappa$

Show $\kappa \geq \aleph_\kappa$ / $\kappa \supseteq \aleph_\kappa$ / Every $\aleph_\lambda \overset{\in}{<} \kappa$ if $\aleph_\lambda < \aleph_\kappa$

$\kappa \not\geq \aleph_\kappa$

i.e. $\lambda < \kappa$

$2^\lambda < \kappa$

Let $\beth^{\aleph_\mu}_\alpha$ be the beth operation but with $\beth^{\aleph_\mu}_0 = \aleph_\mu$ instead of $\aleph_0$. $\beth^{\aleph_\mu}_{\aleph_\mu} \geq \aleph_{\aleph_\mu}$

$:= \bigcup_{\alpha \in \aleph_\mu} \beth^{\aleph_\mu}_\alpha$

$2^{\aleph_\mu} \geq \aleph_{\mu+1}$

$2^{2^{\aleph_\mu}} \geq \aleph_{\mu+2}$

Assume that for any $\beta \in \alpha$, $\beth^{\aleph_\mu}_\beta \geq \aleph_{\mu+\beta}$.

$\lambda = \aleph_\mu$ for some $\mu \leq \lambda$

Case 1 $\alpha = 0$: $\aleph_\mu = \aleph_\mu$.

Case 2 $\alpha = \beta^+$ for some $\beta \in \alpha$: $\beth^{\aleph_\mu}_\alpha := 2^{\beth^{\aleph_\mu}_\beta} > \beth^{\aleph_\mu}_\beta \geq \aleph_{\mu+\beta}$. So, $\beth^{\aleph_\mu}_\alpha \geq \aleph_{(\mu+\beta)^+} = \aleph_{\mu+\alpha}$.

Case 3 α is a limit ordinal: $\beth^{\aleph_\mu}_\alpha := \bigcup_{\beta \in \alpha} \beth^{\aleph_\mu}_\beta \geq \bigcup_{\beta \in \alpha} \aleph_{\mu+\beta} \overset{\text{Thm Schema 8E as } \aleph \text{ is normal}}{=} \aleph_{\sup\{\mu+\beta \mid \beta \in \alpha\}} = \aleph_{\mu+\alpha}$.

since $\beth^{\aleph_\mu}_\beta \geq \aleph_{\mu+\beta}$ for each $\beta \in \alpha$

5. Assume that κ is an inaccessible cardinal. We know $\kappa \leq \aleph_\kappa$ so it suffices to show $\kappa \geq \aleph_\kappa$. To do this, first let $\beth_\alpha^{\aleph_\mu}$ be the typical beth operation but define $\beth_0^{\aleph_\mu} := \aleph_\mu$ instead of $\aleph_0$.

Suppose that for any $\beta \in \alpha$, $\beth_\beta^{\aleph_\mu} \geq \aleph_{\mu+\beta}$.

Case 1 $\alpha = 0$: $\beth_0^{\aleph_\mu} := \aleph_\mu \geq \aleph_\mu$,

Case 2 $\alpha = \beta^+$ for some ordinal $\beta \in \alpha$: $\beth_\alpha^{\aleph_\mu} := 2^{\beth_\beta^{\aleph_\mu}} > \beth_\beta^{\aleph_\mu} \geq \aleph_{\mu+\beta}$. So, $\beth_\alpha^{\aleph_\mu} \geq \aleph_{\mu+\alpha}$.

Case 3 α is a limit ordinal: $\beth_\alpha^{\aleph_\mu} := \sup\{\beth_\beta^{\aleph_\mu} \mid \beta \in \alpha\} \overset{\text{by assumption}}{\geq} \sup\{\aleph_{\mu+\beta} \mid \beta \in \alpha\} \overset{\text{Thm Schema 8E}}{=} \aleph_{\sup\{\mu+\beta \mid \beta \in \alpha\}} \overset{\text{Normality of +}}{=} \aleph_{\mu+\alpha}$.

In any case, $\beth_\alpha^{\aleph_\mu} \geq \aleph_{\mu+\alpha}$. Hence, this is true of every ordinal α by transfinite induction.

Now, select any cardinal $\aleph_\lambda < \aleph_\kappa$ for some ordinal $\lambda \in \kappa$. Consider λ being infinite, Let $\aleph_\mu$ and $\aleph_{\mu'}$ be the least cardinals greater than or equal to and less than or equals to λ respectively. If λ is finite, choose $\beth_0^{\aleph_0}$. By inaccessibility, $\aleph_\mu \leq 2^{\aleph_{\mu'}} < \kappa$. And, $\kappa > \beth_{\aleph_\mu}^{\aleph_\mu} \geq \aleph_{\mu+\aleph_\mu} \overset{\text{Thm 8Q(a)}}{\geq} \aleph_{0+\aleph_\mu} = \aleph_{\aleph_\mu} \overset{\text{Normality}}{\geq} \aleph_\lambda$. Since $\kappa > \aleph_\lambda$ for every cardinal $\aleph_\lambda < \aleph_\kappa$, $\kappa \geq \aleph_\kappa$. Consequently, $\kappa = \aleph_\kappa$. Wherefore, $\operatorname{cf} \aleph_\kappa = \operatorname{cf} \kappa = \kappa = \aleph_\kappa$ and κ is indeed weakly inaccessible. $\square$

16. Ideas

$\lambda \neq \aleph_0$ because $cf\ \aleph_\omega = cf\ \omega = \omega \neq \aleph_\omega$.

