Name Yu Jhao Hong (39) Date

Parent's Signature

Subject Class 4-I

Nothing to be written here

5.

$$(a_1, a_2) + (0,0) = (a_1, a_2)$$
$$t[(a_1, a_2) + (0,0)] = t(a_1, a_2)$$
$$t(a_1, a_2) + t(0,0) = t(a_1, a_2)$$

$$(t+0)(a_1, a_2) = t(a_1, a_2) + 0(a_1, a_2)$$
$$t(a_1, a_2) = t(a_1, a_2) + 0(a_1, a_2)$$

Suppose $t(a_1, a_2) \neq (ta_1, ta_2)$, then;

$$0(a_1, a_2) \neq (0a_1, 0a_2) = (0,0)$$
$$\Rightarrow t(a_1, a_2) + 0(a_1, a_2) \neq t(a_1, a_2) + (0,0)$$
$$\Rightarrow (t+0)(a_1, a_2) \neq t(a_1, a_2)$$
$$\Rightarrow$$

By the property of a vector space V over a field F that:
(VS 8) For $a, b \in F$ and $x = (a_1, a_2) \in V$, $(a+b)x = ax + bx$;

Let $t \in F$,

$$\vec{AB} = \vec{DC} \qquad \vec{DA} = \vec{CB}$$
$$(b_1 - a_1, b_2 - a_2) = (c_1 - d_1, c_2 - d_2)$$
$$b_1 - a_1 = c_1 - d_1 \qquad b_2 - a_2 = c_2 - d_2$$
$$b_1 + d_1 = a_1 + c_1 \qquad b_2 + d_2 = a_2 + c_2$$

$$(x+y)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^{2n-i} y^i$$

$\frac{2n}{2} = n$ $\qquad n+1$

$n-1$ in front $\qquad (n+1)-1 = n$

$2n-n = n$ behind $\qquad 2n+1-(n+1) = n$

Total number of terms: $2n+1$

Given any sequence with an odd number of terms, i.e. $2n+1$ number of terms, the middle term is the $(n+1)$th term as the number of terms before and after it are equal, i.e.:

Number of terms before: $(n+1)-1 = n$

Number of terms after: $2n+1-(n+1) = n$

$$\therefore \binom{n+1}{i} x^{2n-(n+1)} y^{n+1} = \binom{n+1}{i} x^{n-1} y^{n+1}$$

$\forall x \forall y \, (x, y \in V \Rightarrow \exists! z (x+y=z)) \iff \forall x \forall y \left(x, y \in V \Rightarrow \left[\exists z (x+y=z) \wedge \forall z_1 \forall z_2 (x+y=z_1 \wedge x+y=z_2 \Rightarrow z_1 = z_2 \right] \right)$

$\exists z \left[x+y=z \wedge \forall \xi (x+y=\xi \Rightarrow z=\xi) \right]$

If $(v_1+W)+(v_2+W) = \overbrace{z_1}^{(a+b)+W}$ and $(v_1+W)+(v_2+W) = \overbrace{z_2}^{(c+d)+W}$,

$= (v_1+v_2)+W$ $\qquad$ $=(v_1+v_2)+W$

$xRy_1 \wedge xRy_2 \Rightarrow y_1 = y_2$

$4(5)12 \wedge 4(5)-2$

$x_1 = x_2 \Rightarrow (x_1Ry_1 \wedge x_2Ry_2 \Rightarrow y_1 = y_2)$

Conclude: $\quad z_1 = z_2$

$(a+b)+W = (c+d)+W$

If $v+W$ is a subspace of V; then $v+W$ is closed under addition (A1) and for all elements of $v+W$, there exists an additive inverse $-(v+W) \in v+W$. Plus, $\vec{0} \in V$;

For $w=\vec{0}$, $v+\vec{0} \in v+W$

$v+(-v)=\vec{0} \in v+W$

If $v \notin W$ and $v \in V$, then $v+w \notin W$

$x+y$ unique

$x_1+y_1=x_2+y_2$

$x_1=x_2$ and $y_1=y_2$

$av_1+W = av_1 + a\vec{?}_1 + av_1 + (-(av_1))$

$= a\vec{?}_1 + av_1$

$= \omega + av_1$

$\{v_1+v_2+w_1+w_2 \mid$

av_1+w_1

$av_1+a\vec{?}_1+av'_1+a(-v_1)$

v_1+W_1 $\quad$ v_2+W_2

v_1+W_2

$v_1-v_1' \in W$ $\quad$ $v_2-v_2' \in W$

$v_1'-v_1 \in W$ $\quad$ $v_2'-v_2 \in W$

$v+(-v)=-v+(-(-v))=\vec{0}$

$-v+(-v)+(-(-v)) = V$

$\vec{0}+(-(-v)) = v$

$v = -(-v)$

$$\begin{aligned} v_2+W &= \{v_2+w \mid v_2 \in V \text{ and } w \in W\} \\ &= \{v_1+\underset{+w}{v_2}+(-v_2) \mid v_1, v_2 \in V \text{ and } w \in W\} \\ &= \{v_1+w \mid v_1 \in V \text{ and } w \in W\} \\ &= v_1+W \end{aligned}$$

$a(v+\frac{1}{a}w)$

$av+w$

$(v_1+v_2)+W = (v_3+v_4)+W$

$= (v_3+W) + (\quad)$

$(v_1+v_2)+(-(v_3+v_4)) \in W$ $\quad$ $(v_1+W)+(v_2+W)$

$\{v_1+w \mid w \in W\} + \{v_2+w \mid w \in W\} = \{v_3+w \mid w \in W\} + \{v_4+w \mid w \in W\}$

$\{v_1+v_2+w \mid w \in W\} = \{v_3+v_4+w \mid w \in W\}$

$v_1+v_2+w_a = v_3+v_4+w_b$

$v_1+v_2)+(v_3+v_4)+(-(v_3+v_4)) + w_a = (v_3+v_4)+(-(v_3+v_4))+(v_1+v_2)+w_b$

$= \underset{\sim}{0}$

$(1+1)(v_1+v_2) + ~~(-(v_3+v_4))~~ = (v_1+v_2)$

What we want: $v_1+W = v_3+W$ and $v_2+W = v_4+W$

OR

$v_1+W = v_4+W$ and $v_2+W = v_3+W$

$(A^t)_{i,j} = A_{j,i}$ $(-A)_{i,j} = -(A_{i,j})$ $((C^t))_{i,j} = C_{j,i} = C_{i,j}$

$A_{j,i} = -(A_{i,j})$

$A_{i,j} + A_{j,i} = 0$

$x \in M_{n\times n}(F) \iff x \in W_1 + W_2 = \{y+z \mid y \in W_1 \text{ and } z \in W_2\}$

Since W_1 and W_2 are subspaces of $M_{n\times n}(F)$, and $M_{n\times n}(F)$ is a vector space, hence closed under addition (A1); $A + C \in M_{n\times n}(F)$.

Conversely, if $X \in M_{n\times n}(F)$, then we can construct an $E \in W_1$ and $F \in W_2$ such that $X = E + F$:

$\{t \mid$ $F(\varphi(x))$

$E_{i,j} = -(E_{i,j})$

$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$E_{1,1}$	e			$F_{1,1}$	f		
$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$-e$	$E_{2,2}$			f	$F_{2,2}$		
$x_{3,1}$	$x_{3,2}$	$x_{3,3}$	$x_{3,4}$			$E_{3,3}$				$F_{3,3}$	
$x_{4,1}$	$x_{4,2}$	$x_{4,3}$	$x_{4,4}$				$E_{4,4}$				$F_{4,4}$

$\operatorname{ran}\varphi = \{t \mid \exists x(x\varphi t)\} = A/\sim$ $\operatorname{dom}\hat{F} = \{t \mid \exists z(t\hat{F}z)\} = A/\sim$

$F^{-1}(b) = \{t \mid b F^{-1} t\} = \{t \mid \beta F^{-1} t\} = F^{-1}(\beta)$

$(\operatorname{ran}\varphi) \cap (\operatorname{dom}\hat{F}) = A/\sim$ $= \{t \mid tFb\} = \{t \mid tF\beta\}$

$\cancel{\{t \mid \exists x(x\varphi t)\} \cap \{t \mid \exists z(z\hat{F}t)\} = A/\sim}$ $tFb = tF\beta$

$\cancel{\{t \mid \exists x \exists z(x\varphi t \wedge z\hat{F}t)\} = A/\sim}$

$\operatorname{dom}(\hat{F}\circ\varphi)$ $= A/\sim$

$x(\hat{F}\circ\varphi)y$ $x\varphi u \wedge u\hat{F}v$

$\operatorname{im}(\hat{F}\circ\varphi) = \{x \mid \exists u \exists v(x\varphi u \wedge u\hat{G}v)\}$ $x \in A \implies \langle x, [x]_\sim\rangle \in \varphi$

$= \{x \mid \exists u(x\varphi u)\} \cap \{x \mid \exists$ $[x]_\sim \in A$

$= \{x \mid \exists$

$= \operatorname{ran}\varphi \cap \operatorname{dom}\hat{F}$

$E_{i,j} = F_{i,j} - X_{j,i}$ $-E_{i,j} + F_{i,j} = X_{j,i} \implies F_{i,j} = E_{i,j} +$

$E_{i,j} = -F_{i,j} + X_{i,j}$ $E_{i,j} + F_{i,j} = X_{i,j} \implies F_{i,j} = X_{i,j}$

${}_{i,j} - X_{j,i} = -F_{i,j} + X_{i,j} \implies E_{i,j} + X_{j,i} = X_{i,j} - E_{i,j}$

$(1+1)F_{i,j} = X_{i,j} + X_{j,i}$ $(1+1)E_{i,j} = X_{i,j} - X_{j,i}$

$F_{i,j} = \frac{X_{i,j} + X_{j,i}}{1+1}$ $E_{i,j} = \frac{X_{i,j} - X_{j,i}}{1+1}$

$V = W_1 \oplus W_2$ iff (W_1 and W_2 are subspaces of V) $W_1 \cap W_2 = \emptyset$ and $W_1 + W_2 = V$

$\Rightarrow$) $\underbrace{V = W_1 \oplus W_2}$ implies that there exists a unique $x_1 \in W_1$ and $x_2 \in W_2$ such that $x_1 + x_2 \in V$:

Immediately means that $W_1 \cap W_2 = \emptyset$ and $W_1 + W_2 = V$

$= \{x_1 + x_2 \mid x_1 \in W_1 \text{ and } x_2 \in W_2\}$

Let $y_1 \in W_1$ and $y_2 \in W_2$; then $y_1 + y_2 \in V$.

$$x_1 + x_2 = y_1 + y_2 \Rightarrow x_1 = y_1 \text{ and } x_2 = y_2$$

1 $x_1 = y_1 + y_2$ X 2 $x_2 = y_1 + y_2$ X 3 $x_1 = y_1$ and $x_2 = y_2$ 4

$\Leftarrow$) Assume every vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$;

Then if $\underbrace{x_1 + x_2 = y_1 + y_2}_{\in V}$, then $x_1 = y_1$ and $x_2 = y_2$.

$z \in W_1$ and $z \in W_2$

$z + x_2 = y_1 + z$

$n!+1$

$$(F^{-1})^{-1} = \{\langle x,y\rangle \mid x(F^{-1})^{-1}y\}$$
$$= \{\langle x,y\rangle \mid yF^{-1}x\}$$
$$= \{\langle x,y\rangle \mid xFy\}$$
$$= F$$

For all x, y:

$$x(F^{-1})^{-1}y \iff yF^{-1}x$$
$$\iff xFy$$

Assume F is single-rooted, i.e. $(x_1Fy \wedge x_2Fy) \Rightarrow x_1 = x_2$

Then $F^{-1} = \{\langle y,x\rangle \mid xFy\}$

then, $(yF^{-1}x_1 \wedge yF^{-1}x_2) \Rightarrow x_1 = x_2$

$$\exists x(\langle [x]_\sim, b_1\rangle \neq \langle x, b_2\rangle)$$
$$[x]_\sim \neq x \text{ OR } b_1 \neq b_2$$

$$F = \{\langle x,y\rangle \mid xFy\}$$
$$= \{\langle x,y\rangle \mid yF^{-1}x\}$$
$$= \{\langle x,y\rangle \mid x(F^{-1})^{-1}y\}$$

$$\operatorname{ran}(\hat{G} \circ \varphi) = \{t \mid \exists u \exists v(u\varphi v \wedge v\hat{G}t)\}$$
$$= \{t \mid \varphi(u) \in \operatorname{dom}\hat{G} \wedge t \in \operatorname{ran}\hat{G}\}$$
$$= \{t \in \operatorname{ran}\hat{G} \mid \varphi(u) \in \operatorname{dom}\hat{G}\}$$
$$= \{t \in \operatorname{ran}\hat{G} \mid \varphi(u) \in A/\sim\}$$
$$= \operatorname{ran}\hat{G} \neq \operatorname{ran} F$$

$$\forall t(t \in e \iff t \in X \wedge x \in X$$

$$\forall x$$

$$\forall x \forall \sim \forall X \exists e \forall t\ [t \in e \iff (t \in X \wedge x \in X \wedge x \sim t)]$$

$A/\sim \; = \{[x]_\sim \mid x \in A\}$

$y \in A/\sim \iff \exists x(x \in A \wedge [x]_\sim = y)$

$\iff \exists x(x \in A \wedge \{t \mid x \sim t\} = y)$

$\iff \exists x \; x \in A \wedge t \in y \iff$

$t \in [x]_\sim \iff \exists t(x \in A \wedge x \sim t)$.

$\exists x(\cancel{x \in A \wedge}[t \in y \iff \exists x(x \in A \wedge x \sim t)])$

$\exists x(x \in A \wedge [t \in y \iff x \sim t])$.

$\langle u, v\rangle \dot{Q} \langle x, y\rangle \iff u + y = x + v$

$\exists x[x \in A \wedge (P(x,t) \iff \exists x[x \in A \wedge Q(x,t)])]$

$\exists x[x \in A \wedge (P(x,t) \iff [x \in A \wedge Q(x,t)])]$

$\exists x[P(x) \wedge (A(x,t) \iff \ldots$

$2v + 2x = 2y + 2u$.

$(u + y) + (2v + 2x) = (x + v) + (2y + 2u)$

$(u + 2v) + (y + 2x) = (x + 2y) + (v + 2u)$

$2x + 2v \qquad y + 2u$

$u + 2v + y + 2x = x + 2y + v + 2u$.

$\forall x(P(x) \iff Q(x))$

$\forall x(P(x)) \iff \forall x(Q(x))$

$\forall t(t \in e \iff t \in A \wedge x \sim t)$

$\langle u + 2v, v + 2u\rangle Q \langle x + 2y, y + 2x\rangle \quad (u + 2v) + (y + 2x) = (x + 2y) + (v + 2u)$

$\forall x([x]_\sim \in A/\sim \iff \varphi(x) = [x]_\sim)$

$\forall y[y \in A/\sim \iff \exists x(\varphi(x) = y)]$

$\forall y(\exists x(\varphi(x) = y = [x]_\sim) \Rightarrow y \in A/\sim)$

$\forall x(\varphi(x) = [x]_\sim \Rightarrow [x]_\sim \in A)$

$\forall x([x]_\sim \in A/\sim \Rightarrow x \in A \Rightarrow \varphi(x) = [x]_\sim)$

$\forall y\left[y \in A/\sim \iff (y \in \mathcal{P}(\operatorname{ran} R) \wedge \forall x[x \in y \iff \exists t(x \in A \wedge x \sim t)])\right]$

$x_1+2x_2-x_3+x_4=5$

$2x_2-2x_3-4x_4=1 \implies x_2-x_3-2x_4=\frac{1}{2}$

$-x_2+x_3+2x_4=-2$

$\mathrm{span}(S_1)+\mathrm{span}(S_2)=\{x+y \mid x\in\mathrm{span}(S_1)$ and $y\in\mathrm{span}(S_2)\}$

$0=-\frac{3}{2}$

$x_1+2x_2+6x_3=-1$

$-3x_2-11x_3=10$

$-5x_2-17x_3=18$

$x_2+4x_3=-4$

$x_1+2x_2+6x_3=-1$

$x_2+4x_3=-4 \implies x_2=-28$

$x_3=6$

$-5x_2-19x_3=18 \implies 26=18$

Not possible

$\implies x_1 \quad = 19$

$a(x^3+x^2+x+1)+b(x^2+x+1)+c(x+1)=2x^3-x^2+x+3$

$=2x^3-x^2+x+3$

$ax^3+(a+b)x^2+(a+b+c)x+a+b+c$

$a=2$

$a+b=-1$

$a+b+c=1$

$a+b+c=3$

$$\begin{pmatrix}1&2\\-3&4\end{pmatrix}=a\begin{pmatrix}1&0\\-1&0\end{pmatrix}+b\begin{pmatrix}0&1\\0&1\end{pmatrix}+c\begin{pmatrix}1&1\\0&0\end{pmatrix}$$

$$=\begin{pmatrix}a&0\\-a&0\end{pmatrix}+\begin{pmatrix}0&b\\0&b\end{pmatrix}+\begin{pmatrix}c&c\\0&0\end{pmatrix}$$

$$=\begin{pmatrix}a+c&b+c\\-a&b\end{pmatrix}$$

$a+c=1$

$b+c=2$

$-a=-3 \implies a=3$

$b=4$

$3+c=1 \implies 3-2=1$

$4+c=2 \implies c=-2$

$$\mathrm{Span}(S) = \{ a_1u_1 + a_2u_2 + \dots + a_nu_2 \mid a_1, a_2, \dots, a_n \in F \text{ and } u_1, u_2, \dots, u_n \in S \}$$

(A1) $\sum_{k=0}^{n} a_k i_k + \sum_{k=0}^{m} b_k j_k$ $\Rightarrow$ Sum of $n+m$ terms

Finite sum of multiples of vectors in S $\Rightarrow$ By definition in $\mathrm{span}(S)$.

$a_i + b_i \in F$ since fields are closed under addition.

(M1)

$$c \sum_{k=0}^{n} a_k i_k = \sum_{k=0}^{n} c(a_k i_k) = \sum_{k=0}^{n} (ca_k) i_k \in \mathrm{span}(S)$$

$$\text{span}(S) = \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \in F \text{ and } v_i \in S, \text{ for all natural numbers } i \text{ and } n \text{ such that } 1 \leq i \leq n \right\}$$

Exists some vector $s \in \text{span}(S)$ such that $\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{m} b_i u_i = s$

Info: $\sum_{i=1}^{n} a_i v_i = \vec{\underset{\sim}{0}} \implies a_i = 0$ for all i

(Which also means $\vec{\underset{\sim}{0}} \notin S$)

Show: $n = m$; for all i, $a_i = b_i$ and $u_i = v_i$

Proof: If $\vec{\underset{\sim}{0}} \neq s$,

$$\sum_{i=1}^{n} a_i v_i + \left(-\sum_{i=1}^{m} b_i u_i\right) = \vec{\underset{\sim}{0}} \quad \text{where } a_i \neq 0 \text{ and } b_i \neq 0$$

$$\sum_{i=1}^{\max(n,m)} a_i v_i - b_i u_i = \vec{\underset{\sim}{0}} \quad \text{where we define } a_i = 0 \text{ and } v_i = 0 \text{ if } i > n$$

$b_i = 0$ and $u_i = 0$ if $i > m$

$n = m$

For all j, there exists a k such that

$$a_j v_j - b_k u_k = \vec{\underset{\sim}{0}}$$

Simultaenously, $a_j v_j + (-b_k) u_k = \vec{\underset{\sim}{0}} \implies a_j = b_k = 0$

$$a_j v_j = b_k u_k$$

$a_j = b_k = 0$ OR $a_j \neq 0$ and $b_k \neq 0$ but $a_j \neq b_k$. OR $a_j = b_k \neq 0$

$\implies u_k = (b_k^{-1} \cdot a_j)(v_j)$

$\implies v_j = b_k$

If there exists u_k such that $u_k \notin \text{span}\{v_1, v_2, \ldots\}$ then

$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n + b_k u_k = \vec{\underset{\sim}{0}}$

$\implies a_1, a_2, \ldots, a_n, b_k = \vec{\underset{\sim}{0}}$

$$\alpha v_i + \beta u_j = \vec{\underset{\sim}{0}}$$

$$\alpha v_i + \beta (b_k^{-1} \cdot a_j)(u_j) = \vec{\underset{\sim}{0}}$$

$v \in \text{Span}(S_1 \cap S_2) \implies$ exists $u_1, u_2, \dots, u_n \in S_1 \cap S_2$ and $a_1, a_2, \dots, a_n \in F$ such that $v = \sum_{i=1}^{n} a_i u_i$

$\implies v \in S_1$ and $v \in S_2$.

$v \in \text{Span}(S_1) \cap \text{Span}(S_2) \implies v \in \text{Span}(S_1)$ and $v \in \text{Span}(S_2)$

$\sum_{i=0}^{n} a_i x_i \in \text{Span}(S_1)$ and $\sum_{i=0}^{m} b_i y_i \in \text{Span}(S_2)$, where $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in F$;

$x_1, x_2, \dots, x_n \in \text{Span}(S_1)$

and

$y_1, y_2, \dots, y_n \in \text{Span}(S_2)$

Basically, $\sum_{i=0}^{n} a_i x_i = \sum_{i=0}^{m} b_i y_i = v \in \text{Span}(S_1) \cap \text{Span}(S_2)$.

However, it may not be the case that $n = m$ or $a_i = b_i$ for all i or $x_i = y_i$ for all i.
Just as long as they add up to be equivalent.

Examples:

$V = \mathbb{R}^2$ over $\mathbb{R}$

$\text{Span}(\{(0,1),(2,0)\} \cap \{(0,1),(0,2)\}) = \text{Span}(\{(0,1)\})$

$= \{a(0,1) \mid a \in \mathbb{R}\}$

$= \{(0,a) \mid a \in \mathbb{R}\}$

$\text{Span}(\{(0,1),(2,0)\}) \cap \text{Span}(\{(0,1),(0$

$= \{(2b, a) \mid a, b \in \mathbb{R}\} \cap$

$\{(0, c+2d) \mid c, d \in \mathbb{R}\}$

$= \{(0, e) \mid e \in \mathbb{R}\}$

$\forall x \forall y (x, y \in A \wedge x < y \implies f(x) < f(y))$

$\forall x \forall y [(f(x) > f(y) \vee f(x) = f(y)) \implies (x > y \wedge x = y)]$

$\forall x \forall y (x, y \in A \wedge x = y \implies f(x) = f(y) \in A)$

all $x, y \in A$; $f(x) < f(y)$ OR $f(x) = f(y)$ OR $f(y) < f(x)$

Assume $x \neq y$. Then,

1. $x < y$, implying $f(x) < f(y)$
2. $y < x$, implying $f(y) < f(x)$

$f(x) < f(y)$

$\mathrm{span}\left(\bigcup_{i\in I} S_i\right) = W$ where I is a finite set and $S_i = S_j$ iff $i=j$

$v_1, v_2, \dots, v_n \in S_i \implies a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \vec{\underset{\sim}{0}}$

$cv_1, c_2 v_2, \dots, c_n v_n \in W \quad \mathrm{span}(\{v_1, v_2, \dots, v_n\}) = \mathrm{span}(\{c_1 v_1, c_2 v_2, \dots, c_n v_n\})$

If F is a finite field of characteristic p, $\quad F = \{0, 1, 1+1, \dots \sum_{i=1}^{p} 1+1\}$? Probably not

$\forall x (x \in A \implies \neg xRx)$

$\forall x [(xRy \wedge yRz) \implies xRz] \quad (xRy \wedge yRx) \implies xRx$

$\forall x \forall y [x \neq y \implies ((xRy \wedge \neg yRx) \vee (\neg xRy \wedge yRx))]$

If the field F is finite, then it has a nonzero field characteristic.

The statement is true for 0 : $0^3 = 0^3 + 0^3 = 0$!

Assume it is true for some $n \in \mathbb{N}$, then it is true in the $(n+1)$th case as well, because:

$$(n+1)^3 = \sum_{i=1}^{n+1} i^3 - \sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n-1} i^3 + (n+1)^3 - n^3$$

$$n^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n-1} i^3$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{(n-1)(n-1+1)}{2}\right]^2$$

$$\langle a_1,b_1\rangle <_L \langle a_2,b_2\rangle \;\wedge\; \langle a_2,b_2\rangle <_L \langle a_3,b_3\rangle$$

$$\iff \left[a_1 <_A a_2 \vee (a_1=a_2 \wedge b_1 <_B b_2)\right] \wedge \left[a_2 <_A a_3 \vee (a_2=a_3 \wedge b_2 <_B b_3)\right]$$

1. $a_1 <_A a_2$ and $a_2 <_A a_3 \Rightarrow a_1 <_A a_3$ by the transitivity of the linear ordering $<_A$. Therefore, $\langle a_1,b_1\rangle <_L \langle a_3,b_3\rangle$.
2. $a_1 <_A a_2$ and $(a_2=a_3$ & $b_2 <_B b_3) \Rightarrow a_1 <_A a_3$. So, $\langle a_1,b_1\rangle <_L \langle a_3,b_3\rangle$.
3. $(a_1=a_2$ & $b_1 <_B b_2)$ and $a_2 <_A a_3 \Rightarrow a_1 <_A a_3$. Hence, $\langle a_1,b_1\rangle <_L \langle a_3,b_3\rangle$
4. $(a_1=a_2$ & $b_1 <_B b_2)$ and $(a_2=a_3$ & $b_2 <_B b_3) \Rightarrow a_1=a_3$ & $b_1 <_B b_3$ by the transitivity of the linear ordering. Accordingly, $\langle a_1,b_1\rangle <_L \langle a_3,b_3\rangle$.

$$\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\} \Leftarrow \cancel{(x\in A \vee}$$

$$\iff \cancel{[(x\in A \wedge y\in A) \vee x\in B] \wedge [(x\in A \wedge y\in A) \vee}$$

$$\cancel{(x\in A \wedge y\in A) \vee (x\in B \wedge y\in C)}$$

$$\cancel{(x\in A \vee x\in B) \wedge (y\in A \vee y\in C)}$$

Set of all equivalence relations on A: Let's call it S.

$$S = \cancel{\{\{(x,x)\} \mid x\in A\}} \cup \left\{ \overbrace{\{(x,y),(z,z) \mid x,y\in A \wedge x\neq z \wedge y\neq z\}}^{k} \;\middle|\; z\in A \right\}$$

$$\forall A \exists S \forall K \; K\in S \iff K\in \mathcal{P}(A\times A) \wedge \forall k \; k\in K \Rightarrow \exists a\exists b\,(a,b\in A \wedge \langle a,b\rangle \in k)$$

$$\forall z_1,z_2,\ldots,z_n \forall x \forall y \,\exists k\, (k\in K \wedge z_1,z_2,\ldots,z_n,x,y\in A \wedge x\neq z_1 \wedge x\neq z_2 \wedge \ldots \wedge x\neq z_n \wedge y\neq z_1 \wedge y\neq z_2 \wedge \ldots \wedge y\neq z_n$$

$$\Rightarrow (x,x),(y,y),(x,y),(y,x),(z_1,z_1),(z_2,z_2),\ldots,(z_n,z_n)\in k$$

$$2a_1 + a_2 + a_3 = 1$$
$$-a_1 - a_2 + a_3 = -2$$
$$4a_1 + 3a_2 - a_3 = -1$$

$$-a_2 + 3a_3 = -1$$
$$-a_1 - a_2 + a_3 = -2$$
$$-a_2 + 3a_3 = -9$$

$$a_1 + a_2 - a_3 = 2$$
$$a_2 - 3a_3 = 1$$
$$a_2 - 3a_3 = -9$$

Contradiction : $1 \neq -9$.

$$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$$

$$xRy_1 \wedge xRy_2 \Rightarrow y_1 = y_2$$
$$y_1 = f(x) \wedge y_2 = f(x) \Rightarrow y_1 = y_2$$
$$x_1 = x_2 \wedge \quad y_1 = f(x_1) \wedge y_2 = f(x_2) \Rightarrow y_1 = y_2$$
$$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$$

$$\exists u_1 \exists u_2 \dots \exists u_n \exists a_1 \exists a_2 \dots \exists a_n \left(u_1, u_2, \dots, u_n \in S \wedge a_1, a_2, \dots, a_n \in F \wedge (a_1 \neq 0 \vee a_2 \neq 0 \vee \dots \vee a_n \neq 0) \wedge a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{\underset{\sim}{0}} \right)$$

$$\forall u_1 \forall u_2 \dots \forall u_n \forall a_1 \forall a_2 \dots \forall a_n \left(\underbrace{u_1, u_2, \dots, u_n \notin S \vee a_1, a_2, \dots, a_n \notin F \vee (a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0) \vee \neg(a_1u_1 + a_2u_2 + \dots + a_nu_n = \underset{\sim}{0})}_{a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{\underset{\sim}{0}} \Rightarrow \left(u_1, u_2, \dots, u_n \notin S \vee a_1, a_2, \dots, a_n \notin F \vee \neg(a_1 \neq 0 \wedge a_2 \neq 0 \wedge \dots \wedge a_n \neq 0)\right)} \right)$$

$b_n \cdot a_{i,n} x^i$

$$\sum_{i=1}^{m} \left[\sum_{n=1}^{m} (b_n \cdot a_{i,n}) x^i \right]$$

$$\sum_{n=1}^{m} \left(b_n \cdot \sum_{i=1}^{n} a_{i,n} x^i \right) = 0$$

$$\sum_{n=1}^{m} \left[\sum_{i=1}^{n} (b_n \cdot a_{i,n}) x^i \right] = 0$$

$$\sum_{n=1}^{m} \left[\sum_{i=1}^{m} (b_n \cdot a_{i,n}) x^i \right] = 0$$

unless $a_{i,n} = 0$ if $i > n$

$=$

For all i, coefficient of x^i is 0.
i.e. For each and every i,

$b_n \cdot a_{i,n} = 0$ for all n.

$$\forall i \forall n (b(n) \cdot a(\langle i,n \rangle) = 0) \iff \forall i \forall n (b(n) = 0 \vee a(\langle i,n \rangle) = 0)$$

$n \cup \{n\} = m \cup \{m\}$

$\forall x (x \in n \lor x \in \{n\} \iff x \in m \lor x \in \{m\})$

$\forall x (x \in n \lor x = n \iff x \in m \lor x = m)$

$[(x \in n \lor x = n) \land (x \in m \lor x = m)] \lor [(x \notin n \land x \neq n) \land (x \notin m \land x \neq m)]$

$[(x \in n \lor x = n) \land x \in m] \lor [(x \in n \lor x = n) \land x = m]$

$(x \in n \land x \in m) \lor (x = n \land x \in m) \lor (x \in n \land x = m) \lor (x = n \land x = m)$

$\underline{n \in m} \lor n = m$

$m \in n \lor m = n$

Assume $n \in M$, then $n \neq M$.

$\emptyset \in \emptyset$ False.

~~Let $x \neq \emptyset$; then there exists some set $y \neq x$ such that $y \in x$~~

~~$x \in x$~~

~~$x = \{x, y, \ldots\}$~~

$x \in \bigcup x$

$x \in x$

$\{x\} \subseteq x$

$x = x$

$\forall y (y \in x \iff y \in x)$

$y \in x \Rightarrow y \in$

$n = \{\{n\}\}$

$\{n\} \in n$

$n = \{m\} \land \{n\} = m$

$m \in n \land n \in m$

$n \in m$

$n \in \{n\} \in n$

$n \in m \in n$

$\{n\} \cap \{\{n\}\} = \emptyset$

$n \in \{n\} \land n \in \{\{n\}\} \iff n \in \emptyset$

$n \cap \{n\} = \emptyset$

$n \in n \land n = n \iff n \in \emptyset$

$n \in \quad \iff n \in \emptyset$

$\{\{n\}\} \cap \{n\} = \emptyset$

$x \in \{\{n\}\} \land x \in \{n\} \iff x \in \emptyset$

$x = \{n\} \land x = n \iff x \in \emptyset$

$\{n\} \in n \land \{n\} \in \{n\} \iff n \in \emptyset$

$$\exists x \left(x \in a \Rightarrow \underbrace{x \in a^+}_{= \emptyset} \right) \Longleftrightarrow a^+ \neq \emptyset$$

$$\neg \left[\forall x \left(x \in A \Leftrightarrow x \in \emptyset \right) \right]$$

$$\left[(\neg P \vee \neg Q) \wedge (P \vee Q) \right] \Longleftrightarrow \left[(P \Rightarrow \neg Q) \wedge (P \vee Q) \right]$$

$$\exists x \left[(x \notin A \vee x \notin \emptyset) \wedge (x \in A \vee x \in \emptyset) \right]$$

$$\mathcal{P}(n^+) = \mathcal{P}(m^+)$$

$$\mathcal{P}(n \cup \{n\}) = \mathcal{P}(m \cup \{m\})$$

$$\left(x \in \omega' \Longleftrightarrow \left[x \in \omega \wedge \exists y \left(y \in \omega \wedge y^+ = x \right) \right] \right)$$

$$\neg \left[\exists A \exists n \left(A \text{ is inductive} \wedge n \in \omega \wedge |A| = n \right) \right]$$

$$\Longleftrightarrow \forall A \forall n \left(A \text{ is not inductive} \vee n \notin \omega \vee |A| \neq n \right)$$

$$n' = \{ x \in \omega \mid$$

$cu + dv = \underset{\sim}{0} \implies c = d = 0$

Show: $a_1(u+v) + a_2(u-v) = \underset{\sim}{\vec{0}} \implies a_1 = a_2 = 0$

$(a_1+a_2)u + (a_1-a_2)v = \underset{\sim}{\vec{0}}$

Let $a_2 = \frac{c-d}{1+1}$, $a_1 = c - a_2 = d + a_2$

$c = d + (1+1)a_2$

$c = d + c - d$
$= c$

$a_1 + a_2 = 0 \qquad a_1 - a_2 = 0$

$(1+1)a_1 = 0 \qquad a_1 = a_2$

$1+1 = 0$ OR $a_1 = 0$
N.A. $\qquad a_2 = 0$

$d - b = (1+1)a_3$
$d = b + (1+1)a_3$

$a_1 = b - a_2$
$b = a_1 + a_2$

$(1+1)a_2 = d + b - c$
$d = (1+1)a_2 + c - b$
$1 = a_2 + a_3$

$a_2 + c - b = a_3$

$a_1(u+v) + a_2(u-v) = \underset{\sim}{\vec{0}} \implies a_1 = a_2 = 0$

$\implies (a_1+a_2)u + (a_1-a_2)v = \vec{0} \implies a_1\ a_2 = 0$

$b_1 u + b_2 v = \underset{\sim}{\vec{0}}$

$a_2 = c - b - a_3$
$b = c - a_2 - a_3$

$a_1 = b - a_2$
$b = a_1 + a_2$

$c - a_2 - a_3 = a_1 + a_2$
$c = a_1 + (1+1)a_2 + a_3$

$b_1(u+v) + b_2(u+w) + b_3(v+w) = \underset{\sim}{\vec{0}} \qquad a_1 u + a_2 v + a_3 w = \underset{\sim}{\vec{0}} \implies a_1 = a_2 = a_3 = 0$

$(b_1+b_2)u + (b_1+b_3)v + (b_2+b_3)w = \underset{\sim}{\vec{0}}$

$b_1 + b_2 = 0$
$b_1 + b_3 = 0$
$b_2 + b_3 = 0$

$a_1 = b_1 + b_2 \qquad \cancel{a_2 = b_1 + b_3} \qquad a_3 = b_2 + b_3$

$\cancel{b_1 = a_1 - b_2} \qquad \cancel{a_2 = a_1 - b_2 + b_3}$

$\cancel{a_3 = a_1 + (1+1)b_3}$

$\cancel{b_2 + b_3 = a_2 - a_1} \qquad \cancel{b_3 = \frac{a_2 - a_1}{1+1}}$

$\cancel{b_2 = a_2 - a_1 - \frac{a_2 - a_1}{1+1}}$

$b = a_1 + a_2 \qquad c = a_1 + a_3 \qquad d = a_2 + a_3$

$a_1 = b - a_2 \qquad c = b - a_2 + a_3 \qquad d = (1+1)a_2 + c - b$

$a_3 = c - b + a_2 \qquad a_2 = \frac{d+b-c}{1+1}$

$a_1 - b_2$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad \begin{bmatrix} 0 & b_1 \\ c_1 & d_1 \\ e_1 & f_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \\ e_2 & f_2 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 \\ 0 & d_3 \\ e_3 & f_3 \end{bmatrix}, \ldots, \begin{bmatrix} a_6 & b_6 \\ c_6 & d_6 \\ e_6 & 0 \end{bmatrix}$$

$$T : \begin{bmatrix} 0 & c_1 & e_1 \\ b_1 & d_1 & f_1 \end{bmatrix}, \begin{bmatrix} a_2 & c_2 & e_2 \\ 0 & d_2 & f_2 \end{bmatrix}, \begin{bmatrix} a_3 & 0 & e_3 \\ b_3 & d_3 & f_3 \end{bmatrix}, \ldots, \begin{bmatrix} a_6 & c_6 & e_6 \\ b_6 & d_6 & 0 \end{bmatrix}$$

$$ae^{rt} + be^{st} = 0$$

$$ae^{rt} = -be^{st}$$

$$a = -be^{(s-r)t} \qquad e^{\ln c}$$

$$ae^{rt} + be^{st} = ce^{Rt} + de^{St}$$

Let V be a vector space over the field F and $u_1, u_2, \dots, u_n$ be distinct vectors in V.

$\Rightarrow$ Assume $\beta = \{u_1, u_2, \dots, u_k\}$ is a basis for V.

$\Rightarrow$ Span $\beta = V$

$\Rightarrow$ β is linearly independent

If there are some natural $n \le k$ and $m \le k$ with nonzero $a_1, a_2, \dots a_n \in F$, nonzero $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m \in F$

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{m} \bar{a}_i \bar{v}_i = v, \text{ then}$$

$$\sum_{i=1}^{n} a_i v_i - \sum_{i=1}^{m} \bar{a}_i \bar{v}_i = \vec{0}.$$

each v_i is distinct. & each $\bar{v}_i$ is distinct

Show $a_i = \bar{a}_j$ & $v_i = \bar{v}_j$ & $n = m$

Know $\vec{0} \notin \beta$, lest β not linearly independent.

$\Rightarrow$ Either $n = m$ or $n \ne m$. If $n \ne m$, suppose $n > m$ wlog.

Since ... distinct ..., at most m pairs of vectors v_i and $\bar{v}_i$ with $v_i = \bar{v}_i$.

$$\sum_{i \in A} a_i u_i = \sum_{i \in B} b_i u_i = v$$

$$\sum_{i \in A \cap B} (a_i - b_i) u_i + \sum_{i \in A - B} a_i u_i + \sum_{i \in B - A} (-b_i) u_i = \vec{0}$$

distinct

If $A \ne B$, then $A - B$ and or $B - A$ is nonempty.

Let k in the above nonempty set,

then $a_k \ne 0$

$\Rightarrow$ So there is a nontrivial representation of $\vec{0}$ contradicting β lin independent.

$\Rightarrow$ $A = B$ must be true.

$\Rightarrow$ $$\sum_{i \in A \cap B} (a_i - b_i) u_i = \vec{0}$$

Since $u_i \ne \vec{0}$ for all i, and the above is only the trivial $a_i - b_i = 0$

$a_i = b_i$

Conversely, suppose each $v \in V$ can be uniquely expressed as a linear combination of vectors of β.

$$v = \sum_{i=1}^{n} a_i u_i$$

$$\sum_{i=1}^{k} a_k u_k = \vec{0}$$

$\Rightarrow$ the scalars $a_1, a_2, \dots, a_k$ are unique

One possible combination of scalars is $a_1 = a_2 = \dots = a_k = 0$.

Since unique $\Rightarrow$ the above is the only possible combination

$\Rightarrow$ All reps are trivial.

$\Rightarrow$ β is linearly independent ume 1 21 $S_1 \cap S_2 = \emptyset$

Assume S_1 and S_2 are disjoint linearly independent subsets of the vector space V over the field F so that $S_1 \cup S_2$ is linearly dependent. Then there exists some natural k with the vectors $u_1, u_2, \ldots, u_k \in S_1 \cup S_2$ and scalars $a_1, a_2, \cdots a_k \in F$ such that

= the

$$a_1 u_1 + a_2 u_2 + \quad + a_k v_k = \vec{0}.$$

In other words, since each $u_i \in S_1 \cup S_2$ is either in S_1 or S_2, this means that $u_1, u_2, \ldots, u_m \in S_1$; $u_{m+1}, u_{m+2}, \ldots, u_n \in S_2$ and $a_1, a_2, \ldots, a_m, \ldots, a_n$ so

$$\underbrace{a_1 u_1 + a_2 u_2 + \cdots + a_m v_m}_{\neq \vec{0}} + \underbrace{a_{m+1} v_{m+1} + \cdots + a_n v_n}_{\neq \vec{0}} = \vec{0}$$

ithe

for some natural m and n. ~~where $a_1u_1 + \cdots + a_mu_m \neq \vec{0}$ and $a_{m+1}v_{m+1} + \cdots + a_nv_n \neq \vec{0}$ because S_1 and S_2 being both linearly independent.~~

ince

Equivalently, we know that

$$u_1 u_1 + a_2 u_2 + \cdots + a_m v_m = (-a_{m+1}) v_{m+1} + (-a_{m+2}) v_{m+2} + \cdots (-a_n) v_n.$$

su

Thus, as $\sum_{i=1}^{m} a_i v_i \in \text{Span}(S_1)$ and $\sum_{i=m+1}^{n} (-a_i) v_i \in \text{Span}(S_2)$, their intersection $\text{Span}(S_1) \cap \text{Span}(S_2)$ must contain this vector as well.

at,

Hence, $\text{Span}(S_1) \cap \text{Span}(S_2) \neq \{\vec{0}\}$ because $\sum_{i=1}^{m} a_i u_i = \sum_{i=m+1}^{n} (-a_i) v_i \neq \vec{0}$ by virtue of S_1 and S_2 being linearly independent.

Conversely, suppose $\text{Span}(S_1) \cap \text{Span}(S_2) \neq \{\vec{0}\}$ where S_1 and S_2 are still disjoint linearly independent subsets of the vector space V over the field F. Consequently, we see that there exists the natural m and n in the following way:

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{n} b_i v_i \qquad \text{where } a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in F, \; u_1, u_2, \ldots u_m \in S_1 \text{ and } v_1, v_2, \ldots, v_n \in S_2.$$

Thus we simply reverse what we previously did:

$$\sum_{i=1}^{m} a_i u_i - \sum_{i=1}^{n} b_i v_i = \vec{0},$$

$$\sum a_i u_i + \sum (-b_i) v_i = \vec{0}.$$

By the simple reindexing of letting $-b_i = a_{m+i}$ and $v_i = u_{m+i}$, we get that

$$\sum_{i=1}^{~~\text{natural}~~\, m+n} a_i u_i = \vec{0}.$$

Therefore, there indeed exists a nontrivial representation of $\vec{0}$ as a linear combination of vectors in $S_1 \cup S_2$. Hence, $S_1 \cup S_2$ is linearly dependent.

Therefore, $S_1 \cup S_2$ is linearly independent if and only if $\text{Span}(S_1) \cap \text{Span}(S_2) \neq \{\vec{0}\}$. Q.E.D. ☐

$\mathrm{Span}(S) = W$

&

$\forall 5 (5 \subset S \Rightarrow \mathrm{Span}(5) \neq W)$ $\Longrightarrow$ S linearly independent

S lid $\Rightarrow$ $\mathrm{Span}(S) \neq W$ or $\neg(5 \subset S \Rightarrow \mathrm{Span}(5) \neq W)$

$\exists 5 (5 \subset S$ and $\mathrm{Span}(5) = W)$

$S \subseteq W$ (lid)

$S \cup \{v\}$ is lid iff $v \notin \mathrm{Span}(S)$

$\Rightarrow \mathrm{Span}(S \cup \{v\}) =$

$S \subset W$.

$\mathrm{Span}(5) \subseteq W$

Assume $\mathrm{Span}(S) = W$ and for all $5 \subset S$, $\mathrm{Span}(5) \neq W$;

$w \in W \Rightarrow w = \sum_{i=1}^{n} a_i u_i$

$\bar{w} \neq \sum_{i=1}^{n} a_i u_i$

Show for all $a_1, a_2, \ldots, a_k \in F$ and $u_1, u_2, \ldots, u_k \in S$ and $k \in \mathbb{N}$;

$$\sum_{i=1}^{k} a_i u_i \neq \vec{0}$$

Suppose otherwise

exists m

with $\sum_{i=1}^{n} b_j m_j = \vec{0}$

$0x^2 + 2(x) - (2x) \quad 0$

$5 \subset S$

$\text{span}(G) = V \quad G \subseteq V \quad m$ vectors
$L \subseteq V \quad n$ vectors.
$\Longrightarrow$ $m \leq n$ & exists H with $\text{span}(L \cup H) = V$. $H \subseteq G$ $n-m$ vectors

L linearly independent
$\Rightarrow \sum_{i=1}^{n} a_i u_i = \vec{0}$
$\longrightarrow a_i = 0$ for all $i \leq n$

$\text{span}(L) \subseteq V$

Some subset β of G containing $k \leq m$ vectors is a basis of V.

Since $\text{span}(G) = V$ and $L \subseteq V$, $L \subseteq \text{span}(G) \Longrightarrow$ Every vector in L can be written as a linear combination of vectors in G.

If $n > m$, then for any two vectors of L, s.t. $l_1 \neq l_2$,

$$l_1 = \sum_{i=1}^{m} a_i u_i$$

$$l_2 = \sum_{i=1}^{m} b_i u_i$$

where $a_k \neq b_k$ for some natural $k \leq m$.

$$\sum_{i=1}^{k} c_i v_i = \vec{0} \Longrightarrow c_i = 0 \text{ for all } i \leq k$$

$$v = \sum_{i=1}^{k} c_i v_i$$

$$l_1 - l_2 = \sum_{i=1}^{m} (a_i - b_i) u_i = \vec{0}$$

Since $a_k \neq b_k$, the coefficient of u_k, i.e. $a_k - b_k$, must also be nonzero. $\Longrightarrow$ nontrivial rep of $\vec{0}$ as vectors in L

This contradicts our assumption that L is linearly independent.

$$l_j = \sum_{i=1}^{k} a_{ij} v_i \quad \text{where } v_i \in \beta$$

If $m = n$, $\emptyset \subseteq G$.

$$\sum_{j=1}^{n} \left(b_j \cdot \sum_{i=1}^{k} a_{ij} v_i \right)$$

By Theorem 1.5, since $L \subseteq V$, $\text{span}(L) \subseteq V$

21 · $S_i \cap S_j = \emptyset$ · and k

Let V be a vector space over the field F and $u_1, u_2, \dots, u_n$ be distinct vectors in V.

First assume that $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis of V. By definition, every $v \in V$ can be expressed as a linear combination of vectors of β because β generates V. The trickier part is to prove the uniqueness of such a linear combination. When there are two linear combinations that are identical to some $v \in V$, this means that there are some subsets A and B of $\mathbb{N}$ containing natural numbers less than n so that

$$\sum_{i \in A} a_i u_i = \sum_{i \in B} b_i u_i = v;$$

$$\sum_{i \in A \cap B} (a_i - b_i) u_i + \sum_{i \in A - B} a_i u_i + \sum_{i \in B - A} (-b_i) u_i = \vec{0}. \qquad (1)$$

Either $A = B$ or $A \neq B$ must hold. Consider $A \neq B$; then one and only one of $A - B$ ~~and~~ or $B - A$ is nonempty, i.e. there is some natural k in ~~either~~ precisely one of the aforementioned sets, with exactly one of $a_k \neq 0$ or $b_k \neq 0_v$ in each respective case. Notice that coefficient of u_k must thus be nonzero. In other words, there would be a nontrivial representation of $\vec{0}$ as a linear combination of vectors in β. This would contradict our assumption that β is a basis of V — that is, β is linearly independent. Hence, it must be that $A = B$. Consequently, we can state equation (1) now as

$$\sum_{i \in A \cap B} (a_i - b_i) u_i = \vec{0}.$$

Again, by virtue of the fact that the basis β linearly independent, this must be a trivial representation of $\vec{0}$ (as vectors in β). So, $a_i = b_i$. Which means that $\sum_{i \in A} a_i u_i$ is the exact same representation of the vector $v \in V$ as $\sum_{i \in B} b_i u_i$.

Conversely, now suppose that each $v \in V$ can be uniquely expressed as a linear combination of vectors of β. Therefore, the zero vector can also be uniquely written as $\sum_{i=1}^{k} a_i u_i$ for some natural k. Since $a_1 = a_2 = \dots = a_k = 0$ is clearly one such possible combination of scalars and uniqueness is presumed, this must be the only possible combination (of coefficients), which is trivial. Thereupon, β is a basis of V.

Wherefore, $\beta = \{u_1, u_2, \dots, u_n\}$ is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β.

Q.E.D. $\boxed{\sim}$

Section 1.6 Bases and Dimension

1. (a) False. It is its own basis.

(b) True. In fact, a basis must have cardinality less than or equal to that generating set.

(c) False. Counterexample: $P(\mathbb{R})$

(d) False. $\{\hat{\imath}, \hat{\jmath}\}$ and $\{2\hat{\imath}, 2\hat{\jmath}\}$ are bases of $\mathbb{R}^2$.

(e) True. See Corollary 1 of the Replacement Theorem.

(f) False. Since $\{1, x, x^2, \ldots, x^n\}$ is a basis of $P_n(F)$, so $\dim P_n(F) = n+1$.

(g) False. $\dim M_{m\times n}(F) = m\cdot n$ because the set $\{A^{ij} \in M_{m\times n}(F) \mid 0 \le i \le m \ \& \ 0 \le j \le n\}$ is a basis, where A^{ij} is the matrix with $A^{ij}_{ij} = 1$ and $A^{ij}_{i'j'} = 0$ otherwise

(h) True. As S_2 generates V, $|S_2| \ge \dim(V) \ge |S_1|$.

(i) False. That is true iff S is a basis of V. Counterexample: Suppose $V = \mathbb{R}^2$ and $S = \{\hat{\imath}, 2\hat{\imath}, \hat{k}\}$, then $2(\hat{\imath}) + \hat{k} = (2\hat{\imath}) + \hat{k}$.

(j) True. Follows from Theorem 1.11.

(k) True. A subspace S of dimension 0 must be spanned by $\emptyset$ so $S = \{\underset{\sim}{0}\}$. Similarly, a subspace S of dimension $n = \dim V$ must be V itself.

(l) True. See Corollary 2 of the Replacement Theorem.

$\ldots \mathcal{J} \quad L \leq V \quad n$ vectors. $\} \Longrightarrow \} m \leq n$

Self - Proof of Theorem 1.11

W being finite dimensional is an immediate because otherwise there exists a linearly independent subset of V with more than $\dim(V)$ vectors, a contradiction.
For the same reason, we can conclude $\dim(W) \leq \dim(V)$. Let β be a basis of W, then $|\beta| = \dim(V)$ too. Hence, β is also a basis of V.

As such, $V = W$ follows. $\square$

Example 18 Check

$$p(-1,0,1,0,0) + q(-1,0,0,0,1) + r(0,1,0,1,0) = (a,b,c,d,e)$$

$$(-p-q, r, p, r, q) = (a,b,c,d,e)$$

$$\Longrightarrow \begin{cases} a = -p-q \\ b = r \\ c = p \\ d = r \\ e = q \end{cases} \qquad \begin{array}{l} p = -a-q \\ \quad = -a-e \\ p = c \\ r = b = d \\ q = e \end{array}$$

From here, we notice that if $(a,b,c,d,e) = \underset{\sim}{0}$, then $p = q = r = 0$. Hence, this set is indeed linearly independent.

Now fix $p := -(a+e)$; $q := e$ and $r := b$ so (a,b,c,d,e) is an arbitrary vector in W. Which means $b = d$ and $c = -(a+e)$ by the construction of W.
Therefore, $\{(-1,0,1,0,0), (-1,0,0,0,1), (0,1,0,1,0)\}$ generates W:

$$p(-1,0,1,0,0) + q(-1,0,0,0,1) + r(0,1,0,1,0) = (-p-q, r, p, r, q)$$
$$= (a+e-e, b, -(a+e), b, e)$$
$$= (a,b,c,d,e)$$

Indeed, we have now shown that it is a basis of W with cardinality 3. Consequently, $\dim(W) = 3$.

2.(c)

$$a(1,2,-1)+b(1,0,2)+c(2,1,1)=(x,y,z)$$

$$(a+b+2c,\ 2a+c,\ -a+2b+c)=(x,y,z)$$

$$\Rightarrow \begin{cases} a+b+2c = x & \text{— (1)} \\ 2a+c = y & \text{— (2)} \\ -a+2b+c = z & \text{— (3)} \end{cases}$$

(3) + (1):

$$3b+3c = x+z$$
$$b+c = \tfrac{1}{3}x+\tfrac{1}{3}z \quad \text{— (3')}$$

(1) − (3'):

$$a+c = \tfrac{2}{3}x-\tfrac{1}{3}z \quad \text{— (1')}$$

$$\Rightarrow \begin{cases} a+c = \tfrac{2}{3}x-\tfrac{1}{3}z & \text{— (1')} \\ 2a+c = y & \text{— (2)} \\ b+c = \tfrac{1}{3}x+\tfrac{1}{3}z & \text{— (3')} \end{cases}$$

$$\Rightarrow \begin{cases} a+c = \tfrac{2}{3}x-\tfrac{1}{3}z & \text{— (1')} \\ a = -\tfrac{1}{3}x+y+\tfrac{1}{3}z & \text{— (2')} \\ b+c = \tfrac{1}{3}x+\tfrac{1}{3}z & \text{— (3')} \end{cases}$$

$$\Rightarrow \begin{cases} c = \tfrac{4}{3}x-y-\tfrac{2}{3}z \\ a = -\tfrac{2}{3}x+y+\tfrac{1}{3}z \\ b = -x+y+z \end{cases}$$

Once more, this suffices to show that it is indeed a basis.

2. (a)

$$a(1,0,-1) + b(2,5,1) + c(0,-4,3) = (x,y,z)$$

$$(a+2b,\ 5b-4c,\ -a+b+3c) = (x,y,z)$$

$$\begin{cases} a+2b = x \\ 5b-4c = y \\ -a+b+3c = z \end{cases} \qquad \begin{cases} a+2b = x \\ b-\frac{4}{5}c = \frac{1}{5}y \\ 3b+3c = x+z \end{cases} \qquad \begin{cases} a+2b = x \\ b-\frac{4}{5}c = \frac{1}{5}y \\ \frac{27}{5}c = x-\frac{3}{5}y+z \end{cases}$$

$$\begin{cases} a = \frac{19}{27}x - \frac{2}{9}y - \frac{8}{27}z \\ b = \frac{4}{27}x + \frac{1}{9}y + \frac{4}{27}z \\ c = \frac{5}{27}x - \frac{4}{45}y + \frac{5}{27}z \end{cases}$$

Therefore, since span$\{(1,0,-1),(2,5,1),(0,-4,3)\} = \mathbb{R}^3$, and this set contains $3 = \dim \mathbb{R}^3$ vectors, by Corollary 2(a) of the Replacement Theorem, it is a basis for $\mathbb{R}^3$.

(b)

$$a(2,-4,1) + b(0,3,-1),\ c(6,0,-1) = (x,y,z)$$

$$(2a+6c,\ -4a+3b,\ -b-c) = (x,y,z)$$

$$\Rightarrow \begin{cases} 2a + 6c = x \\ -4a+3b = y \\ -b-c = z \end{cases} \qquad \begin{cases} a + 3c = \frac{1}{2}x \\ b+4c = \frac{2}{3}x+\frac{1}{3}y \\ c = \frac{2}{9}x+\frac{1}{9}y+\frac{1}{3}z \end{cases} \qquad \begin{cases} a = -\frac{1}{6}x - \frac{1}{3}y - z \\ b = -\frac{2}{9}x - \frac{1}{9}y - \frac{4}{3}z \\ c = \frac{2}{9}x + \frac{1}{9}y + \frac{1}{3}z \end{cases}$$

Again, this proves that $\{(2,-4,1),(0,3,-1),(6,0,-1)\}$ is a basis of $\mathbb{R}^3$.

2·(e)

$$a(1,-3,-2) + b(-3,1,3) + c(-2,-10,-2) = (x,y,z)$$

$$(a-3b-2c,\ -3a+b-10c,\ -2a+3b-2c) = (x,y,z)$$

$$\Rightarrow \begin{cases} a-3b-2c = x & -(1) \\ -3a+b-10c = y & -(2) \\ -2a+3b-2c = z & -(3) \end{cases}$$

(1) + 3(2):

$$-5a - 22c = x + 2y$$

$$a + \tfrac{22}{5}c = -\tfrac{1}{5}x - \tfrac{2}{5}y \quad -(1')$$

(2) + 3(1'):

$$b - 10c + \tfrac{66}{5}c = y - \tfrac{3}{5}x - \tfrac{6}{5}y$$

$$b + \tfrac{16}{5}c = -\tfrac{3}{5}x - \tfrac{1}{5}y \quad -(2')$$

(3) + 2(1')

$$\underbrace{3b - 2c + \tfrac{44}{5}c}_{3b + \frac{34}{5}c} = -\tfrac{2}{5}x - \tfrac{4}{5}y + z \quad -(3')$$

(3') − 3(2'):

$$-\tfrac{14}{5}c = \tfrac{7}{5}x - \tfrac{1}{5}y + z$$

$$c = -\tfrac{1}{2}x + \tfrac{1}{14}y - \tfrac{5}{14}z \quad -(3'')$$

$$\Rightarrow \begin{cases} a + \frac{22}{5}c = -\frac{1}{5}x - \frac{2}{5}y & -(1') \\ b + \frac{16}{5}c = -\frac{3}{5}x - \frac{1}{5}y & -(2') \\ c = -\frac{1}{2}x + \frac{1}{14}y - \frac{5}{14}z & -(3'') \end{cases}$$

It is clear that $(1') - \frac{22}{5}(3'')$ and $(2') - \frac{16}{5}(3'')$ gives us our desired equations for a and b in terms of x, y, z.