1. $\lambda > \aleph_0$ ✓

Say $\mu < \lambda$, if μ finite, trivial. If $\mu = \aleph_{\mu'}$ for some $\mu' \in \mu \in \lambda$, then $2^\mu \overset{GCH}{=} \aleph_{\mu'+1} < \aleph_\lambda = \lambda$

2. $\mu < \lambda \Rightarrow 2^\mu < \lambda$ ✓

3. By defn of weak inaccessibility, $cf\ \lambda = \lambda$

Proof

(a) $\lambda \neq \aleph_0$ because $cf\ \aleph_\omega = cf\ \omega = \omega \neq \aleph_\omega$.

(b) Let μ be a cardinal less than λ. If μ is finite, 2^μ must be finite and thus also less than λ. So, consider $\mu = \aleph_{\mu'}$ for some ordinal $\mu' \in \mu \in \lambda$. Then $2^\mu \overset{GCH}{=} \aleph_{\mu'+1} < \aleph_\lambda = \lambda$ as λ is a limit ordinal. Therefore, it is guarenteed that $2^\mu < \lambda$.

(c) By the definition of weak inaccessibility, $cf\ \lambda = \lambda$ (which also guareentees λ is a cardinal).

Wherefore, λ is indeed an inaccessible cardinal under the assumption of GCH.

17. Ideas

card $B_i > 0$ given $i \in I$

$A_i \overset{\mathcal{I}}{\leq} B_i$

$\bigcup_{i \in I} A_i \overset{\mathcal{I}'}{\leq} \bigcup_{i \in I} B_i$

$B_0 \{0,1\}$ $\{\langle 0,0\rangle, \langle 1,2\rangle, \cancel{\langle 2,4\rangle}\}$ $\{\langle 0,1\rangle, \langle 1,3\rangle, \cancel{\langle 2,5\rangle}\}$

$B_1 \{2,3\}$ $\{\langle 0,0\rangle, \langle 1,3\rangle, \cancel{\langle 2,4\rangle}\}$ $\{\langle 0,1\rangle, \langle 1,2\rangle, \cancel{\langle 2,5\rangle}\}$

$\cancel{B_2 = \{4,5,6\}}$

$\{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,4\rangle\}$ $\{\langle 0,0\rangle, \langle 1,2\rangle, \langle 2,5\rangle\}$

$\{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,6\rangle\}$

$G : \bigcup_{i \in I} B_i \longrightarrow \bigtimes_{i \in I} B_i$

let l be the least element of I and L be the largest, if it exists

By AC, there exists the well-orders $\sqsubset$ and $<_i$ on I and B_i respectively.

wlog each B_i is disjoint & each B_i has at least 2 elements

Define the lexicographic ordering L by

$b_i L b_j$ iff $i \sqsubset j$ or ($i = j$ and $b_i <_i b_j$)

By AC, select two distinct elements, say a and b, from each B_i and form the bijective map $S_i : 2 \longrightarrow \{a, b\}$

to the set of these elements.

~~By TR~~, define the function G of domain $\bigcup_{i \in I} B_i$ with

$$[G(b_i)](j) = \begin{cases} b_i & \text{if } i = j, \\ S_j(0) & \text{if } i = l \text{ and } i \neq j, \text{ or } i \neq l,\ i \neq L \text{ and } i^+ \neq j, \cancel{\text{or } i = L \text{ and } (j \neq l \text{ or } b_i \neq S_i(0))} \\ S_j(1) & \text{if } i \neq l,\ i \neq L, \text{ and } i^+ = j, \text{ or } i = L \text{ and } j = l. \cancel{\text{ and } b_i = S_i(0)} \end{cases}$$

Suppose $G(b_i) = G(b_k)$

$[G(b_i)](j) = [G(b_k)](j)$

$b_i = b_k$ $\quad S_j(0) = S_j(0)$

$i = l$ & $i \neq j$ / $i \neq l$ & $i^+ \neq j$.

$k = l$ & $k \neq j$ / $k \neq l$ & $k^+ \neq j$.

1 / · Ideas

Let α be the ordinal number of I, which exists by WT.

Assume $\operatorname{card} \bigcup_{i \in \beta} A_i < \operatorname{card} \bigtimes_{i \in \beta} B_i$ for all $\beta \in \alpha$;

Case 1: $\alpha = 0$: $\operatorname{card} \bigcup_{i \in \alpha} A_i = 0 < 1 = \operatorname{card} \bigtimes_{i \in \alpha} B_i$.