Hence, we see that this is also a basis of $\mathbb{R}^3$.

$$a = -2x - \tfrac{5}{7}y + \tfrac{11}{7}z$$

$$b = -\tfrac{11}{5}x - \tfrac{3}{7}y + \tfrac{8}{7}z$$

If $(x,y,z) = (1,0,0)$, $a = -2$, $b = -\frac{11}{5}$, $c = -\frac{1}{2}$

plugging into (1): $\frac{28}{5} = 1$

(2): $\frac{61}{5} = 0$

(3): $\frac{-8}{5} = 0$

2.(d)

$$a(-1,3,1)+b(2,-4,-3)+c(-3,8,2) = (x,y,z)$$

$$(-a+2b-3c,\ 3a-4b+8c,\ a-3b+2c) = (x,y,z)$$

$$\Rightarrow \begin{cases} -a+2b-3c = x & \text{— (1)} \\ 3a-4b+8c = y & \text{— (2)} \\ a-3b+2c = z & \text{— (3)} \end{cases}$$

(2) + 2(1):

$$a + 2c = 2x+y \quad \text{— (2')}$$

(3) + (1):

$$-b-c = x+z$$

$$b+c = -x-z \quad \text{— (3')}$$

2(3') − (1):

$$a+5c = x+2z \quad \text{— (1')}$$

(1') − (2'):

$$3c = -x-y+2z$$

$$c = -\tfrac{1}{3}x - \tfrac{1}{3}y + \tfrac{2}{3}z \quad \text{— (1'')}$$

$$\Rightarrow \begin{cases} c = -\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}z & \text{— (1'')} \\ a+2c = 2x+y & \text{— (2')} \\ b+c = -x-z & \text{— (3')} \end{cases}$$

(2') − 2(1'') and (3') − (1''):

$$\Rightarrow \begin{cases} c = -\frac{1}{3}x - \frac{1}{3}y + \frac{2}{3}z \\ a = \frac{8}{3}x + \frac{5}{3}y - \frac{4}{3}z \\ b = -\frac{2}{3}x + \frac{1}{3}y - \frac{5}{3}z \end{cases}$$

Thus, this is yet again a basis of $\mathbb{R}^3$.

$\dots - (3)$ $\quad\dots = x + 2y$ $\quad (2) + 3(1):$

10.(b) Let $c_0 = -4$, $c_1 = 1$, $c_2 = 3$

$$f_0(x) = \frac{(x-1)(x-3)}{(-4-1)(-4-3)} = \frac{1}{35}(x^2 - 4x + 3), \quad g(c_0) = 24$$

$$f_1(x) = \frac{(x+4)(x-3)}{(1+4)(1-3)} = -\frac{1}{10}(x^2 + x - 12), \quad g(c_1) = 9$$

$$f_2(x) = \frac{(x+4)(x-1)}{(3+4)(3-1)} = \frac{1}{14}(x^2 + 3x - 4), \quad g(c_2) = 3$$

The required polynomial $g : \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} g(x) &= \sum_{i=0}^{2} g(c_i) f_i(x) \\ &= \frac{24}{35}(x^2 - 4x + 3) - \frac{9}{10}(x^2 + x - 12) + \frac{3}{14}(x^2 + 3x - 4) \\ &= -3x + 12 \end{aligned}$$

Check: $g(c_0) = 24$
$g(c_1) = 9$
$g(c_2) = 3$

4. No, by corollary 2 of the Replacement Theorem they do not generate $P_3(\mathbb{R})$ because $\dim(P_3(\mathbb{R})) = 3+1 = 4$ while there are only 3 vectors provided.

5. Since $\dim \mathbb{R}^3 = 3$, no linearly independent subset of $\mathbb{R}^3$ can have more than 3 vectors by the Replacement Theorem. Thus, the provided set containing 4 vectors must not be linearly independent.

7.

$$\frac{22}{57}u_1 + \frac{1}{3}u_2 + \frac{2}{57}u_5 = (1,0,0)$$

$$-\frac{2}{57}u_1 + \frac{1}{3}u_2 + \frac{5}{57}u_5 = (0,1,0)$$

$$\frac{7}{57}u_1 + \frac{1}{3}u_2 + \frac{11}{57}u_3 = (0,0,1)$$

Since the standard basis is contained in $\mathrm{span}\{u_1, u_2, u_5\}$, $\mathrm{span}\{u_1, u_2, u_5\} = \mathbb{R}^3$ so that the set $\{u_1, u_2, u_5\}$ of $\dim \mathbb{R}^3 = 3$ vectors forms a basis for $\mathbb{R}^3$.

10. (a) Let $c_0 = -2$, $c_1 = -1$, and $c_2 = 1$:

$$f_0(x) = \frac{(x+1)(x-1)}{(-2+1)(-2-1)} = \frac{1}{3}(x^2-1), \quad g(c_0) = -6$$

$$f_1(x) = \frac{(x+2)(x-1)}{(-1+2)(-1-1)} = -\frac{1}{2}(x^2+x-2), \quad g(c_1) = 5$$

$$f_2(x) = \frac{(x+2)(x+1)}{(1+2)(1+1)} = \frac{1}{6}(x^2+3x+2), \quad g(c_2) = 3$$

The required polynomial $g: \mathbb{R} \to \mathbb{R}$ is thus given by

$$g(x) = \sum_{i=0}^{2} g(c_i) f_i(x)$$

$$= -2(x^2-1) - \frac{5}{2}(x^2+x-2) + \frac{1}{2}(x^2+3x+2)$$

$$= -4x^2 - x + 8$$

Check: $g(-2) = -6$
$g(-1) = 5$
$g(1) = 3$ ✓

$\ldots)(x-3)$, $g(c_0) = 1.$

10. (d) Let $c_0 = -3$, $c_1 = -2$, $c_2 = 0$, $c_3 = 1$:

$$f_0(x) = \frac{(x+2)(x-0)(x-1)}{(-3+2)(-3-0)(-3-1)} = -\frac{1}{12}(x^3+x^2-2x), \quad g(c_0) = g(-3) = -30$$

$(x^2+x-2)x$

$$f_1(x) = \frac{(x+3)(x)(x-1)}{(-2+3)(-2)(-2-1)} = \frac{1}{6}(x^3+2x^2-3x), \quad g(c_1) = g(-2) = 7$$

$$f_2(x) = \frac{(x+3)(x+2)(x-1)}{(3)(2)(-1)} = -\frac{1}{6}(x^3+4x^2+x-6), \quad g(c_2) = g(0) = 15$$

$(x^2+5x+6)(x-1) = x^3+5x^2+6x-x^2-5x-6 = x^3+4x^2+x-6$

$$f_3(x) = \frac{(x+3)(x+2)(x)}{(1+3)(1+2)(1)} = \frac{1}{12}(x^3+5x^2+6x), \quad g(c_3) = g(1) = 10$$

Thus, the required function $g : \mathbb{R} \to \mathbb{R}$ is given by

$$g(x) = \sum_{i=0}^{3} g(c_i) f_i(x)$$

$$= \frac{5}{2}(x^3+x^2-2x) + \frac{7}{6}(x^3+2x^2-3x) - \frac{5}{2}(x^3+4x^2+x-6) + \frac{5}{6}(x^3+5x^2+6x)$$

$$= 2x^3 - x^2 - 6x + 15$$

Check: $g(c_0) = -30$

$g(c_1) = 7$

$g(c_2) = 15$

$g(c_3) = 10$ ✓

10.(c) Let $c_0 = -2$, $c_1 = -1$, $c_2 = 1$, $c_3 = 3$

$$f_0(x) = \frac{(x+1)(x-1)(x-3)}{(-2+1)(-2-1)(-2-3)} = -\frac{1}{15}(x^3-3x^2-x+3) \quad , \quad g(c_0) = 3$$

$(x^2-1)(x-3) = x^3-3x^2-x+3$

$$f_1(x) = \frac{(x+2)(x-1)(x-3)}{(-1+2)(-1-1)(-1-3)} = \frac{1}{8}(x^3-2x^2-5x+6) \quad , \quad g(c_1) = -6$$

$(x^2+x-2)(x-3) = x^3-3x^2+x^2-3x-2x+6 = x^3-2x^2-5x+6$

$$f_2(x) = \frac{(x+2)(x+1)(x-3)}{(1+2)(1+1)(1-3)} = -\frac{1}{12}(x^3-7x-6) \quad , \quad g(c_2) = 0$$

$(x^2+3x+2)(x-3) = x^3-3x^2+3x^2-9x+2x-6 = x^3-7x-6$

$$f_3(x) = \frac{(x+2)(x+1)(x-1)}{(3+2)(3+1)(3-1)} = \frac{1}{40}(x^3+2x^2-x-2) \quad , \quad g(c_3) = -2$$

$(x+2)(x^2-1) = x^3-x+2x^2-2$

Hence, the required function $g: \mathbb{R} \to \mathbb{R}$ is given by

$$g(x) = \sum_{i=0}^{3} g(c_i) f_i(x)$$

$$= -\frac{1}{5}(x^3-3x^2-x+3) - \frac{3}{4}(x^3-2x^2-5x+6) - \frac{1}{20}(x^3+2x^2-x-2)$$

$$= -x^3+2x^2+4x-5$$

Check: $g(c_0) = 3$

$g(c_1) = -6$

$g(c_2) = 0$

$g(c_3) = -2$ ✓

15. Ideas

$$\{A^{ij} \mid i \neq j\} \cup \overbrace{\{B^{ij} \mid j = i+1\}}^{\beta =} \cup \{B^{n1}\}$$

$$\sum_{i=1}^{n-1} c_i B^{i\,i+1} + c_n B^{n1} = M \quad , \quad \sum_{i=1}^{n} M_{ii} = 0$$

$$\Rightarrow \begin{cases} c_1 - c_n = M_{11} \\ -c_{k-1} + c_k = M_{kk}, \quad k \geq 1 \end{cases}$$

$$c_1 = M_{11}$$

$$\begin{cases} M_{11} & \text{if } k = 1 \\ M_{kk} \\ M_{nn} \end{cases}$$

$$\begin{cases} c_1 & \text{if } k = 1 \\ -c_{k-1} + c_k & \text{if } 1 < k < n \\ c_n & \text{if } k = n \end{cases}$$

Claim: $\sum_{i=1}^{n} M_{ii} + \sum_{j=1}^{k} M_{jj} = c_k \quad \text{if } k \neq n$

$\sum_{i=1}^{n} M_{ii} = c_n$

$\sum_{i=1}^{n} M_{ii} + M_{11} - \sum_{i=1}^{n} M_{ii} = M_{11}$ ✓

$- \cancel{\sum_{i=1}^{n} M_{ii}} - \sum_{j=1}^{k-1} M_{jj} + \cancel{\sum_{i=1}^{n} M_{ii}} + \sum_{j=1}^{k} M_{jj} = M_{kk}$

$c_1 = a$
$-c_1 + c_2 = b$
$-c_2 + c_3 = c$
$-c_3 = d$

$c_1 = a$
$c_2 = a+b$
$c_3 = a+b+c$
$c_3 = -d$

$$\begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} = c_1 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} + c_4 \cancel{\begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}}$$

$c_1 - c_4 = a$
$-c_1 + c_2 = b$
$-c_2 + c_3 = c$
$-c_3 + c_4 = d$

$c_1 - c_4 = a$
$c_2 - c_4 = a+b$
$c_3 - c_4 = a+b+c$
$c_4 = a+b+c+d$

$c_1 = 2a+b+c+d$
$c_2 = 2a+2b+c+d$
$c_3 = 2a+2b+2c+d$
$c_4 = a+b+c+d$

Ideas

17. $M^t = -M$

$M_{ji} = -M_{ij}$

let S^{ij} be the matrix having $\mathbb{1}$ and $-\mathbb{1}$ as its (i,j)th and (j,i)th entry respectively. If $i=j$ let (j,i)th also be $\mathbb{1}$

$\beta := \{ S^{ij} \mid i \leq j \}$

Linear ind trivial

$$\sum_{j=1}^{n} \sum_{i=1}^{j} G_{ij} S^{ij}_{k_1 k_2} = \begin{cases} M_{k_1 k_2} S^{k_1 k_2} & M_{k_1 k_2} \\ -M_{k_1 k_2} S^{k_1 k_2} & M_{k_1 k_2} S^{k_1 k_2} \end{cases}$$

13.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & \text{— (1)} \\ 2x_1 - 3x_2 + x_3 = 0 & \text{— (2)} \end{cases}$$

(2) − 2(1'):

$$(-3+4)x_2 + (1-2)x_3 = 0$$

$$x_2 - x^3 = 0$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & \text{— (1')} \\ x_2 - x_3 = 0 & \text{— (2')} \end{cases}$$

$$\begin{cases} x_1 = x_2 \\ x_2 = x_3 \end{cases}$$

Therefore, this subspace is given by $\{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$. Accordingly, $\{(1,1,1)\}$ is clearly a basis for it.

15. Let A^{ij} be the $n \times n$ matrix with the (i,j)th entry being $\mathbb{1}$ and all other entries $\mathbb{0}$. Also let B^i be the $n \times n$ matrix with the (i,i)th and $(i+1, i+1)$th entries be $\mathbb{1}$ and $-\mathbb{1}$ respectively, $\mathbb{0}$ everywhere else. Lastly, suppose M is a $n \times n$ matrix with trace $\mathbb{0}$ and all nondiagonal entries being $\mathbb{0}$ too. By defining the constants $c_k := \sum_{i=1}^{k} M_{ii}$, we see that for any diagonal matrix M of trace 0:

$$\sum_{i=1}^{n-1} c_i B^i_{kk} = \begin{cases} c_1 & \text{if } k = 1, \\ -c_{k-1} + c_k & \text{if } 1 < k < n, \\ -c_{n-1} & \text{if } k = n. \end{cases}$$

$$= \begin{cases} M_{11} & \text{if } k = 1 \\ M_{kk} & \text{if } 1 < k < n, \\ M_{nn} & \text{if } k = n \end{cases} \qquad \text{because } -\sum_{i=1}^{n-1} M_{ii} = M_{nn} \text{ follows from trace}(M) = 0.$$

$$= M_{kk}.$$

11. For any representation of $\underset{\sim}{0}$ in $\{\underset{\sim}{u}+\underset{\sim}{v}, a\underset{\sim}{u}\}$, that is:

$$c_1(\underset{\sim}{u}+\underset{\sim}{v}) + c_2(a\underset{\sim}{u}) = \underset{\sim}{0}$$
$$(c_1+c_2 a)\underset{\sim}{u} + c_1\underset{\sim}{v} = \underset{\sim}{0}$$

we know that $c_1 + c_2 a = 0$ and $c_1 = 0$ as $\{u,v\}$ is a basis for V. As a, b are nonzero, c_1 and c_2 must be zero instead

We see that $\{\underset{\sim}{u}+\underset{\sim}{v}, a\underset{\sim}{u}\}$ generates V because for any vector $d_1\underset{\sim}{u} + d_2\underset{\sim}{v}$ in V,

$$d_2(\underset{\sim}{u}+\underset{\sim}{v}) + \left(\frac{d_1-d_2}{a}\right)(a\underset{\sim}{u}) = (d_1-d_2)\underset{\sim}{u} + d_2\underset{\sim}{u} + d_2\underset{\sim}{v}$$
$$= d_1\underset{\sim}{u} + d_2\underset{\sim}{v}.$$

As such, $\{\underset{\sim}{u}+\underset{\sim}{v}, a\underset{\sim}{u}\}$ indeed forms a valid basis for V with dimension 2.

Similarly, any vector $d_1\underset{\sim}{u} + d_2\underset{\sim}{v}$ in V can be written as $\left(\frac{d_1}{a}\right)(a\underset{\sim}{u}) + \left(\frac{d_2}{b}\right)(b\underset{\sim}{v})$. Hence, forming a basis for V again.

$\square$

20. (a) Define the function $f : \mathbb{N}_0 \longrightarrow S$ recursively by $f(m)$ being any element of $S - \operatorname{span} f[\![m]\!]$ if it is nonempty, otherwise fix it to be some extraneous object e not belonging to S. All $f(m)$ must be in S given $m \leq n$, lest there exists some least $m \leq n$ for which $S - \operatorname{span} f[\![m]\!] = \varnothing$. This means $S \subseteq \operatorname{span} f[\![m]\!]$ so $\operatorname{span} S = \operatorname{span} f[\![m]\!] = V$, but the generating set $f[\![m]\!]$ having $m < n$ members contradicts $\dim V = n$. Similarly, we see that $f[\![m]\!]$ must always be linearly independent (given it is a subset of S):

$$\sum_{k=0}^{m} c_k f(k) = \underset{\sim}{0}$$

certainly tells us all constants $c_k = 0$ because $\sum_{k=0}^{m-1} d_k f(k) = f(m)$ for no constants d_k. Hence, it follows that $f[\![n]\!]$ is a basis of V since it is a linearly independent set of n vectors. $\square$

(b) In the above proof, we already constructed a subset of S containing exactly n vectors. This suffices to show $|S| \geq n$. $\square$

25. $\dim Z = m+n$ because

$$\sum_{i=1}^{m} c_i(v_i, \underset{\sim}{0}) + \sum_{i=1}^{n} d_i(\underset{\sim}{0}, w_i) = (\underset{\sim}{0}, \underset{\sim}{0})$$

tells us

$$\sum_{i=1}^{m} c_i v_i = \underset{\sim}{0} \quad \text{and} \quad \sum_{i=1}^{n} d_i w_i = \underset{\sim}{0}$$

so that all c_i and d_i must be 0 given $\{v_i \mid i \leq m\}$ and $\{w_i \mid i \leq n\}$ are bases of V and W respectively. It is clear that this set of vectors $\{(v_i, \underset{\sim}{0}) \mid i \leq m\} \cup \{(\underset{\sim}{0}, w_i) \mid i \leq n\}$ spans Z. Hence it is a basis of Z with cardinality $m+n$.

15. We also see that all our B^i are linearly independent, since

$$\sum_{i=1}^{n-1} c_i B^i = \underset{\sim}{0}$$

$$\underset{\sim}{\mathbb{1}} = \begin{cases} c_1 & \text{if } k=1, \\ -c_{k-1}+c_k & \text{if } 1<k<n, \\ -c_{n-1} & \text{if } k=n. \end{cases}$$

Immediately, $c_1 = c_{n-1} = 0$. Assuming $c_{k-1} = \mathbb{1}$, then $c_k = \mathbb{1}$ too for all intermediate cases. So, $c_k = \mathbb{1}$ for any $1 \le k \le n$ as required.
Now, it is clear that $\{A^{ij} \mid i \neq j\} \cup \{B^i \mid i \le n-1\}$ is a linearly independent subset of W that spans W. In other words, this is a basis for W. Accordingly, $\dim(W) = (n^2-n)+(n-1) = n^2-1$ □

17 Let S^{ij} be the $n \times n$ matrix with its (i,i)th entry being $\mathbb{1}$, and if $i \neq j$, its (j,j) being $-\mathbb{1}$. It is straightforward to show that $\{S^{ij} \mid i \le j\}$ is a basis for W. As such, $\dim(W) = \frac{1}{2}n(n+1)$. □

18. Suppose σ_k is the sequence in F with $\sigma_k(k) = \mathbb{1}$ and $\sigma_k(n) = \mathbb{0}$ when $n \neq k$. Now, it is clear that the set $\{\sigma_k \mid k \in \mathbb{N}\}$ is a basis for W, because a linear combination of these vectors only involves some n number of them by definition. And it follows from our construction of σ_k that any resultant vector will only have n number of non zero terms. □

$M = n$ for which S-span $f[\![m]\!] - \alpha$ ... object e not belonging to S. All $f(m)$ ... being any element of S-span $f[\![m]\!]$ if it is nonempty.

29.(a) Ideas

$$\underset{\beta}{\dim(W_1 \overset{\cap}{+} W_2)} \le \underset{\beta_1}{\overset{m_1}{\dim(W_1)}} + \underset{\beta_2}{\overset{m_2}{\dim(W_2)}} \quad \underset{\alpha}{m_3} \qquad n_1 \le m_1, \; n_2 \le m_2$$

$\beta_1 \cap W_2 \subseteq W_1 \cap W_2$ $\quad$ span$(\beta_1 \cap W_2) = W_1 \cap W_2$

$(\beta - W_2) \cup (\beta_1 \cap W_2)$ $\quad$ lest span$((\beta_1 \cap W_2) \cup \alpha) = W_1 \cap W_2$

$\cup (\beta_1 \cap W_2) \cup \alpha$ $\quad$ but then $\alpha \subseteq$ span(β_1). In fact, the coefficient of $\beta_1 - W_2$ must be 0 for any rep of $\underset{\sim}{v} \in \alpha$ because $(\beta_1 - W_2) \cap (W_1 \cap W_2) = \emptyset$. (So if ...

$\underset{\sim}{v} \in \beta_1 \cap W_2$ $\quad$ $\underset{\sim}{v} \in \beta_1 \cap W_2$

Proof (without seeing author's hint)

Suppose β_1 is a basis for W_1. We see that $\beta_1 \cap W_2$ must be a basis of $W_1 \cap W_2$, lest span$(\beta_1 \cap W_2) \subset W_1 \cap W_2$ so there exists some $\alpha \subseteq W_1 \cap W_2$ that extends $\beta_1 \cap W_2$ to generate $W_1 \cap W_2$ (i.e. span$((\beta_1 \cap W_2) \cup \alpha) = W_1 + W_2$). But then $\alpha \subseteq$ span$(\beta_1) (\supseteq W_1 \cap W_2)$ such that $\alpha \subseteq$ span$(\beta ...$

Thus, span$(\beta_1 \cap W_2) = W_1 \cap W_2$ after all. Consequently, $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ is easy to see.

(b) When V is the direct sum of W_1 and W_2, $W_1 \cap W_2 = \{\underset{\sim}{0}\}$ so from (a) we indeed have that $\dim(V) = \dim(W_1) + \dim(W_2)$. The converse holds trivially.

33.(a) By definition, $W_1 \cap W_2 = \{0\}$ so $\beta_1 \cap \beta_2 = \emptyset$ is immediate. It is trivial to see that $\beta_1 \cup \beta_2$ forms a basis for V.

(b) If $\beta_1 \cup \beta_2$ forms a basis for V, then $\dim(V) = \dim(W_1) + \dim(W_2)$. So the result holds from 29.(a). $\square$

35.(a) Consider any member of V/W, i.e. $\underset{\sim}{u} + W$ for some $u \in V$. If $\underset{\sim}{u} \in W$, $\sum_{i=0}^{k} 0 u_i + W = \underset{\sim}{u} + W = W$. Otherwise, $\underset{\sim}{u} \in V - W$ so $\underset{\sim}{u} = \sum_{i=k+1} u_i$ for some constants c_i. Hence, $\underset{\sim}{u} + W = \sum_{i=k+1}^{n} c_i(u_i + W)$. Telling us that span$\{u_{k+1} + W, \dots u_n + W\} = V/W$. We see that the zero vector of V/W is just W, thus if $\sum_{i=k+1} c_i(u_i + W) = W$, $\sum_{i=k+1} c_i u_i = \underset{\sim}{0}$. As such, $c_i = 0$ by the linear independence of $\{u_1, u_2, \dots, u_n\}$. Therefore, $\{u_{k+1} + W, \dots u_n + W\}$ is also linearly independent and wherefore a basis of V/W. $\square$

(b) $\dim(V/W) = \dim(V) - \dim(W)$ by the above result.

26. It has dimension n as the set $\{x^i - a^i + a \mid i \leq n\}$ is a basis for the mentioned subspace.

27. $\dim(W_1 \cap P_n(F)) = m$, where $n = 2m$ in the case of n being even and $n = 2m+1$ if n odd.
and $\dim(W_2 \cap P_n(F)) = n - m$

(b)

28. Ideas

$\beta_{\mathbb{C}} = \{v_1, v_2, \dots, v_n\}$

dim

$\beta_{\mathbb{R}} \ \{v_1, v_2, \dots, v_n, iv_1, iv_2, \dots, iv_n\}$

Proof

We shall see that $\beta_{\mathbb{R}} := \{v_k \mid k \leq n\} \cup \{iv_k \mid k \leq n\}$ forms a basis for V over $\mathbb{R}$ (with

tells Let $\beta_{\mathbb{C}} := \{v_k \mid k \leq n\}$ be a basis of V over $\mathbb{C}$. the same operations of addition, multiplication, et cetera). Consider

$$\sum_{k=1}^{n} c_k v_k + \sum_{k=1}^{n} d_k (iv_k) = \underset{\sim}{0},$$

M

$$\sum_{k=1}^{n} (c_k + d_k i) v_k = \underset{\sim}{0},$$

is n

By the equality of the two sums above, it is easy to see that $\operatorname{span}(\beta_{\mathbb{R}}) =$

we see that $c_k + d_k i$ must be 0 so $c_k = d_k = 0$. $\operatorname{span}(\beta_{\mathbb{C}}) = V$. As such, $\beta_{\mathbb{R}}$ is indeed a basis for V over $\mathbb{R}$ as we claimed, and now, the dimension of V over $\mathbb{R}$ must be $2n$.

$\square$

Exercises

1. (a) False
   (b) False
   (c) False
   (d) True
   (e) True
   (f) True

2. Let (k_n) be the sequence with $k_k = 1$ and $k_n = 0$ if $n \neq k$. The set of all such sequences (k_n) is an infinite linearly independant subset of the subspace of convergent sequences. This suffices to complete the proof (the rest is trivial).

3. We shall see that the set of all powers of π, namely $\{\pi^n \mid n \in \mathbb{N}_0\}$, is a linearly independent subset of V. $\{1\}$ is trivially linearly independent. So suppose $\{\pi^n \mid n \leq k\}$ is linearly independent for a $k \in \mathbb{N}_0$. Then, so must $\{\pi^n \mid n \leq k+1\}$ lest the transcendence of π is contradicted. Therefore, by induction, this is true of every $k \in \mathbb{N}_0$. Consequently, it is easy to see that $\{\pi^n \mid n \in \mathbb{N}_0\}$ must be linearly independent. As such, V must be infinite-dimensional because all finite dimensional vector spaces only contain finite linearly independent subsets.

Self-Proof of Theorem 1.12

Suppose, for the sake of contradiction, that β does not generate V. Then pick any element $\underset{\sim}{y}$ of $S - \operatorname{span}(\beta) \neq \emptyset$, thus $\beta \cup \{\underset{\sim}{y}\}$ must be linearly independent, a contradiction. Therefore, β must generate V. Consequently, it is a basis for V. $\square$

Self-Proof of Theorem 1.13

Let $\mathcal{F}$ be the set of linearly independent subsets of V containing S if they are nonempty and C any chain in $\mathcal{F}$. Then we claim $\bigcup C \in \mathcal{F}$. If not, there exists some vectors $\underset{\sim}{u_1}, \underset{\sim}{u_2}, \dots, \underset{\sim}{u_n} \in C$ and constants $d_1, d_2, \dots, d_n \in \mathbb{F}$ (one of which is nonzero) so $\sum_{i=1}^{n} d_i u_i = \underset{\sim}{0}$. Now, select a member $A_i \in C$ for each $1 \leq i \leq n$ such that $u_i \in A_i$. It is clear that there must be some largest (wrt the $\subseteq$ relation) set A_k for which $u_i \in A_k$ for all $1 \leq i \leq n$. Since $A_k \in \mathcal{F}$, it is linearly independent, a contradiction. As such, we can be certain that $\bigcup C \in \mathcal{F}$. By the Hausdorff Maximal Principle, there is some maximal linearly independent subset of V that contains S. $\square$

LMAO even the use of A_i and A_k is the same as the author's

Corollary

$\emptyset$ is a linearly independent subset of V. So, from Theorem 1.13 we can be certain of the existence of a maximally independent subset of V. Which, by Theorem 1.12, must be a basis for V.

False
False

4. By theorem 1.15, for any basis β_W for W, there exists an extended basis β_V for V with $\beta_W \subseteq \beta_V$

5. Assume, for the sake of contradiction, that even though β is a basis for V, there exists a nonzero vector $\underset{\sim}{vec}$, some other vectors $\underset{\sim}{u_1}, \underset{\sim}{u_2}, \ldots, \underset{\sim}{u_n}$ and $\underset{\sim}{v_{n+1}}, \underset{\sim}{v_{n+2}}, \ldots, \underset{\sim}{v_{n+m}}$ in β with nonzero scalars $c_1, c_2, \ldots c_{n+m}$ such that $\sum_{i=1}^{n} c_i \underset{\sim}{u_i} - \sum_{i=n+1}^{n+m} c_i \underset{\sim}{v_i} = \underset{\sim}{vec}$. But now,

$$\sum_{i=1}^{n} c_i \underset{\sim}{u_i} - \sum_{i=n+1}^{n+m} c_i \underset{\sim}{v_i} = \underset{\sim}{0},$$

which is a nontrivial representation of $\underset{\sim}{0}$ in terms of vectors in β. As such, β is not linearly independent, a contradiction. To summarize, we have shown that if β is a basis for V, then the unique representation of vectors in V is guaranteed. Conversely, now suppose unique representation holds for every vector in V. Then, $\text{span}(\beta) = V$ is immediate, and more importantly, $\underset{\sim}{0}$ has a unique representation too — that must be the trivial representation (empty sum of vectors). Consequently, β is linearly independent. Wherefore the biconditional holds.

6. A similar procedure as used in the proof of Theorem 1.13 can be applied.

7. We claim that $\gamma := [\beta - \text{span}(S_1)] \cup S_1$ gives the desired basis for V. Clearly, $\beta \subseteq \text{span}(\gamma)$ so $\text{span}(\gamma) = V$ is guaranteed. ~~For any vectors~~ ~~$\underset{\sim}{u_i} \in \beta - \text{span}(S_1)$ and $\underset{\sim}{v_i} \in S_1$ with corresponding constants c_i, when~~

$$\cancel{\sum_{i=1}^{n} c_i \underset{\sim}{u_i} + \sum_{i=n+1}^{n+m} c_i \underset{\sim}{v_i} = \underset{\sim}{0},}$$

$$\cancel{\sum_{i=1}^{n} c_i u_i = \sum_{i=n+1}^{n+m} c_i v_i.}$$

~~we shall see that $\sum_{i=1}^{n} c_i \underset{\sim}{u_i} \notin \text{span}(S_1)$ whilst $\sum_{i=n+1}^{n+m} (-c_i) \underset{\sim}{v_i} \in \text{span}(S_1)$, all c_i must be 0.~~

$\text{Span}(\beta \wedge \text{span}(S_1))$

~~if $u_i \in \beta - \text{Span}(S_1)$~~

7. By the previous exercise, taking $S_2 := \beta \cup S_1$, we have a basis γ for V that contains S_1. And so, the desired S is simply $\gamma - S_1$.

16. Let f be any polynomial in $P(\mathbb{R})$ given by $f(x) = \sum_{i=0}^{n} c_i x^i$ for some natural n. Then, $T\left(\sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1}\right) = \sum_{i=0}^{n} c_i x^i = f(x)$. So, surjectivity is certain. However, T is clearly not injective since $T(f(x)+1) = T(f(x))$ even though $f(x)+1 \neq f(x)$.

17. (a) Let β be a basis for V. Thus, since $R(T) = \operatorname{span} T[\beta]$, $\operatorname{rank}(T)$ is maximally $\dim(V) < \dim(W)$ so that $R(T) \subset W$.

(b) Similarly, when β is a basis for V, some subset γ of $T[\beta]$ forms a basis of $R(T)$, where in fact, $\gamma \subset T[\beta]$ must hold left $\dim(V) = \dim(W)$. Which entails that there exists some $x \in \beta$ with $T(x)$ capable of being expressed as $T($some linear combination of vectors in $\gamma)$, removing any hope that T is injective.

~~18. Define $T(x,y) = (x-y, x-y)$, then $T(c(x_1,y_1)) + T(x_2,y_2) = T(cx_1+x_2-cy_1-y_2, cx_1+x_2-cy_1-y_2) = T(cx_1+x_2, cy_1+y_2)$, thus linearity holds. Now notice that $T(x,y)=(0,0)$ implies $x=y$ so $\{(1,1)\}$ is a basis for $N(T)$ and hence an~~

↳ Oh copy read wrongly

18. $T_0(x,y) = (0,0)$ clearly gives such a linear transformation.

19. Let $V = W = \mathbb{R}^2$, $T, U : V \to W$ be defined by $T(x,y) = (x,y)$ and $U(x,y) = 2(x,y)$. Linearity is clear. We see that $N(T) = N(U) = \{(0,0)\}$. Similarly, notice that if $2(x,y) \in R(U)$, $2(x,y) \in R(T)$ because $T(2x,2y) = 2(x,y)$; and when $(x,y) \in R(T)$, $(x,y) \in R(U)$ since $U(\frac{1}{2}(x,y)) = (x,y)$. As such, $R(T) = R(U)$ as desired.

21. (a) We notice that

$$T(c(a_n) + (b_n)) = (ca_{n+1} + b_{n+1}) = cT(a_n) + T(b_n), \text{ and } U(c(a_n)+(b_n)) = (ca_{n-1} + b_{n-1}) = cU(a_n) + U(b_n).$$

Hence, T and U are linear.

(b) For any sequence $(a_n) \in V$, $T(a_{n-1}) = (a_n)$ (where (a_{n-1}) is the sequence $(0, a_1, a_2, a_3, \ldots)$) so that T is surjective. Notice that although $(a_n) := (0, 1, 1, 1, \ldots) \neq (1, 1, 1, 1, \ldots) =: (b_n)$, $T(a_n) = T(b_n)$ such that T is not injective for sure.

(c) Clearly, U is injective. Non-surjectivity holds easily as $(a_n) \notin R(U)$ for any sequence (a_n) with $a_1 \neq 0$. $\square$

22.

Ideas

$T(1,0,0)=a$, $T(0,1,0)=b$, $T(0,0,1)=c$

Proof

Define the scalars $a := T(1,0,0)$, $b := T(0,1,0)$, $c := T(0,0,1)$. Now, $T(x,y,z) = xT(1,0,0)+yT(0,1,0)+zT(0,0,1) = ax+by+cz$. We see that for $T:\mathbb{F}^n \to \mathbb{F}$, the same procedure can be repeated. Finally, consider $T:\mathbb{F}^n \to \mathbb{F}^m$. Define the jth coordinate of $T(e_i)$ to be c_{ij}. Given $x \in \mathbb{F}^n$, we have some constants s_i for which

$$\begin{aligned} T(x) &= \sum_{i=1}^{n} s_i T(e_i) \\ &= \sum_{i=1}^{n}\left(s_i \cdot \sum_{j=1}^{m} c_{ij} e_j\right) \\ &= \sum_{j=1}^{m}\left(\sum_{i=1}^{n} s_i c_{ij}\right) e_j . \end{aligned}$$

This is our analoge, which has now been proven too.

13. The nullspace of T can be
   1. A point (0-dimensional)
   2. A line cutting through $(0,0,0)$ (1-dimensional)
   3. A plane cutting through $(0,0,0)$ (2-dimensional)
   4. The whole of $\mathbb{R}^3$ (3-dimensional)

And in the language of exercise 22, these occur when
1. $a=b=c=0$
2. At least one of a,b,c are nonzero
3. At least 2 of a,b,c are nonzero
4. All of a,b,c are nonzero

respectively.

Self-Proof of Theorem 2.6.

We see that the transformation $T: V \longrightarrow W$ defined by $T(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i w_i$ must be linear by its construction. To prove uniqueness, suppose that $U: V \rightarrow W$ is a linear transformation with $U(v_i) = w_i$. Then it is clear that $U(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i w_i = T(\sum_{i=1}^{n} c_i v_i)$. That is, $U = T$. As such, there indeed only exists exactly one such linear transformation T.

$\square$

$T(cx, y) \neq cT(x, y)$

$T(x_1 + x_2, y_1 + y_2) = T(cx_1, cy_1) + T(x_2, y_2) \neq cT(x_1, y_1) + T(x_2, y_2)$

Exercises

1. (a) T
   (b) F ✓
   (c) T · F [illegible] T might not be linear
   (d) T ✓
   (e) F ✓
   (f) F ✓
   (g) T ✓
   (h) F ✓

9. (a) Counterexample: $10T(10, 1) = (10, 10) \neq (1, 10) = T(100, 10)$.
   (b) Counterexample: $T(1, 1) + T(2, 1) = (1, 1) + (2, 4) = (3, 5) \neq (3, 9) = T(3, 1) = T((1,1) + (2,1))$.
   ⋮

$a(1,1) + b(2,3) = (8,11)$
$a + 2b = 8$, $a + 3b = 11$
$a = 2$, $b = 3$

11. Notice that $(0,1), (1,0) \in \text{span}\{(1,1), (2,3)\}$ so $\{(1,1), (2,3)\}$ forms a basis for $\mathbb{R}^2$. From Theorem 2.6 it follows that there exists a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ with $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. Thus, $T(8,11) = 2T(1,1) + 3T(2,3)$
$= 2(1,0,2) + 3(1,-1,4) = (5,-3,16)$.

$\square$

Let β be a basis for V. Thus, since $R(T) = \operatorname{span} T[\beta]$, $\operatorname{rank}(T)$ is maximally $\dim(V) < \dim(W)$ so that $R(T) \subset W$.

13. Assume, for the sake of contradiction, that S is linearly dependent. Then, $\sum_{i=0}^{k} c_i \underset{\sim}{v_i} = \underset{\sim}{0}$ for some constants c_i, at least one of which is nonzero. We see that this gives rise to a nontrivial representation of the zero vector from W:

$$\sum_{i=0}^{k} c_i \underset{\sim}{w_i} = T\left(\sum_{i=0}^{k} c_i \underset{\sim}{v_i}\right) = T(\underset{\sim}{0}) = \underset{\sim}{0}.$$

However, this contradicts the initial condition that $\{\underset{\sim}{w_i} \mid i \leq k\}$ is linearly independent. As such, we can conclude that, in fact, S being linearly independent must hold true.

14. (a) Suppose T is injective, then $\operatorname{nullity}(T) = 0$ so that the result easily follows. Conversely, consider a linear transformation $T: V \to W$ that carries linearly independent subsets of V into linearly independent subsets of W. Given any nonzero vector $\underset{\sim}{x} \in V$, $\{\underset{\sim}{x}\}$ is linearly independent so $\{T(\underset{\sim}{x})\}$ is too. Telling us $T(\underset{\sim}{x}) \neq \underset{\sim}{0}$, and critically, $\operatorname{nullity}(T) = 0$. Hence, T is injective. Thus, the biconditional holds.

(b) In the case that $T[\![S]\!]$ is linearly independent, exercise 13 of this section immediately informs us that S is linearly independent. Conversely, when S is linearly independent, it follows from (a) that $T[\![S]\!]$ is linearly independent.

(c) From (b), $T[\![\beta]\!]$ is a linearly independent subset of W containing n vectors. This must be a basis of W since the injectivity of T tells us $\operatorname{nullity}(T) = 0$ and its surjectivity says $R(T) = \dim(W)$, so that $\dim(V) = \dim(W) = n$.

15. Notice $cT(f(x)) + T(g(x)) = c\int_0^x f(t)\,dt + \int_0^x g(t)\,dt = \int_0^x cf(t) + g(t)\,dt = T(cf(x) + g(x))$, so linearity of T follows. Suppose $T(f(x)) = T(g(x))$ for any $x \in \mathbb{R}$. Then, $\int_0^x f(t)\,dt = xf(x) = xg(x) = \int_0^x g(t)\,dt$ such that $f(x) = g(x)$ as long as $x \in \mathbb{R}$. In other words, $f = g$ and injectivity holds. Clearly, T cannot be surjective since $T(f(x)) = xf(x)$ lets us know that no constant polynomials exist in $R(T)$, lest $f(x) = \frac{c}{x}$ which is impossible.

... transformation T. ... is clear that $U\left(\sum_{i=1} c_i v_i\right) = \sum_{i=1} c_i v_i = T\left(\sum_{i=1} c_i v_i\right)$. ... uniqueness, suppose that ... That is $U = T$.

Self-Proof$_\gamma$ of the Dimension Theorem: Proof (where β is a basis for V)

First choose a basis γ from the generating set $T[\![\beta]\!]$ for $R(T)$ and notice that for u_j in β but $T(u_j) \notin \gamma$, we can express it as a linear combination of $T(v_i)$'s $\in \gamma$: for some constants c_i:

$$\sum_{i=1}^{n} c_i T(v_i) = T(u_j),$$

thus by linearity

$$T\left(\sum_{i=1}^{n} c_i v_i - u_j\right) = 0.$$

Let $\dim(V) := m$ and $|\gamma| := n$,
We see that the subset $\eta := \{\sum_{i=1}^{n} c_{ij} v_i - u_j \mid j \leq m-n\}$ must be linearly independent. For any constants d_j with

$$\sum_{j=1}^{m-n} d_j\left(\sum_{i=1}^{n} c_{ij} v_i - u_j\right) = 0,$$

$$\left[\sum_{i=1}^{n} v_i \cdot \left(\sum_{j=1}^{m-n} d_j c_{ij}\right)\right] + \sum_{j=1}^{m-n} (-d_j) u_j = 0,$$

As β is a basis for V, each d_j is certainly 0. Similarly, η must also generate $N(T)$. For any $w \in V$, we can express it as a linear combination of vectors in β for some constants a_i and d_j, that is,

$$\begin{aligned} w &= \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m-n} d_j u_j, \\ &= \sum_{j=1}^{m-n} d_j u_j + \left[\sum_{i=1}^{n} v_i \cdot \left(\sum_{j=1}^{m-n} (-d_j) c_{ij}\right)\right] + \left[\sum_{i=1}^{n} v_i \cdot \left(\sum_{j=1}^{m-n} d_j c_{ij}\right)\right] + \sum_{i=1}^{n} a_i v_i. \end{aligned}$$

As such, it again follows from linearity that given $T(w) = 0$,

$$T(w) = \sum_{j=1}^{m-n} (-d_j)\, T\left(\sum_{i=1}^{n} c_{ij} v_i - u_j\right) + \sum_{i=1}^{n}\left(a_i + \sum_{j=1}^{m-n} d_j c_{ij}\right) T(v_i),$$

$$\sum_{i=1}^{n}\left(a_i + \sum_{j=1}^{m-n} d_j c_{ij}\right) T(v_i) = 0.$$

Since the $T(v_i)$'s come together to form a basis γ for W, each $a_i + \sum_{j=1}^{m-n} d_j c_{ij} = 0$. So, $w = \sum_{j=1}^{m-n} (-d_j)\left(\sum_{i=1}^{n} c_{ij} v_i - u_j\right)$.

Indeed, η generates $N(T)$ such that η is now a basis for $N(T)$. Consequently, we notice that $\operatorname{rank}(T) = |\gamma| := n$ and $\operatorname{nullity}(T) = |\eta| = m-n$ where $\dim(V) := m$ so,

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

$\square$

Self - Proof of Theorem 2.4

Assume v_1 and v_2 are distinct vectors in V, then $v_2 - v_1 \neq \underset{\sim}{0}$. Then, $T(v_2 - v_1) \neq \underset{\sim}{0}$ as $N(T) = \{\underset{\sim}{0}\}$ so that $T(\underset{\sim}{v_1}) \neq T(\underset{\sim}{v_1}) + T(\underset{\sim}{v_2} - \underset{\sim}{v_1}) = T(\underset{\sim}{v_2})$.

$\square$

$\dots$ for some constants c_i: ... for u_j in β but $T(u_j) \notin \gamma$, we can express it as

$$\sum_{i=1}^{n} c_i T(v_i) = T(u_j),$$

Self-Proof of the Dimension Theorem

Ideas

~~Choose any nonzero vector $T(v_i) \in R(T)$, where $v_i \in \beta$, then~~ there exists a basis $\gamma \subseteq T[\![\beta]\!]$ for $R(T)$.

$$T\left(\sum_{i=1}^{n} c_i v_i\right) = T(u)$$

$$T\left(\sum_{i=1}^{n} c_i v_i - u\right) = 0$$

$$\sum_{j=1}^{m-n} d_j\left(\sum_{i=1}^{n} c_{ij} v_i - u_j\right) = 0$$

$$\left[\sum_{i=1}^{n} v_i \cdot \left(\sum_{j=1}^{m-n} d_j c_{ij}\right)\right] + \sum_{j=1}^{m-n} (-d_j) u_j = 0$$

As β is a basis for V, each $d_j = 0$.

Therefore the subset $\eta := \{\sum_{i=1}^{n} c_{ij} v_i - u_j \mid j \le m-n\}$ of V forms a basis for $N(T)$.

(Still need prove generation tho)

$$T(w) = 0$$

$$w = \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m-n} d_j u_j$$

$$= \sum_{j=1}^{m-n} d_j u_j + \left[\sum_{i=1}^{n} v_i \cdot \left(\sum_{j=1}^{m-n} (-d_j) c_{ij}\right)\right] + \left[\sum_{i=1}^{n} v_i \cdot \left(\sum_{j=1}^{m-n} d_j c_{ij}\right)\right] + \sum_{i=1}^{n} a_i v_i$$

$$T(w) = T\left[\sum_{j=1}^{m-n} (-d_j)\left(\sum_{i=1}^{n} c_{ij} v_i - u_j\right)\right] + T\left[\sum_{i=1}^{n} v_i \cdot \left(a_i + \sum_{j=1}^{m-n} d_j c_{ij}\right)\right]$$

$$\Longrightarrow \sum_{i=1}^{n} \left(a_i + \sum_{j=1}^{m-n} d_j c_{ij}\right) T(v_i) = 0$$

As $T(v_i) \in \gamma$, $T(v_i) \neq 0$ from linear independence. So,

$$a_i + \sum_{j=1}^{m-n} d_j c_{ij} = 0.$$

Hence, $w = \sum_{j=1}^{m-n} (-d_j)\left(\sum_{i=1}^{n} c_{ij} v_i - u_j\right)$.

Consequently, we ... that

$\dim(R(T)) = |\gamma| = n$ and

$\dim(N(T)) = |\eta| = m-n$ so that

$\dim(V) = \dim(R(T)) + \dim(N(T))$.

$\square$

thus by linearity, $\sum_{i=1} c_i T(v_i) = T(u_j)$, ... v_j in β but $T(u_j) \notin \gamma$, we can express it as

Self - Proof of Theorem 2·1

Firstly, $\underset{\sim}{0}$ must be in both $N(T)$ and $R(T)$ as $T(\underset{\sim}{0}) = 0\,T(x) = \underset{\sim}{0}$ for any nonzero vector $x \in V$ (if $V = \{\underset{\sim}{0}\}$, $T(0 \cdot \underset{\sim}{0}) = T(\underset{\sim}{0})$ tells us $T(\underset{\sim}{0}) = \underset{\sim}{0}$)

Secondly, suppose $u_1, u_2 \in N(T)$ and $T(v_1), T(v_2) \in R(T)$. Then, it is clear that $T(u_1 + u_2) = T(u_1) + T(u_2) = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$ and $T(v_1) + T(v_2) =$

$T(v_1 + v_2)$ so that $u_1 + u_2 \in N(T)$ and $T(v_1) + T(v_2) \in R(T)$. Lastly, closure under scalar multiplication follows in a similar way. Thus, $N(T)$

and $R(T)$ are subspaces of V and W respectively.

$\square$

Self - Proof of Theorem 2·2

Let $\underset{\sim}{u}$ be any member of V. Then it can be expressed as a linear combination of vectors in β, i.e. $\underset{\sim}{u} = \sum_{i=1}^{n} c_i v_i$ for some constants c_i.

Thus, $T(\underset{\sim}{u}) = T\left(\sum_{i=1}^{n} c_i v_i\right) = \sum_{i=1}^{n} c_i T(v_i)$ by property 4 of linear transformations, so $R(T) = \text{span}(\{T(v_1), T(v_2), \dots, T(v_n)\})$ is clear.

$\square$

Example 1

$$T(cf + g) = \begin{pmatrix} cf(1) + g(1) - cf(2) - g(2) & 0 \\ 0 & cf(0) + g(0) \end{pmatrix}$$

$$cT(f) + T(g) = \begin{pmatrix} cf(1) - cf(2) & 0 \\ 0 & cf(0) \end{pmatrix} + \begin{pmatrix} g(1) - g(2) & 0 \\ 0 & cf(0) + g(0) \end{pmatrix}$$

Thus, $T(cf + g) = cT(f) + T(g)$ so T is indeed linear.

34. Consider $K \neq \emptyset$ because otherwise the result is immediately vacuously true. Let $s, r \in K$. We see that $r-s \in N(T)$ because $T(r) = T(r-s) + T(s)$. Thus, for any $r \in K$, $r = (r-s)+s \in \{s\} + N(T)$. In other words, $S \subseteq \{s\} + N(T)$. The converse, that $\{s\} + N(T) \subseteq S$ is trivial as given $r \in N(T)$, $T(s+r) = T(s) + T(r) = b$. Therefore, equality holds. $\square$

35. ~~Define the transformation $T: V \to W$ by $T(x) = f(x)$ for all $x \in \beta$~~ Let $u, v \in V$, there exists unique vectors $x_i \in \beta$ and unique nonzero scalars $a_i \in \mathbb{F}$, by exercise 5 of Chapter 1 Section 1.7, linear combinations of vectors in β; $u = \sum_{i=1}^{n} a_i x_i$ and $v = \sum_{i=n+1}^{m} a_i x_i$ for some $x_i \in \beta$. Now define the transformation $T: V \to W$ by $T(x) = \sum_{i=1}^{n} a_i f(x_i)$. Given any constant $c \in \mathbb{F}$, $cT(u) + T(v) = c\sum_{i=1}^{n} a_i f(x_i) + \sum_{i=n+1}^{m} a_i f(x_i) = T(cu+v)$ is clear. Hence, T is linear. Suppose $U: V \to W$ is another linear transformation with $U(x) = f(x)$ for all $x \in \beta$. Notice $U(u) = \sum_{i=1}^{n} a_i f(x_i) = T(u)$ for each $u \in V$. Hence, $T = U$ which establishes uniqueness. ✓ $\square$

38. rational $T\left(\frac{p}{q}x + y\right) = T\left(\frac{p}{q}x\right) + T(y)$

$T\left(\frac{p}{q}x\right) \overset{?}{=} \frac{p}{q}T(x)$ (Suppose wlog $q > p$)

$T\left(\frac{1}{q}x\right) \overset{?}{=} \frac{1}{q}T(x)$

$T(-x) + T(x) = T(x-x) = 0$

$T(-x) = -T(x)$

$T(x) = T\left(\sum_{i=1}^{q} \frac{1}{q}x\right) = \sum_{i=1}^{q} T\left(\frac{1}{q}x\right) = q \cdot T\left(\frac{1}{q}x\right)$

Proof

Let p and q be natural numbers. We see that $pT(x) = T(px) = T\left(q\left(\frac{p}{q}x\right)\right) = qT\left(\frac{p}{q}x\right)$ by additivity so $\frac{p}{q}T(x) = T\left(\frac{p}{q}x\right)$. Furthermore, $T(x) + T(-x) = 0$, telling us $T(-x) = -T(x)$. This suffices to show $T\left(\frac{p}{q}x\right) = \frac{p}{q}T(x)$ for every rational number $\frac{p}{q} \in \mathbb{Q}$.

39. Since $T((a+bi)+(c+di)) = T((a+c)+(b+d)i) = a+c-(b+d)i = a-bi+c-di = T(a+bi) + T(c+di)$, T is indeed additive. But $T((a+bi)(c+di)) = T(ac-bd+(ad+bc)i) = ac-bd-(ad+bc)i \neq ac+bd-(bc-ad)i = (a+bi)(c-di) = (a+bi)T(c+di)$ in general. So T is not linear. $\square$

$T(cx) \neq cT(x)$ for some $c, x \in \mathbb{R}$

$(a)x \neq cT(x)$

40. $T(x+y) = T(x) + T(y)$ but

neither

$T(x) = x + c$

$T(cx) = cx + c \neq cx + c^2 = cT(x)$

$T(x) = e^x$

$T(cx) = e^{cx} \neq ce^x = cT(x)$

$T(x) = x^3$

$T(cx) = c^3x^3 \neq cx^3 = cT(x)$

$T(x+y) = (x+y)^3 \neq x^3 + y^3 = T(x) + T(y)$

linear

$T(x) = 0$

$T(x) = x$

$T(x) = 9x$

2·(b) Let $\hat{\beta}$ and γ be the standard ordered bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively;

$T(1,0,0) = (2,1)$, $T(0,1,0) = (3,0)$, $T(0,0,1) = (-1,1)$.

Thus, we obtain

$$[T]_\beta^\gamma = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$\square$

3. Closure under Addition: Since $T+U : V \to W$, $T+U \in \mathcal{L}(V,W)$.

Closure under (Scalar) Multiplication: Again, as $aT : V \to W$, $aT \in \mathcal{L}(V,W)$.

(Uniqueness of these functions hold true by the well-definedness of + in W)

(VS 1) – (VS 2) is immediate from the commutativity and associativity of + in W.

(VS 3) $(T+T_0)(x) = T(x) + T_0(x) = T(x) + \underset{\sim}{0} = T(x)$ for all $x \in V$ so $T + T_0 = T$.

(VS 4) $(T + (-1)T)(x) = T(x) - T(x) = \underset{\sim}{0}$ again for each $x \in V$ (from (a)). Hence, $(T-T) = T_0$

(VS 5) $1\,T(x) = T(x)$ for any $x \in V$ and $T : V \to W$.

(VS 6) – (VS 7) are clear from the associativity and distributivity of · over + in W.

(VS 8) is similarly straightforward from the properties of + and · in W.

Consequently, we have verified $\mathcal{L}(V,W)$ to indeed be a vector space.

$\square$

8. Let x, y be vectors in V with the unique representations in β of $\sum_{i=1}^{n} a_i v_i$ and $\sum_{i=1}^{n} b_i v_i$ respectively. Suppose $c \in \mathbb{F}$. Since $cx + y = \sum_{i=1}^{n} (ca_i + b_i) v_i$, we note that $(T(cx+y))_{i1} = ([cx+y]_\beta)_{i1} = ca_i + b_i = c([x]_\beta)_{i1} + ([y]_\beta)_{i1} = c(T(x))_{i1} + (T(y))_{i1}$ for any $1 \le i \le n$. Hence, it follows that $T(cx+y) = cT($

in other words, T is linear.

should be ok

Example 3 Self Attempt

$T(1,0) = (1,0,2)$, $T(0,1) = (3,0,-4)$ so $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$ where β and γ are ...

Self-Proof of Theorem 2.7

(a) Notice that $(aT+U)(c\underset{\sim}{u}+\underset{\sim}{v}) := aT(c\underset{\sim}{u}+\underset{\sim}{v}) + U(c\underset{\sim}{u}+\underset{\sim}{v}) = caT(\underset{\sim}{u}) + aT(\underset{\sim}{v}) + cU(\underset{\sim}{u}) + U(\underset{\sim}{v}) = c(aT+U)(\underset{\sim}{u}) + (aT+U)(\underset{\sim}{v})$ so $aT+U$ is

(b) Just tedious verification.

Self-Proof of Theorem 2.8

Hmm perhaps using $([T+U]_\beta^\gamma)_{ij}$ is fine too since the author does it (was initially concerned with hiding notation)

(a) Let $\beta := \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \ldots, \underset{\sim}{v}_n\}$ and $\gamma := \{\underset{\sim}{w}_1, \underset{\sim}{w}_2, \underset{\sim}{w}_3, \ldots, \underset{\sim}{w}_m\}$, with $T(\underset{\sim}{v}_j) = \sum_{i=1}^{m} a_{ij}\underset{\sim}{w}_i$ and $U(\underset{\sim}{v}_j) = \sum_{i=1}^{m} b_{ij}\underset{\sim}{w}_i$. We see that $(T+U)(\underset{\sim}{v}_j) := T(\underset{\sim}{v}_j) + U(\underset{\sim}{v}_j) = \sum_{i=1}^{m}(a_{ij}+b_{ij})$ ... for all $1 \le i \le n$ and $1 \le j \le m$. As such, $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.

Therefore, $[T+U]_{ij} = a_{ij} + b_{ij} = [T]_{ij} + [U]_{ij}$ (where $[T+U]_{ij}$ is the (i,j)th entry of $[T+U]_\beta^\gamma$)

(b) Similarly, we notice that given any scalar $c \in F$, $(cT)(\underset{\sim}{v}_j) = \sum_{i=1}^{m} ca_{ij}\underset{\sim}{w}_i$. Thus, $[cT]_{ij} = ca_{ij} = c[T]_{ij}$. This suffices to show $[cT]_\beta^\gamma = c[T]_\beta^\gamma$.

Exercises

1. (a) True
   (b) True (L.T.s are uniquely determined by their action on a basis (so the coeffs of the matrix rep are the same))
   (c) False $[T]_\beta^\gamma = \underbrace{\begin{pmatrix} \; \end{pmatrix}}_{|\beta| = m} \Big\} |\gamma| = n$
   (d) True
   (e) True
   (f) False

!! :)

Matrix Repr of Linear Transformations

12 · Ideas

If upper triangular,

$([T]_\beta)_{ij} = 0$ when $i > j$ i.e. $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$ where $a_{ij} = 0$ when $i > j$

so $= \sum_{i=1}^{j} a_{ij} v_i$

when $T(v_j) \in \operatorname{span}\{\cdots\}$,

Proof

Suppose $[T]_\beta$ is upper triangular. Then by definition, there exist (unique) scalars $a_{ij} = 0$ when $i > j$ so $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i = \sum_{i=1}^{j} a_{ij} v_j$, a linear combination of $v_1, v_2, \cdots, v_j$. That is, $T(v_j) \in \operatorname{span}\{v_1, v_2, \cdots, v_j\}$. Conversely, when $T(v_j) \in \operatorname{span}\{v_1, v_2, \cdots, v_j\}$, $T(v_j) = \sum_{i=1}^{j} a_{ij} v_j$ equal to some unique scalars a_{ij} $(1 \le i \le j \le n)$. Further defining $a_{ij} = 0$ for $i > j$, we have $([T]_\beta)_{ij} = a_{ij}$ by definition, which is just 0 if $i > j$. Whence, $[T]_\beta$ is upper triangular indeed. $\square$

11 · Ideas

$T(W) \subseteq W$, ~~let $\gamma = \{v_1, v_2, \cdots, v_k\}$ be an (ordered) basis of W, with $\beta = \{v_1, v_2, \cdots, v_n\}$ being an extension of γ to a basis for V.~~ (ordered) $T(v_j) = \sum_{i=1}^{n} a_{ij} v_j$

$([T]_\beta)_{ij} = a_{ij} = 0$ if $k < i \le n$ and $0 \le j \le k$.

Proof

Let $\{v_1, v_2, \cdots, v_k\}$ be a basis for W and $\beta = \{v_1, v_2, \cdots, v_n\}$ be an extension of it to a basis for V. Given any v_j, there exists unique scalars $a_{ij} \in$ with $T(v_j) = \sum_{i=1}^{n} a_{ij} v_j$. In fact, by the T-invariance of W, if $k < i \le n$ and $0 \le j \le k$, $a_{ij} = 0$ is certain. That is, $([T]_\beta)_{ij} = 0$ holds. $\square$

2·(b) Let β be the standard ordered bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. ... true. ...

16 (a) Clearly the zero element $T_0 : V \to W$ is in S^0. Moreover, $(T+U)(x) = T(x) + U(x) = \underset{\sim}{0}$ and $(cT)(x) = cT(x) = \underset{\sim}{0}$ for $T, U \in \mathcal{L}(V, W)$, $c \in \mathbb{F}$ and all $x \in S$. So, $(T+U), (cT) \in S^0$. Accordingly, S^0 is indeed a subspace of $\mathcal{L}(V, W)$.