Case 2 $\alpha = \beta^+$ for some ordinal $\beta \in \alpha$: $\operatorname{card} \bigcup_{i \in \alpha} A_i = \operatorname{card} \bigcup_{i \in \beta} A_i + \operatorname{card} A_\beta < \cdots$

- Finite trivial
- Infinite ⇒ Just apply the Absorption Law of Cardinal Arithmetic

Case 3 α is a limit ordinal :

$$\bigtimes_{i \in \alpha} B_i = \bigcup_{\beta \in \alpha} \{ f : \alpha \to B \mid f(i) \in B_i \ \& \text{ if } \gamma \ni \beta \text{ then } f(\gamma) = c(B_\gamma) \}$$

18. It is not monotone because, for example, $\aleph_1 < \aleph_\omega$ but $\mathrm{cf}\,\aleph_1 = \aleph_1$ [illegible] $\aleph_\omega$

Also, it is also discontinuous since we have that although $\aleph_1 \in \aleph_2 = \mathrm{cf}\,\aleph_2$ where $\aleph_2 \in \aleph_\omega$, $\aleph_1 \notin \omega = \mathrm{cf}\,\aleph_\omega$.

Accordingly, it mustn't be normal too.

19. Ideas

$$\operatorname{rank} S = \underbrace{\bigcup\{\overbrace{(\operatorname{rank} t)^+}^{\in \kappa} \mid t \in S\}}_{\operatorname{card} S < \kappa}$$

So, $\operatorname{rank} S \in \kappa$.

Proof

Notice that for $t \in S$ we have $\operatorname{rank} t \in \kappa$ and hence $(\operatorname{rank} t)^+ \in \kappa$ as κ is a limit ordinal. Therefore, $\operatorname{rank} S = \bigcup\{(\operatorname{rank} t)^+ \mid t \in S\}$ is the union of a set of ordinals smaller than κ, with cardinality at most $\operatorname{card} S < \kappa$. As κ is regular, this means $\operatorname{rank} S \in \kappa$.

Accordingly, $(\operatorname{rank} S)^+ \in \kappa$ and $S \in V_\kappa$. $\square$

Proof sketch

20. A similar procedure as before. Namely, say $S \subseteq \lambda$ with $\sup S = \lambda$ and $\operatorname{card} S = \mathrm{cf}\,\lambda$, and $T := \{t_\alpha \mid \alpha \in S\}$. Then for each $\beta \in \lambda$: there exists an $\alpha \in S$ with $t_\beta \subseteq t_\alpha$ by monotonicity. Thus, $\sup\{t_\beta \mid \beta \in \lambda\} \subseteq \sup T$. And $T \subseteq \{t_\beta \mid \beta \in \lambda\}$ follows by definition meaning $\sup T \subseteq \sup\{t_\beta \mid \beta \in \lambda\}$. Hence, $\sup T = \sup\{t_\beta \mid \beta \in \lambda\} = t_\lambda$ by continuity. Consequently, $\mathrm{cf}\,\lambda \geq \mathrm{cf}\,t_\lambda$. By repeating a similar (but not identical procedure), we can arrive at $\mathrm{cf}\,\lambda \leq \mathrm{cf}\,t_\lambda$ too. Wherefore, we can conclude that $\mathrm{cf}\,\lambda = \mathrm{cf}\,t_\lambda$.

$\square$

17. Ideas

Assume that when $b_\alpha < b_\beta$, $G(b_\alpha) \neq G(b_\beta)$ for all b

Assume that $b_i \neq b_k$

If $i = l$, i.e. $b_i < b_k$:

i) $k \neq L$: $[G(b_i)](k^+) = S_{k^+}(0) + S_{k^+}(1) = [G(b_k)](k^+)$

ii) $k = L$: $[G(b_i)](i) = b_i \neq b_k = [$

Proof

We first show that no surjections can exist. Suppose, for the sake of contradiction, that there exists some surjective function f from $\bigcup_{i \in I} A_i$ to $\times_{i \in I} B_i$. Then for any $i \in I$, the set $\mathcal{B}_i := B_i - \{[f(a)](i) \mid i \in I \text{ \& } a \in A_i\}$ must be nonempty since card $A_i <$ card B_i. By AC we can define the function g of domain I given by $g(i) \in \mathcal{B}_i$. Now, $g \in \times_{i \in I} B_i$ is certain. Furthermore, $g \notin$ ran f lest $g(i) \notin \mathcal{B}_i$, a contradiction. Consequently, this contradicts the supposed surjectivity of f.

From cardinal comparability, either there exists an injection $f: \bigcup_{i \in I} A_i \longrightarrow \times_{i \in I} B_i$ or an injection $f: \times_{i \in I} B_i \longrightarrow \bigcup_{i \in I} A_i$. The latter mustn't hold, otherwise a surjection from $\bigcup_{i \in I} A_i$ to $\times_{i \in I} B_i$ exists, which has been shown to be impossible. Thus, only the former must be true. As such, card $\bigcup_{i \in I} A_i <$ card $\times_{i \in I} B_i$ is guaranteed. $\square$