(b) Let $T \in S_2^0$. Notice that if $x \in S_1$, $T(x) = 0$ as $x \in S_2$. Hence, $T \in S_1^0$. Thus, $S_2^0 \subseteq S_1^0$ is proven.

(c) Since $V_1 \cup V_2 \subseteq V_1 + V_2$, it is straightforward from (b) that $(V_1 + V_2)^0 \subseteq (V_1 \cup V_2)^0$. Hence, assume $T \in V_1 \cup V_2$ and $x \in V_1 + V_2$. Then for some $v_1 \in V_1$ and $v_2 \in V_2$, $T(x) = T(v_1) + T(v_2) = 0$ so $T \in V_1 + V_2$. Similarly, for $x \in V_1$ and $x' \in V_2$, $T(x) = T(x') = 0$ thus $T \in V_1^0 \cap V_2^0$. And when $U \in V_1^0 \cap V_2^0$, then for $x \in V_1 \cup V_2$, $U(x) = 0$ is clear. Hence, $U \in (V_1 \cup V_2)^0$. In other words, $(V_1 \cup V_2)^0 = (V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

dim(W) need not be finite! Oh nvm it must be finite lol!

17. Ideas $\dim(V) = \dim(W)$ ... $\dim(V) = \dim(W)$. Let $\{v_1, v_2, \ldots, v_n\}$ be any basis of V. Exercise 20 of section 1·6 tells us there exists a basis $\gamma := \{w_1, w_2, \ldots, w_k\}$

By the dimension theorem, $k := \operatorname{rank}(T) \leq \dim(V) = \dim(W)$.

$\subseteq T[\beta]$ for $R(T)$, which can be extended into a basis $\gamma = \{w_1, w_2, \ldots, w_n\}$ of W. Wlog, $w_j = T(v_j)$ for each $1 \leq j \leq k$.

Let η be a basis for $N(T)$ $= \{u_{k+1}, u_{k+2}, \ldots, u_n\}$.

$$\sum_{i=1}^{n} c_i u_i = \underset{\sim}{0}$$

$$\sum_{i=1}^{k} c_i T(u_i) = \underset{\sim}{0}$$

$c_i = 0$ (for $1 \leq i \leq k$)

$T(u_j) = T(u_j)$

Proof

Let $n := \dim(V) = \dim(W)$ and $k := \operatorname{rank}(T)$. By Theorem 2·2 and exercise 20 of section 1·6, there exists a basis $= \{T(u_1), T(u_2), \ldots, T(u_k)\}$ with $\{u_1, \ldots, u_k\} \subseteq V$ any linearly independent. Now let $\{u_{k+1}, u_{k+2}, \ldots, u_n\}$ be a basis for $N(T)$. Given any representation of 0 in terms of $\beta = \{u_1, \ldots, u_n\}$, i.e. $\sum c_i u_i = 0$, we have $\sum_{i=1}^{k} c_i T(u_i) = 0$ by linearity so $c_i = 0$ when $1 \leq i \leq k$. As such, $\sum_{i=1}^{n} c_i u_i = \sum_{i=k+1}^{n} c_i u_i = 0$, thus $c_i = 0$ for all i. Accordingly, β is a basis for V. Consequently, for any extension of $\{T(u_1), T(u_2), \ldots, T(u_k)\}$ to a basis γ for W, $([T]_\beta^\gamma)_{ij} = 0$ whenever $i \neq j$ (because $T(u_j)$ can be uniquely expressed in terms of γ as either itself if $1 \leq j \leq k$, or 0 if $k < j \leq n$).

$\square$

9. Let $k \in \mathbb{R}$; notice $T(k(a+bi)+(c+di)) = (ka+c)-(kb+d)i = kT(a+bi)+T(c+di)$. This suffices to show T is linear.
$T(1)=1$ and $T(i)=-i$. As such, $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

10. $([T]_\beta)_{ij} = \mathbb{1}$ if $i=j$ or $i=j+1$, otherwise it is simply $\mathbb{0}$. In matrix form, this is represented as

$$[T]_\beta = \underbrace{\left.\begin{pmatrix} \mathbb{1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbb{1} & 0 & 0 & \cdots & 0 \\ 0 & \mathbb{1} & \mathbb{1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbb{1} & \mathbb{1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & & \mathbb{1} \end{pmatrix}\right\} n}_{n}$$

IF $T(v_j) := v_j + v_{j+1}$

Too late in the night read -1 into $+1$ lol.

But conceptually I think I'm fine.

For $T(v_j) = v_j + v_{j-1}$,

$$\begin{pmatrix} \mathbb{1} & \mathbb{1} & 0 & \cdots & 0 \\ 0 & \mathbb{1} & \mathbb{1} & \cdots & 0 \\ 0 & 0 & \mathbb{1} & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & \mathbb{1} \\ 0 & 0 & \cdots & & \mathbb{1} \end{pmatrix}$$

Yeah this way of writing these types of matrices is probably clearer.

Ight imma head out zzz

Self - Proof of Theorem 2.9

Let $x, y \in V$ and $c \in \mathbb{F}$. We see that $UT(cx+y) = U(cT(x) + T(y)) = cU(T(x)) + U(T(y))$ by the linearity of U and T. Hence, UT is indeed linear.

$= c(UT)(x) + (UT)(y)$

Self - Proof of Theorem 2.10

(a) $(T(U_1+U_2))(x) = T(U_1(x) + U_2(x)) = (TU_1)(x) + (TU_2)(x)$ and $((U_1+U_2)T)(x) = U_1(T(x)) + U_2(T(x)) = (U_1T)(x) + (U_2T)(x)$

(b) $(T(U_1U_2))(x) = T(U_1(U_2(x))) = ((TU_1)U_2)(x)$.

(c) Trivial as $\mathcal{L}(V,W)$ is a vectorspace with I being its identity vector (Ex 6 of Section 2.2)

(d) $a(U_1U_2)(x) = (aU_1)(U_2(x)) = a(U_1(U_2(x))) = U_1((aU_2)(x))$.

$\square$

Choice of Basis in Theorem 2.11

Let

$$T(v_j) = \sum_{i=1}^{\dim(W)} a_{ij} h_i$$

$$= \sum_{i=1}^{\dim(W)} a_{ij} \left(\sum_{k=1}^{\dim(W)} b_{ki} w_k\right)$$

$$= \sum_{k=1}^{\dim(W)} \left(\sum_{i=1}^{\dim(W)} a_{ij} b_{ki}\right) w_k$$

$$([T]_\alpha^\beta)_{kj} = \sum_{i=1}^{\dim(W)} a_{ij} b_{ki}$$

$$\cancel{U(h_j) = \sum_{k=1}^{\dim(W)} b_{kj} U(w_k)}$$

$$\cancel{= \sum_{k=1}^{\dim(W)} b_{kj} \sum_{i=1}^{\dim(Z)} c_{ik} z_i}$$

$$= \sum_{i=1}^{\dim(Z)} \left(\sum_{k=1}^{\dim(W)} b_{kj} c_{ik}\right) z_i$$

$$\cancel{([U]_{\beta'}^\gamma)_{ij} = \sum_{k=1}^{\dim(W)} b_{kj} c_{ik}}$$

$$\cancel{{}_{ik} = \sum_{j=1}^{\dim(W)} b_{jk} c_{ij}}$$

$$\cancel{([U]_\beta^\gamma [T]_\alpha^\beta)_{ij} = \sum_{k=1}^{\dim(W)} \overbrace{([U]_\beta^\gamma)_{ik}}^{c_{ik}} \overbrace{([T]_\alpha^\beta)_{kj}}^{\sum_{i=1}^{\dim(W)} a_{ij} b_{ki}}}$$

$i \to k$
$j \to j$
$k \to k$

$$([U]_{\beta'}^\gamma [T]_\alpha^{\beta'})_{ij} = \sum_{k=1}^{\dim(W)} \left(\sum_{j=1}^{\dim(W)} b_{jk} c_{ij}\right) a_{kj}$$

Defn of matrix multiplication suffices to prove this lol

Self-Proof of Theorem 2.13

(a) $(u_j)_{i1} = \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{n} A_{ik}(v_j)_{k1}$. $\square$

(b) $(v_j)_{i1} = B_{ij} = \left(\sum_{k=1} B_{ik} \cdot 0\right) + B_{ij} \cdot 1 = \sum_{k=1}^{n} B_{ik}(e_j)_{i1}$

Self Proof of Theorem 2.14

Idea

$$[T(u)]_\gamma = \left[\sum_{i=1}^{n} a_i T(v_i)\right]_\gamma \qquad \left(\overbrace{[T]_\beta^\gamma}^{B}\right)_{ij} = b_{ij}$$

$$= \left[\sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_{ji} w_j\right] \qquad \left(\overbrace{[u]_\beta}^{z}\right)_{i1} = a_i$$

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_i b_{ji}\right) w_j \qquad \left([T]_\beta^\gamma [u]_\beta\right)_{i1} = \sum_{k=1}^{n} b_{ik} a_k$$

$$j1 \qquad b_{jk} a_k$$

$$\begin{tikzcd} V \arrow[r, "u_1"] \arrow[d, "T"'] & \mathbb{F}^n \arrow[d, "{[T]_\beta^\gamma}"] \\ W & \mathbb{F}^m \arrow[l, "u_2"'] \end{tikzcd}$$

Proof

Assume $n := \dim(V)$, $m := \dim(W)$, $\beta = \{v_1, v_2, \dots v_n\}$, $\gamma = \{w_1, w_2, \dots w_m\}$. We know there exists scalars a_k and b_{ij} for which $u = \sum_{k=1}^{n} a_k v_i$ and $T(v_k) = \sum_{i=1}^{m} b_{ik} w_i$. Thus, notice that

$$[T(u)]_\gamma = \left[\sum_{k=1}^{n} a_k T(v_k)\right]_\gamma$$

$$= \left[\sum_{k=1}^{n} a_i \sum_{i=1}^{m} b_{ik} w_i\right]_\gamma$$

$$= \left[\sum_{i=1}^{m}\left(\sum_{k=1}^{n} a_i b_{ik}\right) w_i\right]_\gamma$$

whose $(i,1)$th entry is just $\sum_{k=1}^{n} b_{ik} a_i$. Similarly, first noting $([T]_\beta^\gamma)_{ik} = b_{ik}$ and $([u]_\beta)_{i1} = a_i$, we see that $([T]_\beta^\gamma [u]_\beta)_{i1} = \sum_{k=1}^{n} b_{ik} a_k$. Whence, since $[T(u)]_\gamma$ and $[T]_\beta^\gamma [u]_\beta$ are both $m \times 1$ column matrices / vectors in $\mathbb{F}^n$ with identical entries, they are equivalent. $\square$

Self-Proof of Theorem 2.15.

(b) If $A = B$, $L_A = L_B$ is clear. So, consider $L_A = L_B$. Then $L_A(e_j) = Ae_j$ gives the jth column of A since $(Ae_j)_{i1} = \sum_{k=1}^{n} A_{ik}(e_j)_{k1} = A_{ij}$, which is also the jth column of B. Since this holds for any column of A and B, $A = B$. Accordingly, the biconditional holds.

Alt
Assume $A \neq B$. Then $A_{ij} \neq B_{ij}$ for some natural i, j. Then $L_A(e_j)$ again is the jth column of A which differs from that of B at least in its $(i,1)$th entry. So, $L_A \neq L_B$.

(a) Again, $L_A(e_j) = Ae_j$ is the jth column of A so it is just $\sum_{i=1}^{n} A_{ij} e_i$. Hence, $([L_A]_\beta^\gamma)_{ij} = A_{ij}$. Since they are both $m \times n$ matrices, equality holds.

(c) This is just Theorem 2.12

(d) It follows from Theorem 2.14 that $T(v) = [T(v)]_\gamma = [T]_\beta^\gamma [u]_\beta = L_{[T]_\beta^\gamma}(u)$. Thus, $T = L_{[T]_\beta^\gamma}$ where uniqueness must hence hold by use of (b).

(e) Notice that

$$\begin{aligned}
[L_{AE}(v)]_{ij} &= [(AE)v]_{ij} \\
&= \sum_{k=1}^{n} (AE)_{ik} v_{kj} \\
&= \sum_{k=1}^{n} \left(\sum_{l=1}^{n} A_{il} E_{lk} \right) v_{kj} \\
&= \sum_{l=1}^{n} A_{il} \left(\sum_{k=1}^{n} E_{lk} v_{kj} \right) \\
&= \sum_{l=1}^{n} A_{il} (Ev)_{lj} \\
&= [A(Ev)]_{ij} \\
&= [L_A L_E(v)]_{ij}.
\end{aligned}$$

Therefore, $L_{AE} = L_A L_E$

(f) By Theorem 2.12, $L_{I_n}(v) = I_n v = v = I_{F^n}(v)$, so equality holds.

□

Self-Proof of Theorem 2.16 / E. 19

See self-proof of Thm 2.15 (e).

Exercises

1\. (a) False

(b) True

(c) False

(d) ~~False (depends on basis)~~ True

Uhh what was I thinking? If you're representing the same basis vectors ($Iv(u)=u$) in terms of the same basis vectors, we obviously should have the identity matrix I!

(e) False

(f) False

(g) False, V is not necessarily some $\mathbb{F}^n$.

(h) ~~True~~ False $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(i) True

(j) True

$\boxed{\text{Ideas}}$ (f) Let A be a $m \times n$ matrix with $A^2 = I$,

$$(Ae_j)_{i1} = \sum_{k=1}^{n} A_{ik}(e_j)_{k1} = A_{ij}$$

$\begin{pmatrix} 0 \\ \vdots \\ A_{ij} \\ \vdots \\ 0 \end{pmatrix}$ ith

$$[(AA)(e_j)]_{i1} = [A(Ae_j)]_{i1} \quad \& \quad Ie_j = e_j$$

$$= \sum_{k=1}^{n} A_{ik}(Ae_j)_{k1}$$

$$= A_{ij}^2$$

Case 1 $i=j$: $A_{ij}^2 = 1 \quad (= (Ie_j)_{i1})$

$$A_{ij}^2 - 1 = 0$$

$$(A_{ij}+1)(A_{ij}-1) = 0$$

Check: $A_{ij}^2 + A_{ij} - A_{ij} - 1 = A_{ij}^2 - 1$

$A_{ij}+1 = 0$ or $A_{ij} - 1 = 0$ (fields have no zero divisors)

$A_{ij} = -1$ $\quad$ $A_{ij} = 1$

Case 2 $i \neq j$: $A_{ij}^2 = 0, \ (= (Ie_j)_{i1})$

$$A_{ij} = 0.$$

Therefore, A need not be $\pm I$. A suitable example is choosing $\mathbb{F} = \mathbb{R}$ and the matrix A to be such that $A_{11} = -1$ but agreeing with I everywhere else. Then $[(AA)e_1]_{11} = A_{11}^2 = 1 = (e_1)_{11} = (Ie_1)_{11}$, even though $A \neq I$.

(a) .

Exercise 14

(a) We know that for some constants a_j, $z = \sum_{j=1}^{p} a_j e_j$. So by the distributivity of matrix multiplication over sums (Corollary of Thm 2.12) and Theorem 2.13 (b),

$$Bz = B\left(\sum_{j=1}^{p} a_j e_j\right) = \sum_{j=1}^{p} a_j Be_j = \sum_{j=1}^{p} a_j v_j .$$

(b) Ideas

$$\cancel{u_j = Av_j = \left(\sum_{i=1}^{n} w_i\right) v_j = \sum_{i=1}^{n} w_i v_j}$$

Now $u_j = \sum_{i=1}^{n} a_i w_i$ with $a_i = (v_j)_{i1}$

$(u_j)_{ij} = \sum_{k=1}^{n} (A_{ik})(B_{kj})$

$u_j = \sum_{k=1}^{n} w_k$

$$\left(\sum_{k=1}^{n} (v_j)_{k1} w_k\right)_{i1} = \sum_{k=1}^{n} (v_j)_{k1} (w_k)_{i1} = \sum_{k=1}^{n} B_{kj} A_{ik} = (u_j)_{i1}$$

Proof

Let w_k be the kth column of A. We see that $\left(\sum_{k=1}^{n} (v_j)_{k1} w_k\right)_{i1} = \sum_{k=1}^{n} (v_j)_{k1} (w_k)_{i1} = \sum_{k=1}^{n} B_{kj} A_{ik} = (u_j)_{i1}$. Therefore, the two $m \times 1$ column matrices, u_j and $\sum_{k=1}^{n} (v_j)_{k1} w_k$, are equal.

(c) Ideas

$wA = (A^t w^t)^t = \left(\sum_{j=1}^{p} a_j (\alpha_j)^t\right)^t = \sum a_j \alpha_j$

α_j^t jth column of A^t

α_j jth row of A

Proof

Again, $w = \sum_{j=1}^{p} a_j e_j$ for some constants a_j; so $w^t = \sum_{j=1}^{p} a_j (e_j)^t$. Let α_j be the jth row of A. From (a) it now holds that $wA = (A^t w^t)^t = \left(\sum_{j=1}^{p} a_j (\alpha_j)^t\right)^t = \sum_{j=1}^{p} a_j \alpha_j$.

(d) Notice the analogous result that $\left(\sum_{k=1}^{n} (\alpha_j)_{1k} v_k\right)_{1i} = \sum_{k=1}^{n} (\alpha_j)_{1k} (v_k)_{1i} = \sum_{k=1}^{n} A_{jk} B_{ki} = (u_j)_{1i}$. Hence, we similarly have that $u_j = \sum_{k=1}^{n} (\alpha_j)_{1k} v_k$.

Letting α_j be the jth row of A as before, but now v_j and u_j be the respective jth rows of AB and B, we

2. (a)

$$A(2B+3C) = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\left[\begin{pmatrix}2&0&-6\\8&2&4\end{pmatrix}+\begin{pmatrix}3&3&12\\-3&-6&0\end{pmatrix}\right] = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}5&3&6\\5&-4&4\end{pmatrix} = \begin{pmatrix}20&-9&18\\5&10&8\end{pmatrix}$$

$$(AB)D = \left[\begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}1&0&-3\\4&1&2\end{pmatrix}\right]\begin{pmatrix}2\\-2\\3\end{pmatrix} = \begin{pmatrix}13&3&3\\-2&-1&\cancel{-10}\ -8\end{pmatrix}\begin{pmatrix}2\\-2\\3\end{pmatrix} = \begin{pmatrix}29\\\cancel{-32}\ -26\end{pmatrix}$$

$$A(BD) = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\left[\begin{pmatrix}1&0&-3\\4&1&2\end{pmatrix}\begin{pmatrix}2\\-2\\3\end{pmatrix}\right] = \begin{pmatrix}1&3\\2&-1\end{pmatrix}\begin{pmatrix}-7\\12\end{pmatrix} = \begin{pmatrix}29\\-26\end{pmatrix}$$

Indeed, we again notice $(AB)D = A(BD)$, the associativity of matrix multiplication.

(b)

$$A^t = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}, \quad A^tB = \begin{pmatrix}2&-3&4\\5&1&2\end{pmatrix}\begin{pmatrix}3&-2&0\\1&-1&4\\5&5&3\end{pmatrix} = \begin{pmatrix}23&19&0\\27&-2 \to -1&14 \to 10\end{pmatrix}$$

$$BC^t = \begin{pmatrix}3&-2&0\\1&-1&4\\5&5&3\end{pmatrix}\begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}12\\16\\29\end{pmatrix} \quad CB = \begin{pmatrix}4&0&3\end{pmatrix}\begin{pmatrix}3&-2&0\\1&-1&4\\5&5&3\end{pmatrix} = \begin{pmatrix}27&7&9\end{pmatrix}$$

$$CA = \begin{pmatrix}4&0&3\end{pmatrix}\begin{pmatrix}2&5\\-3&1\\4&2\end{pmatrix} = \begin{pmatrix}20&26\end{pmatrix}$$

12. (a) If T is not injective, then $N(T) \neq \{0_V\}$ so $T(0_V) = T(x)$ for some nonzero vector x in V. Thus, $(UT)(0_V) = (UT)(x) = 0_Z$.

OR

Since UT is injective, $N(UT) = \{0_V\}$ so $N(T) = \{0_V\}$. As such T must be injective. It is clear that assuming T injective, UT is injective iff $U \restriction R(T)$. In fact, U itself need not be injective, only the restriction of U to $R(T)$, i.e. $U \restriction R(T)$.

(b) If UT is surjective, then for each $z \in Z$ there exists $x \in V$ with $(UT)(x) = U(T(x)) = z$. So, for each $z \in Z$ there is some vector $y \in W$, for example $T(x)$, with $U(y) = z$. Accordingly U must be surjective. On the other hand, T need not be surjective itself. In fact, $R(T) \geq \dim(Z)$ can be enough. For instance, consider $T: \mathbb{F}^2 \to \mathbb{F}^2$ and $U: \mathbb{F}^2 \to \mathbb{F}$ defined by $T(e_1) = e_1$ and $T(e_2) = 0_{\mathbb{F}^2}$, $U(e_1) = 1$ and $U(e_2) = 1$. Then $(UT)(e_1) = 1$, from which the surjectivity of UT is clear.

(c) Consider any sets A, B, C and bijections $T: A \to B$ and $U: B \to C$. Given $(UT)(a) = (UT)(a')$, $T(a) = T(a')$ by U's injectivity, so $a = a'$ similarly by T's injectivity. Any element c of C also has a preimage $b \in B$ for which $U(b) = c$ by U's surjectivity. Lastly, from T's surjectivity, it holds that $b = T(a)$ for some $a \in A$. Therefore, the existence of an element $a \in A$ for which $(UT)(a) = c$ is guaranteed. As such, UT itself is bijective.

$\square$

13. We see that

$$
\begin{aligned}
tr(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= tr(BA).
\end{aligned}
$$

And $tr(A) = tr(A^t)$ follows easily from the fact that $A_{ii} = (A^t)_{ii}$. $\square$

15. Ideas (To see which presentation to use)

Let A be $m\times n$, B be $n\times r$;

$$\sum_{k=1}^{n} A_{ik}B_{kj} \quad (u_p)_{i1} = \sum_{k=1}^{n} A_{ik}(v_p)_{k1} = \sum_{k=1}^{n} A_{ik}\cdot\sum_{\ell=1}^{K} c_\ell (v_{j\ell})_{k1}$$

$$(u_{j\ell})_{i1} = \sum_{s=1}^{n} A_{is}\underbrace{B_{sj\ell}}_{(v_{j\ell})_{s1}}$$

$$= (\alpha_i v_p)_{11} \quad = \sum_{\ell=1}^{K} c_\ell \sum_{k=1}^{n} A_{ik}(v_{j\ell})_{k1} = \sum_{\ell=1}^{K} c_\ell (u_{j\ell})_{i1}$$

$$= \sum_{\ell=1}^{K} c_\ell(\alpha_i v_{j\ell})_{11} = \sum_{\ell=1}^{K} c_\ell(\alpha_i v_{j\ell})_{11}$$

12

$$= \sum_{\ell=1}^{K} c_\ell (u_{j\ell})_{ij\ell}$$

$\boxed{B}$ (Ans): Use Theorem 2.13(a).

$$u_p = Av_p$$
$$= A(c_1v_{j_1} + c_2v_{j_2} + \cdots + c_kv_{j_k})$$
$$= c_1Av_{j_1} + c_2Av_{j_2} + \cdots + c_kAv_{j_k}$$
$$= c_1u_{j_1} + c_2u_{j_2} + \cdots + c_ku_{j_k}$$

Proof

Let A and B be $m\times n$ and $n\times r$ matrices respectively, so AB exists. We see that

$$(u_p)_{i1} = \sum_{s=1}^{n} A_{is}(v_p)_{s1} = \sum_{s=1}^{n} A_{is}\cdot\sum_{\ell=1}^{k} c_\ell(v_{j\ell})_{s1} = \sum_{\ell=1}^{k} c_\ell\sum_{k=1}^{n} A_{ik}(v_{j\ell})_{k1} = \sum_{\ell=1}^{k} c_\ell(u_{j\ell})_{i1}.$$

Therefore, $u_p = \sum_{\ell=1}^{k} c_\ell u_{j\ell}$ indeed. $\square$

17. Ideas

When $T=T^2$, $T(x) = T(T(x))$ $\quad W_1\cap N(T) = \{0\}$ because for nonzero $v\in N(T)$, $T(v) = 0 \neq v$

If some $v_j\in\beta$ is not in $R(T)$,

$$\sum_{i=1}^{\operatorname{rank}(T)} c_iT(v_i) = T(v_j)$$

$$\left(\sum_{i=1}^{\operatorname{rank}(T)} c_iv_i\right) - v_j \in N(T)$$

W_1 a subspace since it's clearly closed under $+,\cdot$ and $T(0)=0$ so $0\in W_1$

$W_1\oplus N(T) = V$

$T(w+n) = w$

$T(T(w+n)) = T(w)$
$= T(w+n)$

Proof

First assume that $T=T^2$. Then $W_1\cap N(T) = \{0\}$ is immediate. Let $\alpha = \{v_1, v_2, \dots, v_r\}$ be a basis of $R(T)$ and $\beta = \{v_1, v_2, \dots, v_n\}$ its extension to V. We see that if $v_j\in\beta-\alpha$, $T(v_j) = \sum_{i=1}^{r} c_iv_i$ for some scalars c_i so $T(v_j) = \sum_{i=1}^{r} c_iT(v_i)$ and $v_j - \sum_{i=1}^{r} c_iT(v_i)\in N(T)$. Notice that for $T(x)\in R(T)$, $T(T(x)) = T(x) \in$ … Thus, $R(T) = W_1$. Hence, $W_1\oplus N(T) = V$ since it contains β. By this fact, we know that for any $u\in V$, there exists $y\in W_1$ and $x\in N(T)$ with $u = y+x$. Accordingly, $T(u) = y$. Therefore, T is indeed a projection on W_1 along $N(T)$. Conversely, suppose T is a projection on W_1 along $N(T)$. Then, again, for $u\in V$ there is $y\in W_1$ and $x\in N(T)$ with $u = y+x$. As such, $T(T(u)) = T(y) = T(y) + T(x) = T(u)$. Which informs us $T = T^2$. $\square$

16. (a) Ideas

rank(T) = rank(T^2): $R(T^2) \subseteq R(T)$ $\longrightarrow$ $R(T^2) = R(T)$

$\alpha = \{v_1, v_2, \ldots, v_r\}$

$T(v_j) = \sum_{i=1} c_i v_i$

$v_j = \sum_{i=1}^{n} a_i T(v_i)$ $\quad v_j = \sum_{i=1}^{n} a_i T(v_i)$

Else, say $R(T) \cap N(T) \neq \{0\}$, … exists … $v \in V$ … both of them:

$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$ for unique a_{ij}

$$T(T(v_j)) = \sum_{i=1}^{n} a_{ij} T(v_i) = \sum_{i=1}^{n}\left(a_{ij} \sum_{k=1}^{n} a_{ki} v_k\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ij} a_{ki}\right) v_k$$

rank$(T^2) \le$ rank$(T) - 1$

$$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i = \sum_{i=1}^{n} a_{ij} \sum_{k=1}^{n} b_{ki} T(v_k) \text{ as } v_i \in R(T^2) = \sum_{k=1}^{n} T\left(\sum_{i=1}^{n} a_{ij} b_{ki} v_k\right)$$

Proof

Assume, for the sake of contradiction, that there exists some nonzero $v \in R(T) \cap N(T)$. $\{v\}$ can be extended to form a basis β for $R(T)$. ~~Which must also be a basis for $R(T^2)$~~ Since $T[\beta]$ spans $R(T^2)$, $T(v) = 0$ means that rank$(T^2) \le$ rank$(T) - 1$. A contradiction.

Furthermore, $R(T^2)$ being a subspace of $R(T)$ having equal dimensions tells us $R(T^2) = R(T)$.

Now let $\{v_1, v_2, \ldots, v_r\}$ be a basis for $R(T)$ $= R(T^2)$, and $\beta = \{v_1, v_2, \ldots, v_n\}$ being its extension to V. We know $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$ for (unique) scalars a_{ij}. In fact, as $v_i \in R(T^2)$, $T(v_j) = \sum_{i=1} a_{ij} \sum_{k=1} b_{ki} T(v_k) = T\left(\sum_{k=1}^{r} \sum_{i=1}^{n} a_{ij} b_{ki} v_k\right)$. So, $v_j - \sum_{k=1} \sum_{i=1} a_{ij} b_{ki} v_k \in N(T)$; which is nonzero for any $j > r$. It is hence clear that $V = R(T) \oplus N(T)$.

(b) Ideas

$R(T^k) \oplus N(T^k)$

$R(T^k) \cap N(T^k) = \{0\}$ $\quad R(T^k) + N(T^k) = V$

$A := \bigcap_{i=1}^{\infty} (R(T^k) \cap N(T^k))$ $\quad$ If $A \neq \{0\}$, exists nonzero $v \in$ all $R(T^k)$ and $N(T^k)$

A being an intersection of subspaces is a vectorspace itself

Let α be a basis of A containing v, β its extension to V.

$N(T^2) - N(T) \subseteq R(T)$ as $T^2 : R(T) \to R(T)$

$R(T) \cap N(T) \supseteq R(T^2) \cap N(T^2) \supseteq \cdots$

$R(T) \supseteq R(T^2) \supseteq R(T^3) \supseteq \cdots$ nonincreasing sequence ~~Since V is finite dimensional~~, there must exist some least dimension $\lambda = \dim(R^k)$ for some positive integer k of $\{\dim(R^i) \mid i \in \mathbb{Z}^+\}$.

$R(T^k) = R(T^{2k})$ by leastness.

Proof

Since $R(T) \supseteq R(T^2) \supseteq R(T^3) \supseteq \cdots$, there must exist some least rank(T^k) of $\{\text{rank}(T^i) \mid i \in \mathbb{Z}^+\}$ for some positive integer k. Therefore, rank(T^k) = rank(T^{2k}) so by (a), $V = R(T^k) \oplus N(T^k)$. $\square$

$= \sum c_\ell (u_{i,\ell})$ ..

- When $k=0$, $[T^0]_\beta = I_V = I_n = ([T]_\beta)^0$ (where $n := \dim(V)$) as desired. Assume $[T^k]_\beta = ([T]_\beta)^k$. By Theorem 2.11, $[T^{k+1}]_\beta = [T^k T]_\beta = [T^k]_\beta [T]_\beta$ $= ([T]_\beta)^k [T_\beta] = ([T_\beta])^{k+1}$. Therefore, $[T^k]_\beta = ([T]_\beta)^k$ indeed holds for any nonnegative integer k. $\square$

Corollary 1

This is just a special case of Theorem 2.18

Corollary 2

Again, the fact that A is invertible iff L_A is, is straightforward from Theorem 2.18. Since Theorem 2.15 says $(L_A)(L_{A^{-1}}) = L_{AA^{-1}} = I$ (or the other way around),

so $(L_A)^{-1} = L_{A^{-1}}$. [Or just notice $(L_A)^{-1} = (L_A)^{-1}[(L_A)(L_{A^{-1}})] = [(L_A)^{-1}(L_A)](L_{A^{-1}}) = L_{A^{-1}}$]

Exercise 13

For reflexivity, just take the identity map. Similarly, for symmetry and transitivity, we simply use the inverse and composition of isomorphisms, respectively.

Example 3

$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ & $\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

Self-Proof of Theorem 2.19

~~When there is an isomorphism $T: V \to W$, Theorem 2.4 tells us nullity $(T) = 0$. Furthermore,~~

When $V \cong W$, the Corollary to Theorem 2.17 tells us $\dim(V) = \dim(W)$. Conversely, consider $\dim(V) = \dim(W)$. Then let $\beta := \{v_1, v_2, \dots, v_n\}$ and $\gamma := \{w_1, w_2, \dots, w_n\}$ be bases of V and W respectively. Define the linear transformation $T: V \to W$ by $T(v_i) = w_i$. Now, $\text{rank}(T) = \dim(W) = \dim(V)$ is clear, so by Theorem 2.5, T is bijective. And hence, is an isomorphism.

Self-Proof of Theorem 2.20

Ideas

Linearity: $\Phi_\beta^\gamma(cT + U) = c\Phi_\beta^\gamma(T) + \Phi_\beta^\gamma(U)$ by Theorem 2.8

Injectivity:

$$\Phi_\beta^\gamma(T) = \Phi_\beta^\gamma(U)$$
$$T(v_j) = U(v_j) \quad \text{for any } 1 \le j \le n$$
$$T = U$$

(possible by Thm 2.6)

Surjectivity: Let $A \in M_{m\times n}(F)$. Define the l.t. $T: V \to W$ by

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

Proof

By Theorem 2.8, Φ_β^γ must be linear. Let $\beta := \{v_1, v_2, \dots, v_n\}$ and $\gamma := \{w_1, w_2, \dots, w_m\}$. Injectivity must also hold because when $\Phi_\beta^\gamma(T) = \Phi_\beta^\gamma(U)$, $T(v_j) = U(v_j)$ for any $1 \le j \le n$, so $T = U$ by Theorem 2.6. Similarly, surjectivity holds because of Theorem 2.6 asserting the existence of a linear transformation $T: V \to W$ for which $T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$ given any matrix $A \in M_{m\times n}(\mathbb{F})$. Hence, $\Phi_\beta^\gamma(T) = A$. As such Φ_β^γ is indeed an isomorphism.

Corollary

$M_{m\times n}(\mathbb{F})$ is clearly of dimension mn so by Theorem 2.19, so must $\mathcal{L}(V, W)$.

Self-Proof of Theorem 2.17

Ideas

$T: V \to W \quad T^{-1}: W \to V$

$T^{-1}(cw_1 + w_2) = T^{-1}(cT(v_1) + T(v_2)) = T^{-1}(T(cv_1 + v_2)) = cv_1 + v_2 = cT^{-1}(w_1) + T^{-1}(w_2)$

Corollary

Proof ~~If V is finite dimensional,~~ By Theorem 2.5, $\dim(V) = \text{rank}(T) = \dim(W)$ as $R(T) = W$. □

Example 2

$\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus, the inverse of $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$ is indeed $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$.

Self-Proof of Theorem 2.18

Ideas

If T is invertible, let u_j be the jth column of $[T]_\beta^\gamma [T^{-1}]_\beta^\gamma$ and v_j the jth column of $[T]_\beta^\gamma$ ~~$([T]_\beta^\gamma [T^{-1}]_\beta^\gamma)_{ij}$~~ $= \sum_{k=1}^{n} ([T]_\beta^\gamma)_{ik} ([T^{-1}]_\beta^\gamma)_{kj}$

$u_j = [T]_\beta^\gamma v_j$

$[T]_\beta^\gamma [T^{-1}]_\beta^\gamma = [TT^{-1}]_\beta^\gamma = [I_V]_\beta^\gamma = I_n$

$A := [T]_\beta^\gamma, \quad B := A^{-1}$

Define $U: W \to V$ with $U(w_j) = \sum_{i=1}^{n} B_{ij} v_i$

$T(U(w_j)) = \sum_{i=1}^{n} B_{ij} T(v_i)$
$= \sum_{i=1}^{n} B_{ij} \sum_{k=1}^{n} A_{ki} w_k = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} A_{ik} B_{kj}\right) w_i = \sum_{i=1}^{n} (I_n)_{ij} w_i$
$= w_j$

$[T^{-1}]_\beta^\gamma [T]_\beta^\gamma = \ldots$

Proof Consider T being invertible.

Let $\dim(V) = n$. By Theorem 2.11, $[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [TT^{-1}]_\beta = [I_V]_\beta = I_n$. Similarly, $[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = I_n$, so equality holds. Furthermore, we have now shown that $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. Conversely, when $A := [T]_\beta^\gamma$ has some inverse B, define $U: W \to V$ with $U(w_j) = \sum_{i=1}^{n} B_{ij} v_i$ where $\beta = \{v_1, v_2, \ldots, v_n\}$, $\gamma = \{w_1, w_2, \ldots, w_n\}$. Now we see that

$T(U(w_j)) = \sum_{k=1}^{n} B_{kj} T(v_k) = \sum_{k=1}^{n} B_{kj} \sum_{i=1}^{n} A_{ik} w_i = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} A_{ik} B_{kj}\right) w_i = \sum_{i=1}^{n} (I_n)_{ij} w_i = w_j$.

It can be shown similarly that $U(T(v_j)) = v_j$. It is now clear that $U = T^{-1}$. □

Note: Oops I thought we didn't have this fact (quite simple to prove anyways) but we already know on page 81 that $[U]_\beta^\gamma = [T]_\beta^\gamma$ [illegible]

Proof of Theorem 2.21

Follows similarly as Theorem 2.20.

Exercises

1. (a) ~~True~~ False, T is not said to be invertible / bijective.
(b) True
(c) False
(d) False
(e) True
(f) False
(g) True
(h) True
(i) True

2. (c) Ideas: / checking

$3a_1 - 2a_3 = 3b_1 - 2b_3$ $\quad 3a_1 + \cancel{4a_2} = 3b_1 + \cancel{4b_2}$ $\quad a_1 = b_1$

$3a_1 - 2a_3 = \alpha$ $\quad a_2 = \beta$ $\quad 3a_1 + 4a_2 = \gamma$

$\gamma - 4\beta - 2a_3 = \alpha$ $\quad a_3 = \frac{\gamma - 4\beta - \alpha}{2}$

$3a_1 + 4\beta = \gamma$ $\quad a_1 = \frac{\gamma - 4\beta}{3}$

Proof
By solving simultaneous equations, we can easily verify T to be bijective, i.e. invertible. So, yes T is invertible.

(d) T cannot be invertible since $T(1) = T(2) = 0$ tells us T is not injective.

Core

This

Cor.

Any

So

Exe

For

4. Proof

By the associativity of matrix multiplication shown in Theorem 2.16,

$$(AB)(B^{-1}A^{-1}) = [(AB)B^{-1}]A^{-1} = [A(BB^{-1})]A^{-1} = AA^{-1} = I.$$

It is clear that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$ since $(A^{-1})^{-1} = A$. Hence, AB is indeed invertible, and in fact, $(AB)^{-1} = B^{-1}A^{-1}$.

5. From page 89, we know $(AB)^t = B^tA^t$ for any suitable matrices A and B (for which AB exists). As such,

$$A^t[(A^{-1})^t] = [A^{-1}A]^t = I \quad \text{and} \quad [(A^{-1})^t]A^t = (AA^{-1})^t = I.$$

So, A^t is indeed invertible, with $(A^t)^{-1} = (A^{-1})^t$.

6. If A is invertible and $AB = O$, $A^{-1}(AB) = A^{-1}O$ so $(A^{-1}A)B = B = O$ by associativity (again, see Theorem 2.16).

7. (a) Suppose, for the sake of contradiction, that $A^2 = O$ but A is invertible. Then, $A^2A^{-1} = A = OA^{-1} = O$. However, AA^{-1} is now O, a contradiction.

(b) Presume A is invertible, and $AB = O$. Then $A^{-1}(AB) = I_nB = B = A^{-1}O = O$. Thus, if $AB = O$ and $B \neq O$, then A can't be invertible.

8. See the self-proof of these corollaries.

9. (a) If one of them, say A, isn't invertible, then by Theorem 2.5, $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is neither injective nor surjective. Therefore, L_{AB} cannot be surjective either. Thence implying AB is not invertible.

(b) $(a_1\ a_2\ a_3)\begin{pmatrix} a_1^{-1} \\ a_2^{-1} \\ a_3^{-1} \end{pmatrix} = (1)$ which is its own inverse.

10. (a) Since $AB = I_n$, it is its own inverse so exercise 9 says A and B are invertible.

(b) As A and B are invertible, $(AB)B^{-1} = A = I_nB^{-1} = B^{-1}$.

(c) For linear transformations $T : V \to W$, $U : W \to Z$, let $n := \dim(V) = \dim(Z)$ and $m := \dim(W)$. We cannot generalize this to cases where $n \neq m$ because (a) wouldn't always hold. As an example, let $V := \mathbb{R}^3$, $W := \mathbb{R}^4$, $Z = \mathbb{R}^3$, and $T(a,b,c) := (a,b,c,0)$, $U(a,b,c,d) = (a,b,c)$. Then $UT = I_{\mathbb{R}^3}$, which is its own inverse, while U is not invertible, for instance $U(1,1,1,0) = U(1,1,1,1) = (1,1,1)$. So consider $n = \dim(V) = \dim(W) = \dim(Z)$ and further assume $V = Z$ so $UT = I_V$. For an analogous result of (a) and (b) to make sense, UT is invertible similarly to exercise 9; Theorem 2.5 informs us U and T must both be invertible. So, as in (b), $U = (UT)T^{-1} = I_VT^{-1} = T^{-1}$. And U^{-1} and T^{-1} exist as shown above.

23. Notice $T(c\sigma+\sigma') = \sum_{i=0}^{n}(c\sigma+\sigma')(i)x^i = c\sum_{i=0}^{n}\sigma(i)x^i + \sum_{i=0}^{n}\sigma'(i)x^i = cT(\sigma)+T(\sigma')$, thus T is linear. Let $Q(x) = \sum_{i=0}^{n} c_i x^i \in P(F)$. Thus, the sequence σ with $\sigma(i) = c_i$ is the unique sequence so that $T(\sigma) = Q(x)$. Accordingly, T is invertible and hence an isomorphism.

22. Ideas

$$T(f) = T(f')$$

$$\sum_{i=0}^{n} a_i c_k^i = \sum_{i=0}^{n} b_i c_k^i \quad \text{for any } 0 \le k \le n$$

$$\sum_{i=0}^{n} (a_i - b_i) c_k^i = 0$$

$$a_{n+1} = \sum_{i=0}^{n+1} b_i c_k^{i-n-1} - \sum_{i=0}^{n} a_i c_k^{i-n-1} = \sum_{i=0}^{n} (b_i - a_i) c_k^{i-n-1} + b_{n+1}$$

$$a_{n+1} - b_{n+1} = \sum_{i=0}^{n} (b_i - a_i) c_k^{i-n-1}$$

$$(a_{n+1} - b_{n+1}) c_k^{n+1} = \sum_{i=0}^{n} (b_i - a_i) c_k^i$$

$$\sum_{i=0}^{n} (b_i - a_i) c_k^i = \sum_{i=0}^{n} (b_i - a_i) c_{k+1}^i$$

$$\sum_{i=0}^{n} (b_i - a_i)(c_k^i - c_{k+1}^i)$$

12. See self-proof of Theorem 2.21

15. Assume that T is an isomorphism. Thus, by Theorem 2.2, $W = \operatorname{span}(T[\![\beta]\!])$. Since $\dim(V) = \dim(W) = n$, we can be sure that $T[\![\beta]\!]$ is linearly independent, and hence, must be a basis for W. For the converse, see the proof / self-proof of Theorem 2.19. □

16. Linearity: $\Phi(cA+B) = B^{-1}(cA+B)B = (cB^{-1}A + I_n)B = cB^{-1}AB + B = c\Phi(A) + \Phi(B)$

Injectivity: Presume $\Phi(A) = \Phi(A')$. Then, $A = B(B^{-1}AB)B^{-1} = B(B^{-1}A'B)B^{-1} = A'$.

Hence, by Theorem 2.5, bijectivity is certain.

Consequently, Φ is indeed an isomorphism as expected.

17. (a) By Theorem 2.1, $T[\![V_0]\!] = \operatorname{ran}(T\restriction V_0)$ is a subspace of W (as $T\restriction V_0$ is still linear).

(b) Clearly, the linear transformation $T\restriction V_0 : V_0 \longrightarrow T[\![V_0]\!]$ is an isomorphism, so $\dim(V_0) = \dim(T[\![V_0]\!])$ by Theorem 2.19. □

20. First notice that $\operatorname{rank}(L_A\phi_\beta) = \operatorname{rank}(L_A)$ since $\operatorname{ran}(L_A\phi_\beta) = \operatorname{ran}(L_A)$ since ϕ_β is an isomorphism. Similarly, $\operatorname{rank}(T\phi_\beta) = \operatorname{rank}(T)$ as exercise 17 tells us $\operatorname{rank}(\phi_\gamma\restriction T[\![\beta]\!]) = \operatorname{rank}(T)$. Hence, from the commutative diagram that is Fig 2.2, $\operatorname{rank}(L_A) = \operatorname{rank}(L_A\phi_\beta) = \operatorname{rank}(\phi_\gamma T) = \operatorname{rank}(T)$. Using the rank-nullity / dimension theorem, $\operatorname{rank}(L_A) + \operatorname{nullity}(L_A) = n = \operatorname{rank}(T) + \operatorname{nullity}(T)$ so $\operatorname{nullity}(T) = \operatorname{nullity}(L_A)$. □

21. For $\sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij} = T_0$, $\sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij}(v_k) = \sum_{i=1}^{m} c_{ik}w_i = 0$ for each v_k so c_{ik} is always 0 by γ's linear independence. Hence, so is $\{T_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$. Let $U \in \mathcal{L}(V, W)$. For every v_k, $U(v_k) = \sum_{i=1}^{m} c_{ik}w_i = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij}(v_k)$. Therefore by Theorem 2.6, $U = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}T_{ij}$. As such, $\{T_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is indeed a basis for $\mathcal{L}(V, W)$. Since $T_{ij}(v_j) = w_i$, in the (i, j)th entry we have $[T_{ij}]_\beta^\gamma$ being 1. Everywhere else, its entries are zero because $T_{ij}(v_k) = 0_W$ if $k \ne j$. Accordingly, $[T_{ij}]_\beta^\gamma = M^{ij}$ as expected. It is clear that $\{M^{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m\times n}(\mathbb{F})$. As such, exercise 15 tells us Φ_β^γ is an isomorphism.

22. Ideas

When $n=0$, immediately true

Assume true for n, then

$$\frac{\sum_{i=0}^{n+1} a_i x^i}{x-r} = a_{n+1} x^n + \frac{\left[(a_{n+1} r + a_n) x^n + \sum_{i=1}^{n-1} a_i x^i\right]}{x-r}$$

$$\begin{array}{r|l} & a_{n+1} x^n \\ x-r & a_{n+1} x^{n+1} + a_n x^n + \dots + a_0 \\ & -(a_{n+1} x^{n+1} - a_{n+1} r x^n) \\ & (a_{n+1} r + a_n) x^n + \dots + a_0 \end{array}$$

$$= a_{n+1} x^n + q(x) + \frac{k}{x-r} \quad \text{by ih}$$

$$p(x) = (x-r) q(x) + k$$

$$p(c_i) = 0 \rightarrow p(x) = (x-r) q(x)$$

Proof

Lemma 1

Notice that for any zeroth degree polynomial $c \in P_n(\mathbb{F})$ and any $r \in \mathbb{F}$, $c = 0(x-r) + c$ obviously. Now assume that for any $p(x) \in P_n(\mathbb{F})$ of degree $m \le n$ and each $r \in \mathbb{F}$, there exists $q(x) \in P_n(\mathbb{F})$ of degree $m-1$, and scalar $k \in \mathbb{F}$ so $p(x) = (x-r) q(x) + k$. As such, given any $(m+1)$th degree polynomial $p(x) \in P_n(\mathbb{F})$, similarly we have $p(x) = (a_{n+1} x^n)(x-r) + (a_{n+1} r + a_n) x^n + \sum_{i=1}^{n-1} a_i x^i = (a_{n+1} x^n + q(x))(x-r) + k$, where $\deg(a_{n+1} x^n + q(x)) = m$ as expected. Hence, this is true of all polynomials in $P_n(\mathbb{F})$. $\diamond$

Lemma 2

Suppose $p(x) \in P_n(\mathbb{F})$ and $x = r$ is one of its roots. Thus, by Lemma 1, $p(r) = (r-r) q(x) + k = 0$ so $k = 0$. Implying that $p(r) = (x-r) q(x)$. $\diamond$

Linearity (of T) is clear from definition, as $(af+g)(c_i) = af(c_i) + g(c_i)$. Theorems 2.4 and 2.5 informs us that it suffices to show nullity$(T) = 0$. Accordingly, suppose $T(f) = 0$. Then given $0 \le i \le n$, $f(c_i) = 0$. Therefore, from Lemma 2, it holds that $f(x) / \prod_{i=0}^{n} (x-c_i) \in P_n(\mathbb{F})$. As a result, $\deg(f) - (n+1) \ge 0$ (as a polynomial) lest $\deg(f) = -1$ and f is the zero polynomial, but then $\deg(f) \ge n+1 > n$, a contradiction. Hence, nullity$(T) = 0$. Which tells us that, indeed, T is an isomorphism. $\square$

Self-Proof of Theorem 2.22

(a) This holds by Theorem 2.18 since I_V has an inverse, namely itself.

(b) By Theorem 2.14, $[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$

Self-Proof of Theorem 2.23

By Theorems 2.11 and 2.18, $[T]_{\beta'} = [I_V]_\beta^{\beta'}[T]_{\beta'}^{\beta} = [I_V]_\beta^{\beta'}[T]_\beta[I_V]_{\beta'}^{\beta} = \left([I_V]_{\beta'}^{\beta}\right)^{-1}[T]_\beta[I_V]_{\beta'}^{\beta} = Q^{-1}TQ.$

Example 2

and $T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + 3\begin{pmatrix}1\\-1\end{pmatrix}$ so $[T]_\beta = \begin{pmatrix}3&1\\-1&3\end{pmatrix}$ indeed.

$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}$

$T\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}2\\14\end{pmatrix} = 4\begin{pmatrix}2\\4\end{pmatrix} - 2\begin{pmatrix}3\\1\end{pmatrix}$ and $T\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}8\\6\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} + 2\begin{pmatrix}3\\1\end{pmatrix}$ hence $[T]_{\beta'} = \begin{pmatrix}4&1\\-2&2\end{pmatrix}$.

Example 3

$T(0,1) = \frac{1}{5}T\big(2(1,2)+(-2,1)\big) = \frac{2}{5}(1,2) + \frac{1}{5}(2,-1) = \frac{1}{5}(4,3)$

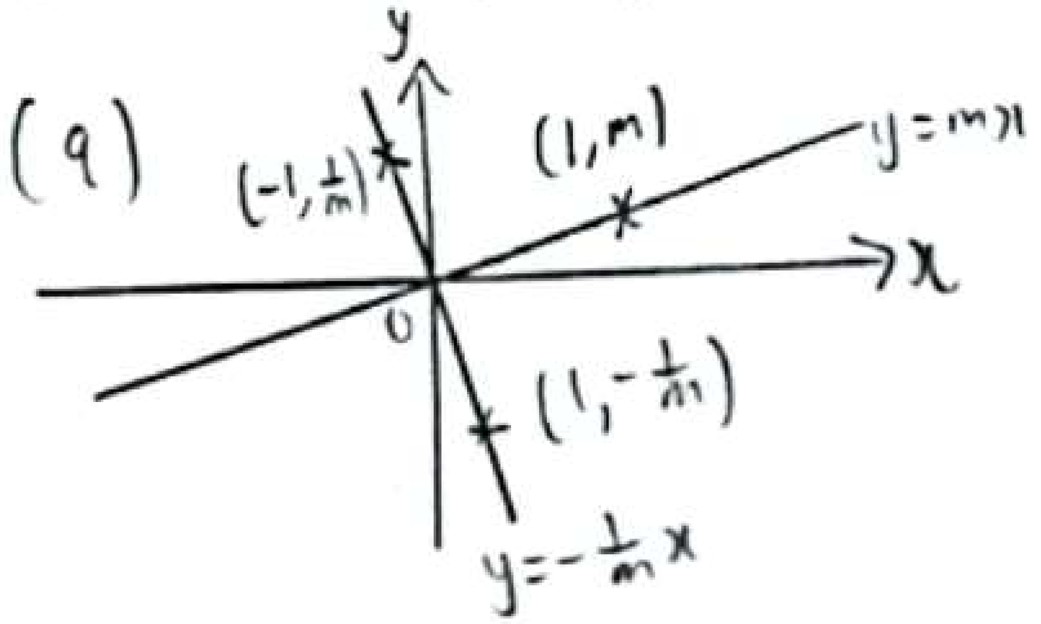
$T(1,0) = \frac{1}{5}T\big((1,2)-2(-2,1)\big) = \frac{1}{5}(1,2) - \frac{2}{5}(2,-1) = \frac{1}{5}(-3,4)$

so $[T]_\beta = \frac{1}{5}\begin{pmatrix}-3&4\\4&3\end{pmatrix}$

24. The equivalence relation $\sim$ on V defined by $u \sim v$ iff $u - v \in N(T)$ is being induced, where T respects $\sim$ (i.e. if $u \sim v$ then $T(u) = T(v)$). Hence, the necessary results are straightforward from Theorem I. □

25. Notice $\Psi(cf+g) = \sum_{s \in S, (cf+g)(s) \neq 0} (cf+g)(s) \cdot s = \sum_{s \in S, (cf+g)(s) \neq 0} cf(s)s + g(s)s = c \sum_{s \in S, (cf+g)(s) \neq 0} f(s)s + \sum_{s \in S, (cf+g)(s) \neq 0} g(s)s = c\Psi(f) + \Psi(g)$ because if $(cf+g)(s) = 0$, $cf(s) = -g(s)$ so they still 'cancel' out even if $f(s)$ or $g(s)$ is nonzero. Hence, linearity is certain. As such, now consider $v \in V$. For some unique vectors $u_i \in S$ and scalars $c_i \in \mathbb{F}$, $v = \sum_{i=1}^{n} c_i u_i$ by exercise 5 of section 1.7. Thus, define the unique function $f : S \longrightarrow \mathbb{F}$ with $f(u_i) = c_i$ and $f(w) = 0$ otherwise. Accordingly, $\Psi(f) = \sum_{s \in S, f(s) \neq 0} f(s)s = \sum_{i=1}^{n} c_i u_i = v$. Consequently, Ψ is indeed a bijection, wherefore an isomorphism when paired with linearity. □

7. (a)



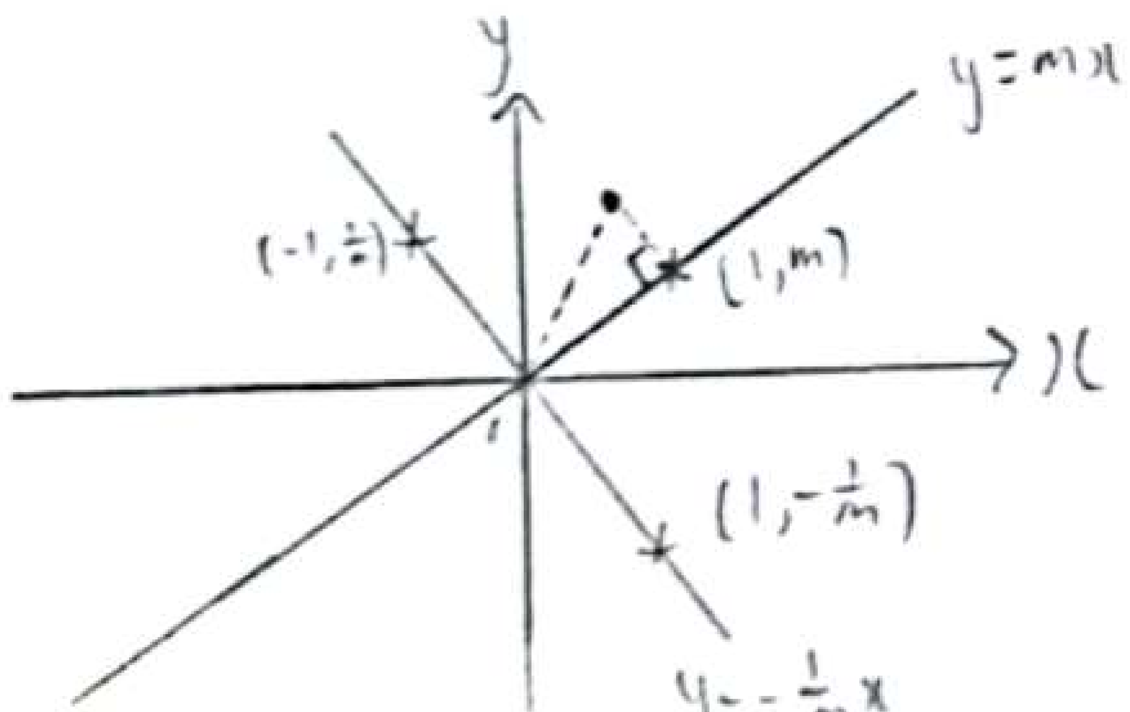
Define the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ representing the reflection of $\mathbb{R}^2$ about L by $T(1,m) = (1,m)$ and $T(-1, \frac{1}{m}) = (1, -\frac{1}{m})$. In other words, we have that $[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $\beta' := \{(1,m), (1, -\frac{1}{m})\}$. And let $\beta := \{(1,0),(0,1)\}$. So the change of coordinate matrix Q that changes β'-coordinates to β coordinates is given by $\begin{pmatrix} 1 & 1 \\ m & -\frac{1}{m} \end{pmatrix}$. Further, since $(1,0) = \frac{1}{m^2+1}[(1,m) + m^2(1, -\frac{1}{m})]$ and $(0,1) = \frac{m}{m^2+1}[(1,m) - (1, -\frac{1}{m})]$, $Q^{-1} = \frac{1}{m^2+1}\begin{pmatrix} 1 & m \\ m^2 & -m \end{pmatrix}$. Therefore, by Theorem 2.23,

$$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \frac{1}{m^2+1}\begin{pmatrix} 1 & -1 \\ -m & -\frac{1}{m} \end{pmatrix}\begin{pmatrix} 1 & m \\ m^2 & -m \end{pmatrix} = \frac{1}{m^2+1}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}.$$

Consequently, $T(x,y) = \frac{1}{m^2+1}(-m^2x + 2my + x,\ m^2y + 2mx - y)$

To verify, let's compute $T(1,m) = \frac{1}{m^2+1}(-m^2 + 2m^2 + 1,\ -m^3 - 2m + m) = (1, m)$
and $T(1, -\frac{1}{m}) = \frac{1}{m^2+1}(-m^2 - 1,\ m + \frac{1}{m}) = (-1, \frac{1}{m})$ as expected.

(b)



The projection $T: \mathbb{R}^2 \to \mathbb{R}^2$ on L along the line perpendicular to L is given by $T(1,m) = (1,m)$ and $T(1, -\frac{1}{m}) = (0,0)$. Notice $T(1,0) = T\left(\frac{1}{m^2+1}(1,m) + \frac{m^2}{m^2+1}(1, -\frac{1}{m})\right) = \frac{1}{m^2+1}(1,m)$ and $T(0,1) = T\left(\frac{m}{m^2+1}(1,m) - \frac{m}{m^2+1}(1, -\frac{1}{m})\right) = \frac{m}{m^2+1}(1,m)$.

Hence, $[T]_\beta = \frac{1}{m^2+1}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$. In other words, $T(x,y) = \frac{1}{m^2+1}(x + my,\ mx + m^2y)$.

To check, again we compute $T(1,m) = \frac{1}{m^2+1}(1 + m^2,\ m^2 + m^3) = (1,m)$ and
$T(1, -\frac{1}{m}) = \frac{1}{m^2+1}(1-1,\ m - m) = (0,0)$ as necessary.

Exercises

1. (a) False, it should be $[x'_j]_\beta$

(b) True

(c) True

3. (d) False

(e) True

2. (a)
Note $(a_1, a_2) = a_1 e_1 + a_2 e_2$ and $(b_1, b_2) = b_1 e_1 + b_2 e_2$. Thus, the required matrix is $Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$.

(c)
We see that $e_1 = 3(2,5) + 5(-1,-3)$ and $e_2 = -(2,5) - 2(-1,-3)$. So, the change of coordinate matrix necessary is $Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$

5. First, to find the change of coordinate matrix Q that changes β'-coordinates to β-coordinates, notice that $1+x = 1(1) + 1(x)$ and $1-x = 1(1)+(-1)(x)$; therefore $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Furthermore, since $T(1) = 0$ and $T(x) = 1$, $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Accordingly, Theorem 2.23 informs us that
$$[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$
To verify this result, we find that $T(1+x) = 1 = \frac{1}{2}(1+x) + \frac{1}{2}(1-x)$ and $T(1-x) = -1 = -\frac{1}{2}(1+x) - \frac{1}{2}(1-x)$ such that $[T]_{\beta'} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

6. (a)
Notice $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 11\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Hence, $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$.
Similarly, $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Now we wish to find Q^{-1} so $QQ^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a+2c & b+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Solving the corresponding simultaneous equations, we find that $a = 2$, $b = c = -1$ and $d = 1$. In other words, $Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ such that $Q^{-1}AQ = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$ as expected.

(c)
We see that $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. We now want some Q^{-1} with $QQ^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+d+g & b+e+h & c+f+i \\ a+g & b+h & c+i \\ a+d+2g & b+e+2h & c+f+2i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Again solving the corresponding simultaneous equations, $a = 1$, $b = 1$, $c = -1$, $d = 1$, $e = -1$, $f = 0$, $g = -1$, $h = 0$, $i = 1$. Therefore, $Q^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and now
$$[L_A]_\beta = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}.$$
lmao happened to just nice do the 2 parts where the ans are provided, not that I necessarily mind checking.

(b) For a linear operator T on a finite n-dimensional vector space V, we can define $tr(T) := tr\,[T]_s$, where s is the standard ordered basis for $\mathbb{F}^n$. Well-definedness comes by the fact that $[T]_s$ is unique as the action of T on a fixed basis s is fixed.

1. (a) By Theorem 2.11, $RQ = [I]_\beta^\gamma [I]_\alpha^\beta = [I]_\alpha^\gamma$, which by definition is the change of coordinate matrix that changes α-coordinates into γ-coordinates

(b) This is straightforward from Theorem 2.18.

2. Straightforward from Theorem 2.23.

3. Define the invertible linear transformation $T : V \longrightarrow V$ by $T(x_j) = \sum_{i=1}^{n} Q_{ij} x_i = x'_j$. Since T is a surjection, Theorem 2.2 tells us span $\beta' = V$. Furthermore, rank$(T) = n$ so β' is certainly a basis for β'. Now, $[I_V]_{\beta'}^{\beta} = Q$ too as a result.

4. Ideas

$V := \mathbb{F}^n$ $\quad \beta := \{e_1, e_2, \dots, e_n\}$ $\quad \beta'$ as in 13. $\quad Q$

$W := \mathbb{F}^m$ $\quad \gamma := \{e_1, e_2, \dots, e_m\}$ $\quad \gamma'$ $\quad P$

$B = P^{-1}AQ$

$[T]_{\beta'}^{\gamma'} = [I_W]_{\gamma}^{\gamma'} [T]_{\beta}^{\gamma} [I_V]_{\beta'}^{\beta}$

Proof

One possible choice is to let $V := \mathbb{F}^n$, $W := \mathbb{F}^m$, $\beta := \{e_1, e_2, \dots, e_n\}$, and $\gamma := \{e_1, e_2, \dots, e_m\}$. Define $x'_j := \sum_{i=1}^{n} Q_{ij} e_i$ and $\beta' := \{x'_j \mid 1 \le j \le n\}$, $y'_j := \sum_{i=1}^{m} P_{ij} e_i$ and $\gamma' := \{y'_j \mid 1 \le j \le m\}$. By exercise 14, Q is the change of coordinate matrix changing β'-coordinates to β-coordinates and P the change of coordinate matrix that changes γ' coordinates to γ coordinates. Now, $[L_A]_{\beta'}^{\gamma'} = [I_W]_{\gamma}^{\gamma'} [L_A]_{\beta}^{\gamma} [I_V]_{\beta'}^{\beta} = P^{-1}AQ = B$.

8. Ideas

$$P^{-1}[T]_\beta^\gamma Q = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta = [I_W]_\gamma^{\gamma'}[TI_V]_{\beta'}^\gamma = [T]_{\beta'}^{\gamma'}$$ by Theorem 2.11

Proof

We notice that $[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'}[TI_V]_{\beta'}^\gamma = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta = P^{-1}[T]_\beta^\gamma Q$ by Theorems 2.11 and 2.18. □

9. Reflexivity

This is immediate since $B = I_n B I_n = I_n^{-1} B I_n$, i.e. B is similar to itself.

Symmetry

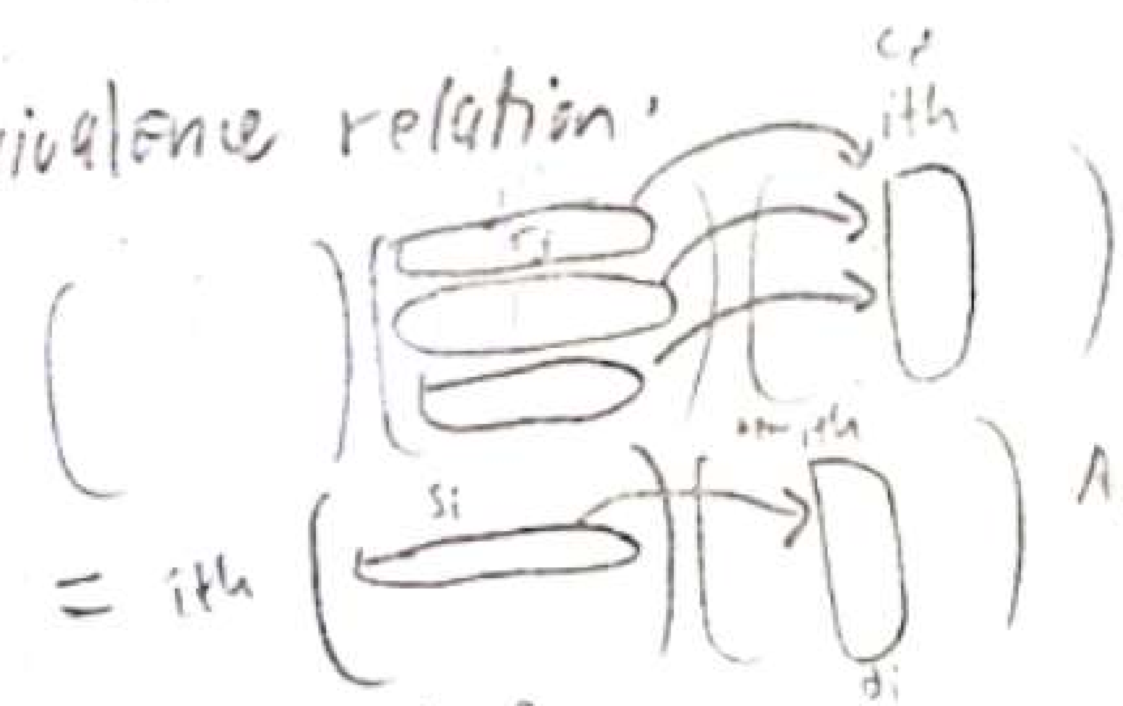
When B is similar to A, that is $B = Q^{-1}AQ$ for some invertible $n \times n$ matrix Q, $A = QBQ^{-1} = (Q^{-1})^{-1}B(Q^{-1})$. So, A is similar to B too.

Transitivity

If A is similar to B and B is similar to C, in other words there exists $n \times n$ invertible matrices P and Q such that $A = P^{-1}BP$ and $B = Q^{-1}CQ$, $A = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP)$. Which also means A is similar to C.

then since $B = PAP^{-1}$, $PAP^{-1} = Q^{-1}CQ$. Therefore,

With all three properties satisfied, similarity of $n \times n$ matrices indeed induces an equivalence relation. □



10. (a) Ideas

$$QB = AQ$$

$$Q^{-1}Q = I_n$$

$$\sum_{k=1}^n (Q^{-1})_{ik}Q_{ki} = 1$$

$$B_{ii} = \sum_{k=1}^n (Q^{-1})_{ik}(AQ)_{ki} = \sum_{k=1}^n (Q^{-1})_{ik}\sum_{\ell=1}^n A_{k\ell}Q_{\ell i} = \sum_{k=1}^n\sum_{\ell=1}^n (Q^{-1})_{ik}A_{k\ell}Q_{\ell i}$$

$$\sum_{i=1}^n\sum_{k=1}^n\sum_{\ell=1}^n (Q^{-1})_{ik}A_{k\ell}Q_{\ell i} = \sum_{k=1}^n\sum_{\ell=1}^n \underbrace{\sum_{i=1}^n Q_{\ell i}(Q^{-1})_{ik}}_{(I_n)_{\ell k}} A_{k\ell} = \sum_{k=1}^n\sum_{\ell=1}^n A_{k\ell}(I_n)_{\ell k} = \sum_{k=1}^n (AI_n)_{kk} = \sum_{k=1}^n A_{kk}$$

= ith

$A_{ii} = s_i \cdot d_i$

$s_i \sum_{j=1} r_j c_i$

Proof

Since A and B are similar, $A = Q^{-1}BQ$ for some invertible $n \times n$ matrix Q. Hence, $tr(A) = \sum_{i=1}^n\sum_{k=1}^n (Q^{-1})_{ik}(BQ)_{ki} = \sum_{i=1}^n\sum_{k=1}^n (Q^{-1})_{ik}\sum_{\ell=1}^n B_{k\ell}Q_{\ell i} = \sum_{i=1}^n\sum_{k=1}^n\sum_{\ell=1}^n (Q^{-1})_{ik}B_{k\ell}Q_{\ell i} = \sum_{k=1}^n\sum_{\ell=1}^n\sum_{i=1}^n B_{k\ell}Q_{\ell i}(Q^{-1})_{ik} = \sum_{k=1}^n\sum_{\ell=1}^n B_{k\ell}(I_n)_{\ell k} = \sum_{k=1}^n (BI_n)_{kk} = tr(B)$.

$B = \begin{pmatrix} c_j \end{pmatrix}$ c_j is the action of B on e_j

Ideas for alternate proof

$[L_B]_\gamma = [I_n]_s^\gamma[L_B]_s^s[I_n]_\gamma^s$

$tr[L_B]_\gamma = tr[L_B]_s = tr(B)$

$\gamma = \{w_1, w_2, \dots\}$, $s = \{e_1, e_2, \dots, e_n\}$

$L_B(w_j) = \sum_{i=1}^n c_{ij}w_i$ jth of $L_B(w_j)$ is $c_{jj}w_j$

Self - Proof of Theorem 2.26

Linearity is clear so since the previous Lemma informs us that $\text{nullity}(\psi) = 0$, which means ψ is injective by Theorem 2.4. Furthermore, for any $G : V^* \longrightarrow \mathbb{F}$, letting $\beta := \{v_1, v_2, \dots, v_n\}$ be a basis for V, we can define $x := \sum_{i=1}^{n} G(f_i) v_i$ such that $\hat{x}(f_i) = G(f_i)$. By Theorem 2.6, $\hat{x} = G$. In other words, $\psi(x) = G$ and ψ is surjective. Now, we can be certain it is an isomorphism. $\square$

Yeah Or just use Thm 2.5, (directly) as expected, but this way is a bit funner

Self - Proof of Corollary

Let $\beta^* := \{f_1, f_2, \dots, f_n\}$ be a basis for V^* and $\beta^{**} := \{g_1, g_2, \dots, g_n\}$ the dual basis of β^*. Using Theorem 2.26, define $u_i := \psi^{-1}(g_i)$ for $1 \leq i \leq n$. Which must be a basis for V by Theorem 2.2. Furthermore, we notice that since $\psi(u_i) = g_i$, $f_j(u_i) = g_i(f_j)$, which is 1 iff $i = j$ and 0 otherwise. As such, it is clear that β^* is the dual basis of $\beta := \{u_1, u_2, \dots, u_n\}$. $\square$

Self-Proof of Theorem 2.24

Ideas
$$f(x_i) = f(x_i)\overbrace{f_i(x_i)}^{1} \qquad \sum_{i=1}^{n} f(c_i x_i) = \sum f(x_i)\cdot c_i = \sum f(x_i) f_i(c_i x_i)$$

Proof

Given $v \in V$, we see that for some scalars $c_i \in \mathbb{F}$, $f(v) = \sum_{i=1}^{n} f(c_i x_i) = \sum_{i=1}^{n} f(x_i)\cdot c_i = \sum_{i=1}^{n} f(x_i) f_i(c_i x_i) = \sum_{i=1}^{n} f(x_i) f_i(v)$. Therefore, $f = \sum_{i=1}^{n} f(x_i) f_i$ holds.

As such, β^* spans V^*. And hence, must be a basis for V^* (as $|\beta^*| = n$). □

Self-Proof of Theorem 2.25

$n := \dim(V)$, $m := \dim(W)$

ideas $T^t(g_j) = g_j T = \sum_{i=1}^{n} (g_j \circ T)(x_i) f_i \qquad \left([T^t]_{\gamma^*}^{\beta^*}\right)_{ij} = (g_j \circ T)(v_i) \qquad T(v_j) = \sum_{i=1}^{n} \left([T]_\beta^\gamma\right)_{ij} w_i$

$$= g_j\left(\sum_{k=1}^{n} \left([T]_\beta^\gamma\right)_{ki} w_k\right) = \left([T]_\beta^\gamma\right)_{ji}$$

Linearity certainly holds since $T^t(cg + g') = (cg+g')T = cgT + g'T$.

Proof

Let $\beta := \{v_1, v_2, \dots, v_n\}$, $\beta^* := \{f_1, f_2, \dots, f_n\}$, $\gamma := \{w_1, w_2, \dots, w_m\}$, $\gamma^* := \{g_1, g_2, \dots, g_m\}$. By Theorem 2.24, $T^t(g_j) = g_j T = \sum_{i=1}^{n} (g_j \circ T)(v_i) f_i$. Accordingly, $\left([T]_{\gamma^*}^{\beta^*}\right)_{ij} = (g_j \circ T)(v_i) = g_j\left(\sum_{k=1}^{n} \left([T]_\beta^\gamma\right)_{ki} w_k\right) = \left([T]_\beta^\gamma\right)_{ji}$, so that $[T^t]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t$. □

Self-Proof of Lemma

Ideas: If $x \neq 0$, $f_i(x) \neq 0$ for some i $\qquad f(x) = 0$ for all $f \in V^*$

Proof: Let $\beta := \{u_1, u_2, \dots, u_n\}$ be a basis for V and suppose $x \in V$ is nonzero. Then there exists scalars $c_i \in \mathbb{F}$ with $x = \sum_{i=1}^{n} c_i u_i$ where some $c_j \neq 0$. As such, $\hat{x}(f_j) = f_j(x) = c_j \neq 0$. □

Guess for what Theorem 2.26 is about: The transformation $X : V \to V^{**}$ defined by $X(x) = \hat{x}$ is an isomorphism.

Ideas $V^{**} = \mathcal{L}(\mathcal{L}(V, \mathbb{F}), \mathbb{F})$ $\qquad G \in V^{**} \Rightarrow G : V^* \to \mathbb{F}$

$\underset{\hat{x}}{\overset{\Psi}{}}$ $\qquad$ for all f, $G(f) = c \overset{!}{=} f(x)$ for some x?

Counter-ing. Not necessarily. We can define $G(f_i) = 0$ (where $\mathbb{F} = \mathbb{R}$) so $f_i(x)$

$X(x) = 0$

$\hat{x} = 0$-function

$\hat{x}(f) = 0$ for any $f \in V^*$

$X : V \to V^{**}$, $X(x) = \hat{x}$

By Lemma, nullity$(X) = 0$. Theorem 2.4 says X is injective.

let $G : V^* \to \mathbb{F}$ $\qquad$ want to find x so $f_i(x) = G(f_i)$

$f : V \to \mathbb{F}$ $\qquad (c\widehat{x+y})(f) = f(cx+y)$

$= cf(x) + f(y)$ by linearity

$= c\hat{x}(f) + \hat{y}(f)$

$G(f_i) = c_i \qquad x := \sum_{i=1}^{n} G(f_i) u_i$ so $\hat{x}(f_i) = f_i(x) = G(f_i)$ Surjectivity

Exercises

1\. (a) False

T (b) False. If $\dim(V) = n$, then a linear functional $T: V \to \mathbb{F}$ would have a $1 \times n$ matrix representation.

T (c) False, true only if V is of finite dimension.

T (d) False. Let V be the vector space of all sequences $s: \mathbb{N} \to \mathbb{N}$ with a finite number of nonzero entries, over the field of itself, $\mathbb{N}$. Then it is clear that $\dim(V) = \aleph_0$. Since for finite spaces W, $\dim(W^*) = \dim(W) < \aleph_0$; for infinite spaces Z, $\dim(Z^*) > \dim(Z) \geq \aleph_0$, so there exists no vector space whose dual is isomorphic to V (or any other space of dimension $\aleph_0$).

"Assume all vector spaces are finite dimensional" oops

(e) False

(f) True

(g) True

(h) False.

2\. (a) This is a linear functional

(c) This is also a linear functional

(f) This is a linear functional as well

3\. (b) $f_1(a + bx + cx^2) = a$, $f_2(a + bx + cx^2) = b$, $f_3(a + bx + cx^2) = c$

(a) Notice

$$
\begin{aligned}
f_1(1,0,1) &= f_1(e_1) + f_1(e_3) = 1 \\
f_1(1,2,1) &= f_1(e_1) + 2f_1(e_2) + f_1(e_3) = 0 \\
f_1(0,0,1) &= f_1(e_3) = 0
\end{aligned}
$$

Solving similarly for all 3, we have that $f_1(x,y,z) = x - \frac{1}{2}y$, $f_2(x,y,z) = \frac{1}{2}y$, $f_3(x,y,z) = -x + z$ using a calculator.

(Just change the coefficients on the right hand side, [0 1 0] and [0 0 1] for f_2 and f_3)

4. We know $\dim(V^*) = \dim(V) = 3$ so proving $\operatorname{span}\{f_1, f_2, f_3\} = V^*$ suffices. To do this, we just need to show there exists linear functionals $g_1, g_2, g_3 \in V^*$ with $g_1(x,y,z) = x$, $g_2(x,y,z) = y$, $g_3(x,y,z) = z$. Notice that for g_1,

$$a(x-2y) + b(x+y+z) + c(y-3z) = x,$$

$$(a+b)x + (-2a+b+c)y + (b-3c)z = x.$$

Thus by comparing, $a+b = 1$, $-2a+b+c = 0$, $b-3c = 0$; Solving these simultaneous equations, we have that $g_1 = \frac{2}{5}f_1 + \frac{3}{5}f_2 + \frac{1}{5}f_3$, $g_2 = -\frac{3}{10}f_1 + \frac{3}{10}f_2 + \frac{1}{10}f_3$, $g_3 = -\frac{1}{10}f_1 + \frac{1}{10}f_2 - \frac{3}{10}f_3$ by repeating a similar procedure for g_2 and g_3.

Now, for each linear functional f on V, $f = f(1,0,0)g_1 + f(0,1,0)g_2 + f(0,0,1)g_3$ which can be expressed as a linear combination of f_1, f_2, f_3.

Accordingly, $\{f_1, f_2, f_3\}$ is a basis for V^*.

$\{f_1, f_2, f_3\}$ is the dual basis of some $\beta = \{(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)\}$, so we have the following equations

$$\begin{array}{lll} a_1 - 2a_2 = 1 & b_1 + b_2 + b_3 = 1 & c_2 - 3c_3 = 1 \\ 2a_1 - 2a_2 = 0 \ (\text{if } a_1 \neq 0) & 2b_1 + b_2 + b_3 = 0 \ (\text{if } b_1 \neq 0) & 2c_2 - 3c_3 = 0 \ (\text{if } c_2 \neq 0) \\ & b_1 + 2b_2 + b_3 = 0 \ (\text{if } b_2 \neq 0) & \end{array}$$

Therefore, $a_1 = a_2 = -1$, $b_1 = b_2 = -1$ and $b_3 = 3$, $c_2 = -1$ and $c_3 = -\frac{2}{3}$. We need β to span $\mathbb{R}^3$ so for any $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ there should exist $a, b, c \in \mathbb{R}$ with

$$a(-1,-1,a_3) + b(-1,-1,3) + c(c_1,-1,-\tfrac{2}{3}) = (\alpha, \beta, \gamma),$$

$$\begin{array}{lll} -a-b+cc_1 = \alpha & -a-b-c = \beta & aa_3 + 3b - \frac{2}{3}c = \gamma \\ (c_1+1)c = \alpha - \beta & a+b = \frac{\beta-\alpha}{c_1+1} - \beta & aa_3 + \frac{3\beta-3\alpha}{c_1+1} - 3\beta - 3a + \frac{2}{3}\left(\frac{\beta-\alpha}{c_1+1}\right) = \gamma \\ c = \frac{\alpha-\beta}{c_1+1} & & (a_3-3)a = \gamma + 3\beta + \frac{11}{3}\left(\frac{\alpha-\beta}{c_1+1}\right) \end{array}$$

Hence, one possibility for β is $\{(-1,-1,1), (-1,-1,3), (1,-1,-\frac{2}{3})\}$. For which $\{f_1, f_2, f_3\}$ is the dual basis of, by construction. $\square$

7. (a) $T^t(f)(a+bx) = fT(a+bx) = f(a-2a-2b,\ a+b) = f(-a-2b,\ a+b) = -a-2b-2(a+b) = -3a-4b$

(b) Let $\gamma^* := \{g_1, g_2\}$ be the dual basis of $\gamma := \{(1,0),(0,1)\}$ and $\beta^* := \{f_1, f_2\}$ the dual basis of $\beta := \{1, x\}$.
There exists scalars $a, b \in \mathbb{R}$ with $T^t(g_1) = af_1 + bf_2$ with $T^t(g_1)(1) = -1$ and $T^t(g_1)(x) = -2$ so that $a=-1$, $b=-2$.
Similarly, $T^t(g_2) = a'f_1 + b'f_2$ such that $T^t(g_2)(1) = 1$ and $T^t(g_2)(x) = 1$. Therefore, $a'=1$ and $b'=1$.
Consequently, $[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$.

(c) Notice $T(1) = (-1, 1)$ and $T(x) = (-2, 1)$. As such, $[T]_\beta^\gamma = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$. Indeed, as we expected, $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^t = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$.

8. Ideas

$u := (a,b,c)$ $\quad v := (d,e,f)$

$\alpha a + \beta d = 0 \quad \alpha b + \beta e = 0 \quad \alpha c + \beta f = 0 \quad \Rightarrow \alpha = \beta = 0$

$\alpha = -\frac{\beta d}{a} \quad -\frac{\beta bd}{a} + \beta e = 0$

$a\beta e = \beta bd$

$a = \frac{bd}{e}$

$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(r,s,t) := su + tv$

$f(su+tv) = 0 \quad \beta := \{u, v, w\}$

$f(u) = f(v) := 0, \quad f(w) := 1$

$(\mathbb{R}^3)^* := \mathcal{L}(\mathbb{R}^3, \mathbb{R})$

required vector / LFAL $f : \mathbb{R}^3 \to \mathbb{R}$

Proof

Define the required linear functional on $\mathbb{R}^3$ by $f(u) = f(v) := 0$ and $f(w) := 1$, where w is a vector with $\beta := \{u, v, w\}$ being a basis for $\mathbb{R}^3$.
Then, for every $a,b,c \in \mathbb{R}$, $f(au+bv+cw) = 0$ implies $c = 0$. Hence, $N(f) = \{su+tv \mid s,t \in \mathbb{R}\}$. $\square$

) Fc
) Fc
Th
So
) Fc
) T
T
Fa
Th
Th
Th
f
N
f
f
f

9. Ideas

Assume T linear, want $f_i : \mathbb{F}^n \to \mathbb{F}$ Define $f_j(e_i) = ([T]_s^{s'})_{ji}$ $\sum_{j=1}^{m} f_i(\sum_{i=1}^{n} c_i e_i) =$

$T(\sum_{i=1}^{n} c_i e_i) = \sum_{i=1}^{n} c_i T(e_i) = \sum_{i=1}^{n} c_i \sum_{j=1}^{m} ([T]_s^{s'})_{ji} e_j$

or: $f_j(x) := g_j(T(x))$

Proof

Assume that T is linear, then define the linear functional on $\mathbb{F}^n$, f_j, by $f_j(e_i) = ([T]_s^{s'})_{ji}$ where s and s' are the standard ordered bases for $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively. So, $T(\sum_{i=1}^{n} c_i e_i) = \sum_{i=1}^{n} c_i \sum_{j=1}^{m} ([T]_s^{s'})_{ji} e_j = \sum_{j=1}^{m} (\sum_{i=1}^{n} c_i f_j(e_i)) e_j = \sum_{j=1}^{m} f_j(\sum_{i=1}^{n} c_i e_i) e_j = (f_1(\sum_{i=1}^{n} c_i e_i), f_2(\sum_{i=1}^{n} c_i e_i), \dots, f_m(\sum c_i e_i))$

Conversely, if $T(x) = (f_1(x), f_2(x), \dots, f_m(x))$, $T(ax+y) = \sum_{j=1}^{m} f_j(ax+y) e_j = a \sum_{j=1}^{m} f_j(x) e_j + \sum_{j=1}^{m} f_j(y) e_j$ $= aT(x) + T(y)$. Which means T is linear.

So, the biconditional holds. $\square$

10. (a) Ideas

$g_i(x^j) = \delta_{ij}$ $\quad f_i(x^j) = c_i^j$ $\quad \sum_{k=1}^{n} a_k f_k(x^j)$

$g_i = \sum_{k=1}^{n} a_k f_k$ $\quad \sum_{k=1}^{n} a_{ik} c_k^i = 1$ $\quad \sum_{k=1}^{n} a_{ik} c_k^j = 0$ if $i \neq j$

$\sum_{i=1}^{n} a_i f_i = T_0$

$\sum_{i=1}^{n} a_i p(c_i) = 0$ for all $p \in P_n(F)$

$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j c_i^j = 0$ $\forall$ for all $b_j \in \mathbb{F}$ $\quad p(c_i) = \sum_{j=1}^{n} b_j c_i^j$

let $k := |\{1 \le i \le n \mid a_i \neq 0\}|$, $b_j :=$

$G(x^j) = c_i^j$

$e_i \sim g_i$

$\sum_{i=0}^{n} c_i^j e_i \sim \sum_{i=0}^{n} c_i^j g_i =: G$

$V = P_n(\mathbb{F}) \longrightarrow \mathbb{F}^{n+1} \longrightarrow V^* = P_n(\mathbb{F})^*$

$\{1, x, \dots, x^n\}$ $\quad \{(1,1,\dots,1), (c_0, c_1, \dots, c_n), \dots, (c_0^n, c_1^n, \dots, c_n^n)\}$

$f(c_i) = f_i(f(x))$

Proof

Let $\gamma := \{1, x, \dots, x^n\}$, $\bar{\gamma} := \{e_1, e_2, \dots, e_{n+1}\}$ — the standard ordered bases for $P_n(\mathbb{F})$ and $\mathbb{F}^{n+1}$ — and $\gamma^* := \{g_0, g_1, \dots, g_n\}$ be the dual basis of γ. Notice that Exercise 22 of section 2.5 tells us that $\bar{\gamma}' := \{(1,1,\dots,1), (c_0, c_1, \dots, c_n), \dots (c_0^n, c_1^n, \dots, c_n^n)\}$ is also a basis for $\mathbb{F}^{n+1}$. By defining the isomorphism $T : \mathbb{F}^{n+1} \to V^*$ with $T(e_{i+1}) = g_i$, we see that $T[\![\gamma^*]\!]$ forms a basis for V^*. Consider any one of its members; $T(\sum_{k=0}^{n} c_k^j e_{k+1}) = \sum_{k=0}^{n} c_k^j g_k$ and note $\sum_{k=0}^{n} c_k^j g_k(x^i) = c_i^j x^i = f_j(x^i)$ for any $0 \le i \le n$. As such, $T(\sum_{k=0}^{n} c_k^j e_{k+1}) = f_j$, meaning $T[\![\gamma^*]\!] = \{f_0, f_1, \dots, f_n\}$. $\square$

Ideas

$T^t(f) := fT \quad T^{tt}(g) := gT^t$

$T^{tt}(g)(f) = gT^t(f) = gfT$

$\psi_2 T(v) = \widehat{T(v)} \qquad W^* \to \mathbb{F} \qquad \widehat{T(v)}(f) = f(T(v))$

$T^{tt}\psi_1(v) = T^{tt}(\hat{v}) = \hat{v}T^t \qquad W^* \to V^* \to \mathbb{F}$

$\hat{v}T^t(f) = \hat{v}(fT) \qquad fT : V \to W \to F$

$= fT(v)$

$\psi_2 T(v) = \widehat{T(v)} = \hat{v}T^t := T^{tt}(\hat{v}) := T^{tt}\psi_1(v) = T^{tt}\psi_1(v)$

Proof

We see that for any $v \in V$ and $f \in W^*$, $[\widehat{T(v)}](f) := f(T(v)) = fT(v) = \hat{v}(fT) := \hat{v}T^t(f)$, so $\psi_2 T(v) = \widehat{T(v)} = \hat{v}T^t := T^{tt}(\hat{v}) := T^{tt}\psi_1(v) = T^{tt}\psi_1(v)$ for each $v \in V$. The diagram commutes, in other words.

And hence, $\psi_2 T = T^{tt}\psi_1$ as expected since the above holds for each $v \in V$. $\square$

2. Ideas

$\beta^* := \{f_1, f_2, \cdots, f_n\}$, $\beta^{**} := \{g_1, g_2, \cdots, g_n\}$

$\psi(v_i) = \hat{v}_i : V^* \to \mathbb{F}$, $\quad \hat{v}_i(h) = h(v_i)$

$\hat{v}_i(f_j) = f_j(v_i) = \delta_{ji} = \delta_{ij} = g_i(f_j) \qquad f_i(v_j) = \delta_{ij}, \quad g_i(f_j) = \delta_{ij}$

$V \to \mathbb{F} \qquad V^* \to \mathbb{F}$

Proof

Let $\beta^* := \{f_1, f_2, \cdots, f_n\}$ and $\beta^{**} := \{g_1, g_2, \cdots, g_n\}$ be the dual bases of β and β^* respectively. Consider any $1 \le i \le n$ and $1 \le j \le n$, then $[\psi(v_i)](f_j) := \hat{v}_i(f_j) := f_j(v_i) := \delta_{ji} = \delta_{ij} := g_i(f_j)$ so $\psi(v_i) = g_i$. Consequently,

3. (a) Since $T_0(x) = 0$ for all $x \in V$, $T_0 \in S^0$. Furthermore, for any two $f, g \in S^0$, $c \in \mathbb{F}$, and $x \in S$; $(f+g)(x) = f(x) + g(x) = 0$ and $cf(x) = c \cdot 0 = 0$. Thus, $f+g$ and cf are both in S^0. Hence S^0 is a subspace of V^*.

(b) Ideas

$x \in V - W \quad \beta := \{x, x_1, x_2, \cdots, x_n\}$

$f_0 \in S$

Proof

Let $\beta := \{x, x_1, x_2, \cdots, x_n\}$ be a basis for V and $\beta^* := \{f_0, f_1, \cdots, f_n\}$ be its dual basis. Since $x \in V - W$, $f_0 \in W^0$ where $f_0(x) := 1 \neq 0$.

$\mathrm{span}(\psi(S)) = \mathrm{span}\{\hat{x} \mid x \in S\} =$

any $g \in V^*$ $\quad f(g) = \sum_{i=1}^{n} c_i \hat{v}_i(g)$ $\quad \beta$ be a basis of V extended from S

$= \sum_{i=1}^{n} c_i g(v_i) \quad h_j(v_i) = \delta_{ji}$ $\quad \beta^*$ be its dual basis $= \{h_1, h_2, \cdots, h_n\}$

(c) Ideas $(S^0)^0 = \{f \in V^{**} \mid f(g) = 0$ for all $g \in S^0\}$ $\quad \hat{x}(g) = g(x) = 0$

$(\subseteq V^*)$

$f : V^* \to \mathbb{F} \qquad f(h_{n+j}) = 0$ for $1 \le j \le m-n \qquad \mathrm{span}\{\hat{v}_i \mid 1 \le i \le n\}$

$\sum_{i=1}^{n} f(h_j)\hat{v}_i(h_j) = f(h_j)$

Proof

Let $\{v_1, v_2, \cdots, v_n\}$ be the least subset of S whose span includes S, $\beta := \{v_1, v_2, \cdots, v_m\}$ be its extension to a basis for V, and $\beta^* := \{h_1, h_2, \cdots, h_m\}$ the dual basis of β. For $f \in (S^0)^0$, that is, any $f : V^* \to \mathbb{F}$ with $f(g) = 0$ for each $g \in S^0$, $\sum_{i=1}^{n} f(h_j)\hat{v}_i(h_j) = f(h_j)$ if $1 \le j \le n$ (immediate) and $\sum_{i=1}^{n} f(h_j)\hat{v}_i(h_j) = 0 = f(h_j)$ otherwise (anyway). Hence, $f = \sum_{i=1}^{n} f(h_j)\hat{v}_i$. So, we have that $\mathrm{span}(\psi(S)) \supseteq (S^0)^0$. Conversely, when $x \in S$, $\hat{x}(g) = g(x) = 0$ for every $g \in S^0$. Thus, $\mathrm{span}(\psi(S)) \subseteq (S^0)^0$. And accordingly, equality holds.

10.(b) By the Corollary to Theorem 2.26, there exists a basis $\beta := \{p_0(x), p_1(x), p_2(x), \ldots, p_n(x)\}$ for $P_n(\mathbb{F})$, which $\{f_0, f_1, \ldots, f_n\}$, as defined in (a), is the dual basis of. Consider another basis $\gamma := \{q_0(x), q_1(x), q_2(x), \ldots, q_n(x)\}$ for which $\{f_0, f_1, \ldots, f_n\}$ is also the dual basis of.

Accordingly, $p_i(c_j) = f_j(p_i(x)) = \delta_{ij}$ for each $0 \le i \le n$ and $0 \le j \le n$. So $\frac{p_i(x)}{\prod_{i \ne j} x - c_j} = \frac{q_i(x)}{\prod_{i \ne j} x - c_j} = \overset{(\text{some})}{k} \in \mathbb{F}$ by Lemma 2 of (my proof of) exercise 22 in section 2.5. (In fact,

Then, $p_i(c_j) = \underset{= \delta_{ij}}{q_i(c_j)}$

as $p_i(c_i) = q_i(c_i) = 1$, $k = 1$.) Hence, $p_i = q_i$ for every $0 \le i \le n$, thereby establishing uniqueness.

(c) By defining the polynomial as suggested, $q(c_i) = \sum_{j=0}^{n} a_j \delta_{ji} = a_i$ by (b). Uniqueness clearly holds from the fact that β, as defined above in (b), is a basis for $P_n(\mathbb{F})$.

(d) Let $p(x) \in V = P_n(\mathbb{F})$. Then there exists scalars a_i with $p(x) = \sum_{i=0}^{n} a_i p_i(x)$ by virtue of β being a basis. Hence, we see that $p(c_i) = a_i p_i(c_i) = a_i$.

Thus, it holds that $p(x) = \sum_{i=0}^{n} p(c_i) p_i(x)$.

(e) ~~Ideas~~

$\sum_{i=0}^{n} p(c_i) \int_a^b p_i(t)\,dt = \int_a^b \sum_{i=0}^{n} p(c_i) p_i(t)\,dt = \int_a^b p(t)\,dt$

Direct

Simply notice that $\int_a^b p(t)\,dt = \int_a^b \sum_{i=0}^{n} p(c_i) p_i(t)\,dt = \sum_{i=0}^{n} p(c_i) \int_a^b p_i(t)\,dt = \sum_{i=0}^{n} p(c_i) d_i$ by (d) and properties of the definite integral.

We see that when $n = 1$, $d_0 = \int_a^b \frac{t-b}{a-b}\,dt = \left[\frac{t^2 - 2bt}{2(a-b)}\right]_a^b = \frac{b-a}{2}$ and $d_1 = \int_a^b \frac{t-a}{b-a}\,dt = \frac{b-a}{2}$, so

$\int_a^b p(t)\,dt = p(c_0)d_0 + p(c_1)d_1 = \frac{b-a}{2}[p(a) + p(b)]$ [for $p(t)$ of degree at most $n = 1$]

Similarly, in the case where $n = 2$, $d_0 = \int_a^b \frac{t - \frac{a+b}{2}}{a - \frac{a+b}{2}} \cdot \frac{t-b}{a-b}\,dt = \frac{b-a}{6}$, $d_1 = \int_a^b \frac{t-a}{\frac{a+b}{2} - a} \cdot \frac{t-b}{\frac{a+b}{2} - b}\,dt = \frac{2}{3}(b-a)$, and $d_2 = \int_a^b \frac{t-a}{b-a} \cdot \frac{t - \frac{a+b}{2}}{b - \frac{a+b}{2}}\,dt = \frac{b-a}{6}$

by wolfram alpha. Hence, $\int_a^b p(t)\,dt = p(c_0)d_0 + p(c_1)d_1 + p(c_2)d_2 = \frac{b-a}{6}p(a) + \frac{2}{3}(b-a)p\left(\frac{a+b}{2}\right) + \frac{b-a}{6}p(b) = \frac{b-a}{6}\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right]$ as expected.

[for $p(t)$ of degree at most $n = 2$]

$\square$

(Newton-Cotes formulas)

3. (d) Ideas
As $W_1^\circ = W_2^\circ$, i.e. $\{f: V \to \mathbb{F} \mid f(x) = 0 \ \forall x \in W_1\} = \{f: V \to \mathbb{F} \mid f(x) = 0 \ \forall x \in W_2\}$
If $x \in W_1$, $f(x) = 0$ for all $f \in W_2^\circ$ $\Rightarrow$

OR: Suppose $W_1 \neq W_2$, and wlog, that $\exists x (x \in W_1 \ \& \ x \notin W_2)$
$\beta := \{x, x_1, x_2, \dots, x_m\}$ a basis for V
$f(x) := 1$, $f(x_i) := 0$
$\Rightarrow f(y)$ always 0 for $y \in W_2$, $f \in W_1$
$\Rightarrow f(x) = 1$, $f \notin W_1$

Proof
Assume $W_1 \neq W_2$, and without loss of generality, that there exists $x \in W_1 - W_2$. Let $\beta' := \{x_1, x_2, \dots, x_k\}$ be a basis for W_2, and $\beta := \{x, x_1, x_2, \dots, x_m\}$ its extension to a basis for V. Define $f: V \to \mathbb{F}$ with $f(x) = 1$ and $f(x_i) = 0$ for every $1 \leq i \leq m$. As such, $f \in W_2$ but $f \notin W_1$. Therefore, $W_1^\circ \neq W_2^\circ$. The converse is trivial. Hence, the biconditional holds.

(e) By exercise 16(c), this must be true.

$\square$

14. Ideas $\underbrace{W}_{m}$
$\beta := \{v_1, v_2, \dots, v_n\}$ a basis for V, $\beta^* := \{\dots\}$ its dual
$j \geq 1$: $f_{n+j} \in W^\circ$, while $1 \leq i \leq m$: $f_i \notin W^\circ$ since $f_i(v_i) = 1 \neq 0$

Proof Let $\{v_1, v_2, \dots, v_n\}$ be a basis for W, $\beta := \{v_1, v_2, \dots, v_m\}$ its extension to a basis for V, and $\beta^* := \{f_1, f_2, \dots, f_m\}$ the dual basis of β. Then, for $1 \leq j \leq m-n$ and $1 \leq i \leq m$, $f_{n+j} \in W^\circ$ while $f_i \notin W^\circ$. That is, ~~$W^\circ \cap \beta^* = \{f_{n+1}, f_{n+2}, \dots, f_m\}$ and~~ $\dim(W^\circ) = m-n$ as $f = \sum_{i=n+1} f(v_i) f_i$ for $f \in W^\circ$ by Thm 2.24. So, $\dim(W) + \dim(W^\circ) = n + (m-n) = m = \dim(V)$.

$\square$

$V \xrightarrow{T} W \xrightarrow{f} \mathbb{F}$

15. Ideas $T^t(f) = fT$ $\quad N(T^t)$ $\quad (R(T))^\circ := \{f: W \to \mathbb{F} \mid \underbrace{f(w) = 0 \text{ for all } w \in R(T)}_{f(T(v)) = 0 \text{ for all } v \in V}\}$
$fT = T_0$
$fT(\cdot) = 0 \ \forall v \in V$

Proof
When $f \in N(T^t)$, then $fT(v) = 0$ for every $v \in V$. In other words, $f(w) = 0$ given $w \in R(T)$. Thus, $f \in (R(T))^\circ$ and implying $N(T^t) \subseteq (R(T))^\circ$. Conversely, say $f \in (R(T))^\circ$. Then, as aforementioned, $fT(v) = 0$ for any $v \in V$. Accordingly, $T^t(f) = fT = T_0$, telling us $f \in N(T^t)$ and $(R(T))^\circ \subseteq N(T^t)$.
Hence, equality holds.

$\square$

16. Ideas

A is $m \times n \iff A^t$ is $n \times m$

$Ax,\ x = \left(\ \right)\}^n \qquad A^t x,\ x = \left(\ \right)\}^m$

$L_A : \mathbb{F}^n \to \mathbb{F}^m \qquad L_{A^t} : \mathbb{F}^m \to \mathbb{F}^n$

$[L_A^t]_s = A^t \qquad L_A^t : (\mathbb{F}^m)^* \to (\mathbb{F}^n)^*$

$L_A^t(f) := f L_A \qquad f(Ax)$

$f L_A(e_j) = f\left[\sum_{i=1}^{m} (A^t_{ij}) e_i\right]$

Example

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$f : \mathbb{F}^m \to \mathbb{F} \qquad f \restriction R(L_A) = T_0$

$N(L_A^t) = (R(L_A))^0$

$\text{rank}(L_A) + \dim(R(L_A)^0) = m$

$\text{rank}(L_A) + \text{nullity}(L_A^t) = m$

$\text{rank}(L_A) + [m - \text{rank}(L_A^t)] = m$

$\text{rank}(L_A^t) = \text{rank}(L_A)$

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "L_A"] \arrow[d, "\varphi_n"'] & \mathbb{F}^m \arrow[d, "\varphi_m"] \\ (\mathbb{F}^n)^* \arrow[d, "\phi_n"'] & (\mathbb{F}^m)^* \arrow[l, "L_A^t"'] \arrow[d, "\phi_m"] \\ \mathbb{F}^n & \mathbb{F}^m \arrow[l, "L_{A^t}"'] \end{tikzcd}$$

$\phi_n L_A^t = L_{A^t} \varphi_m$

Proof

First, notice that since $N(L_A^t) = (R(L_A))^0$ according to exercise 15, this means $\text{rank}(L_A) + \text{nullity}(L_A^t) = m$ by exercise 14. As such, $\text{rank}(L_A^t) = \text{rank}(L_A)$. Furthermore, by letting $S_n := \{e_i \mid 1 \le i \le n\}$ (the standard ordered basis for $\mathbb{F}^n$), we notice $\phi_{S_n} L_A^t = L_{A^t} \phi_m$ because $[L_A^t]_{S}^{S} = A^t$. In other words, $\text{rank}(L_A) = \text{rank}(L_A^t) = \text{rank}(L_A)$. $\square$

18. Ideas

$\Phi(f) = f_0$ (the $f_0 : S \to \mathbb{F}$ with $f(s) = 0$)

$f_S = f_0$

$\forall t \in S,\ f_S(t) = 0$

$f = T_0$

Proof

Suppose $\Phi(f) = f_0$, the zero function from $f_0 : S \to \mathbb{F}$ with $f(t) := 0$. Then $f_S = f_0$ so, for each $t \in S$, $f_S(t) = 0$. Since S is a basis for V, $f = T_0$, the zero transformation $T_0 : V \to \mathbb{F}$. By Theorems 2.4 & 2.5, Φ is guaranteed to be an isomorphism. □

19. (a) Let $\gamma := \{v_1, v_2, \ldots, v_n\}$ be a basis for W, $\beta := \{v_1, v_2, \ldots, v_m\}$ be its extension to a basis for V. Define $f : V \to \mathbb{F}$ by $f(v_i) := g(v_i)$ for $1 \le i \le n$, $f(v_{n+1}) = a$, and $f(v_i) = 0$ otherwise (for $n+2 \le i \le m$). Then $f(v_{n+1}) = a$ where v_{n+1}'s existence is assured by W being a proper subspace of V, and $f(x) = g(x)$ for all $x \in W$ is clear.

(b) Let $g(x)$ be the zero transformation from W into $\mathbb{F}$. Then set $a := 1$ and define f and the relevant terms as above, in (a). It is clear that $f(x) = 0$ for all $x \in W$ but $f(v_{n+1}) = 1$ tells us f is nonzero.

20. (a) If T is surjective, suppose $T^t(f) = T^t(g)$. Then for every $w \in W$, there exists $v \in V$ so that $f(w) = fT(v) = [T^t(f)](v) = [T^t(g)](v) = gT(v) = g$. Hence, $f = g$ which informs us that T^t is injective. Now consider T not being surjective, i.e. there exists $y \in W$ with $y \notin T[V]$. So, letting $\beta := \{w_1, \ldots, w_k\}$ be a basis for $R(T)$ and $\gamma := \{y, w_1, w_2, \ldots, w_\lambda\}$ be a basis for W (allowed by Exercise 7 of Ch 1.7), we can define $f(y) = 1$ and $f(w_\alpha) = 0$ for all $\alpha \in \lambda$. Thus, for the zero linear functional $g_0 : W \to \mathbb{F}$, $fT(v) = g_0T(v) = 0$ (as $T(v)$ being a linear combination of vectors in β) for any $v \in V$, even though $f \ne g_0$ by virtue of $f(y) := 1 \ne 0 = g_0(y)$. Wherefore, the biconditional must hold. □

First ... FLA

17. Idea, $T: V \to V$, W subspace V, $T^t: V^* \to V^*$

W is T-invariant i.e. $T[W] = W$, show $T^t[W^0] = W^0$, find $T^t(f) = fT = g$

Let γ be basis for W ($1 \to n$), β its extension to a basis for V ($1 \to m$)

Let $g \in W^0$, i.e. $g: V^* \to \mathbb{F}$ where $g(w) = 0 \ \forall w \in W$

$g := \sum_{i=n+1}^{m} g(w_i) f_i = \sum_{i=1}^{m} g(w_i) f_i$

$f := \sum_{i=1}^{m} c_i f_i$

$fT := \sum_{i}^{m} c_i f_i T$

$c_i f_i(T(w_i)) = g(w_i)$

$a_i c_i = g(w_i)$

$c_i = \dfrac{g(w_i)}{\underbrace{f_i(T(w_i))}_{\text{must be non-zero!}}}$

$W \xrightarrow{T} [W$

$W^0 \xrightarrow{T^t} [W^0$

since $W = R(T_W) = \operatorname{span} T[\gamma]$

$T(w) \in W$

$V \to V \to \mathbb{F}$

$\Longrightarrow$ Oops mixed up the defn of T-invariance

Show $T^t(f) \in W^0$. Let $f \in W^0$, $T^t(f) = fT$

$\forall w \in W$ $fT(w) = 0$ as $T(w) \in W$

W^0 T^t-invariant

$fT(w) = 0$, $f := \sum_{i=n+1}^{m} f_i$ $\Longrightarrow T(w) \in W$ as if $T(w) \notin W$ for some w, then $f(T(w)) \neq 0$

Proof

Assume W is T-invariant and let $f \in W^0$. Then for each $w \in W$; since $T(w) \in W$, $[T^t(f)](w) = fT(w) = 0$. Hence, $T^t(f) \in W^0$ and W^0 is T^t-invariant. Now let $\gamma := \{v_1, v_2, \dots, v_n\}$ be a basis for W, $\beta := \{v_1, v_2, \dots, v_m\}$ its extension to a basis for V, and $\beta^* := \{f_1, f_2, \dots, f_m\}$ the dual basis of β. Supposing W^0 is T^t-invariant, we can define $f := \sum_{i=n+1}^{m} f_i$. Given $w \in W$, $fT(w) = 0$ so this means $T(w)$ is a linear combination of γ. Thus, $T(w) \in W$ and W is T-invariant. Therefore, the bi-conditional holds. $\square$

[The case where V and/or W are infinite dimensional follows similarly: $f_\alpha T(w) \underset{\text{still}}{=} 0$ for $k \le \alpha \le \lambda$, where $\gamma := \{v_\alpha \mid \alpha \in K\}$ and $\beta := \{v_\alpha \mid \alpha \in \lambda\}$]

$U_1U_2U_3 = U_1U_3U_2 = U_2U_1U_3 = U_3U_2U_1 = \dots$

q. Ideas

When $n=1$, clear

Assume that for any bijection $q:\{1,2,\dots,n\} \to \{1,2,\dots,n\}$, we have

for every set $\{U_1, U_2, \dots, U_n\}$ of pairwise commutative linear operators

Proof

Lemma : Order Invariancy

We want to first prove that for any bijection $q:\{1,2,\dots,n\} \longrightarrow \{1,2,\dots,n\}$, we have $U_1U_2\cdots U_n = U_{q(1)}U_{q(2)}\cdots U_{q(n)}$.

When $n=1$, the result follows immediately. So assume it holds for $n \in \mathbb{N}$. Then, for any bijection $q:\{1,2,\dots,n+1\} \longrightarrow \{1,2,\dots,n+1\}$, with $q(k) = n+1$

for some k, define the bijection $p:\{1,2,\dots,n\} \longrightarrow q[\{1,2,\dots,n\}]$ by having $p(1)$ be the least $q(i)$, and $p(n+1)$ be the least $q(i) > p(n)$, where

$1 \le k \le n$. For the commutative linear operator on V given by $T_i := U_{p(i)}$, $U_{q(1)}\cdots U_{q(n)} = T_{p^{-1}q(1)}\cdots T_{p^{-1}q(n)} = T_1 \cdots T_n$ by the above assumption

since $p^{-1}q$ is a bijection from $\{1,2,\dots,n\}$ to $\{1,2,\dots,n\}$. Thus, we now have that $U_{q(1)}\cdots U_{q(n+1)} = U_1U_2\cdots U_{q(n+1)-1}U_{q(n+1)+1}U_{q(n+1)+2}\cdots U_{n+1}U_{q(n+1)}$

$= U_1U_2\cdots U_{q(n+1)-1}U_{q(n+1)+1}U_{q(n+1)+2}\cdots U_{q(n+1)}U_{n+1}$ by pairwise commutativity. If $q(n+1) = q(n+1)$, the result is clear. Otherwise, repeat

the above reindexing to obtain $U_{q(1)}\cdots U_{q(n+1)} = U_1\cdots U_{n+1}$. So, the $(n+1)$th case must hold true. Accordingly, it is true for all $n \in \mathbb{N}$ by induction.

[Implicit assumption of general associativity in the qns and lemma above but that is straightforward.]

Consequently, when $U_i(v) = 0_V$ for any $v \in V$, $(U_1\cdots U_n)(v) = (U_1U_2\cdots U_{i-1}U_{i+1}U_{i+2}\cdots U_n)U_i(v) = 0_V$ as expected. So, the claim that

$N(U_i) \subseteq N(U_1U_2\cdots U_n)$ certainly is true.

$\square$

$f_1(x_1, x_2) = x_1 \cdot x_2$

$f_{n+1}(x_1, \dots, x_{n+1}) = f_n(x_1, \dots, x_n) \cdot x_{n+1}$

Self-check of Theorem 2.29

$f'(t) = D(e^{ct}) = D(e^{at+ibt}) = D(e^{at}[\cos(bt) + i\sin(bt)]) = ae^{at}[\cos(bt) + i\sin(bt)] + e^{at}[-b\sin(bt) + ib\cos(bt)]$

$= e^{at}[a\cos(bt) - b\sin(bt) + ia\sin(bt) + ib\cos(bt)]$

$= (a+bi)e^{at}[\cos(bt) + i\sin(bt)]$

$= ce^{ct}$

Self-Proof of Theorem 2.30

Ideas

AP: $p(t) = t + a_0 = 0$

$y'' = -a_0 y'$ $\quad y' = -a_0 y$ $\quad \Rightarrow \quad -a_0 y' = a_0^2 y$

$= a_0^2 y$ $\qquad y' = -a_0 y$

$\int -\frac{1}{a_0 y} y' \, dt = \int 1 \, dt$

$\int -\frac{1}{a_0 y} dy = t + c_0$

$-\frac{1}{a_0}\ln(y) = t + c_0$

$\ln(y) = -a_0 t + c_1$

$y = e^{-a_0 t + c_1}$

$= Ce^{-a_0 t}$ $\quad y' = -Ca_0 e^{-a_0 t}$

$-Ca_0 e^{-a_0 t} + Ca_0 e^{-a_0 t} = 0$

Proof

Notice $y'' = -a_0 y' = -a_0(-a_0 y) = a_0^2 y$ so $y' = -a_0 y$. Thus, integrating both sides, we have $\int -\frac{1}{a_0 y} y' \, dt = \int -\frac{1}{a_0 y} dy = \int 1 \, dt$. Which simplifies to $y = Ce^{-a_0 t}$, for any real constant $C \in \mathbb{R}$. Conversely, if $y = Ce^{-a_0 t}$ for some $C \in \mathbb{R}$, $y' = -Ca_0 e^{-a_0 t}$ so $y' + a_0 y = 0$ as expected.

Hence, it is clear that $\{e^{-a_0 t}\}$ is a basis and the solution space is of dimension 1.

Unfortunately only proves the result for functions $\mathbb{R} \to \mathbb{R}$. While we need it for functions $\mathbb{R} \to \mathbb{C}$.

Checking commutativity of operators $D - c_i I$: As expected,

$(D - c_i I)(D - c_j I) = D^2 - D(c_j I) - (c_i I)D + (c_i I)(c_j I) = D^2 - D(c_i I) - D(c_j I) + (c_i I)(c_j I) = (D - c_i I)(D - c_j I)$.

Self - Proof of Theorem 2·32 Let N_j be the number of times $D-c_jI$ is repeated in the product $(D-c_1I)\cdots(D-c_nI)$

Ideas

Claim: $\{t^{i_j}e^{c_jt} \in \mathbb{C}^* \mid 1 \le j \le n \ \& \ 1 \le i_j \le N_j - 1\}$

$S :=$

When $n=1$, clear

Assume true for $n \in \mathbb{N}$.

$$\underbrace{(D-c_1I)\cdots(D-c_nI)}_{\mathbb{D}_n}(D-c_{n+1}I)(y) = y_0$$

$$D\underbrace{(D-c_1I)\cdots(D-c_nI)(y)}_{z} = c_{n+1}\underbrace{(D-c_1I)\cdots(D-c_nI)(y)}_{z}$$

$$D(z) = c_{n+1}z$$

$$z' - c_{n+1}z = 0$$

$$z = Ae^{c_{n+1}t} \quad \text{for some } A \in \mathbb{C}$$

$$\Rightarrow \overbrace{\underbrace{(D-c_1I)\cdots}_{r(D)}(D-c_nI)(y)}^{q(D)} = Ae^{c_{n+1}t} \quad \text{——} (\star)$$

$$y' - c_n y = Ae^{c_{n+1}t}$$

Let $u_p = \alpha e^{c_{n+1}t}$, $u_p' = \alpha c_{n+1}e^{c_{n+1}t}$ Let $C := \prod_{i=1}^{n} \frac{A}{c_{n+1}-c_i}$

$$\alpha(c_{n+1}-c_n)e^{c_{n+1}t} = Ae^{c_{n+1}t}$$

$$\alpha := \frac{A}{c_{n+1}-c_n}, \quad y := \frac{A}{c_{n+1}-c_n}e^{c_{n+1}t} \to \text{Just 1 possible soln}$$

c_k repeated roots:

$$(D-c_1I)^{m_1}\cdots(D-c_kI)^{m_k} = 0$$

$$(D-c_1I)^{m_1}\cdots(D-c_kI)^{m_k}(y) = Ae^{c_1t} \quad \text{—} (\star)$$

$$(D-c_1I)^{m_1-1}\cdots(D-c_kI)^{m_k}$$

If all repeated,

$$(D-cI)^{n+1}(y) = 0$$

$$(D-cI)(t^ne^{ct}) = nt^{n-1}e^{ct} + ct^ne^{ct} - ct^ne^{ct}$$

$$= nt^{n-1}e^{ct}$$

$$(D-cI)^n(t^ne^{ct}) = n!e^{ct}$$

$$\Rightarrow (D-cI)^{n+1}(t^ne^{ct}) = 0$$

Again, $g + t^ne^{ct}$ should be a soln

Suppose g is a soln. By $(\star)$, ...

$$(D-c_1I)(t^{m_1-1}e^{c_1t}) = (m_1-1)t^{m_1-2}e^{c_1t} + c_1t^{m_1-1}e^{c_1t} - c_1t^{m_1-1}e^{c_1}$$

$$\sum_{j=1}^{k}\sum_{i=1}^{m_j-1} a_{ij}t^ie^{c_jt}$$

$$t^{m_1-1}e^{c_1t}$$

$$a_1e^{3t} + a_2te^{3t} = e^{5t}$$

$$a_1e^{3a_1a_2} + a_1a_2^2e^{3a_1a_2} = e^{5a_1a_2}$$

$$a_1 + a_1a_2^2 = e^{2a_1a_2}$$

$$b_1 + b_2 + \cdots + b_N = 1$$

$$a_{ij}e^{c_j}$$

Claim: The solution space of $q(D)(y)$ is $S \cup \{Ce^{c_{n+1}t}\}$ (if c_{n+1} is different from each c_i, $1 \le i \le n$)

Suppose f is a solution, i.e. ~~$(D-c_1I)\cdots(D-c_nI)(f) = 0$~~. Then $f + Ce^{c_{n+1}t}$ must also be a soln. Hence, $(D-c_1I)\cdots(D-c_nI)(f + Ce^{c_{n+1}t}) = Ae^{c_{n+1}t}$

$(D-c_1I)\cdots(D-c_nI)(f) = 0$. By assumption, f is a linear combination of members of S.

reduces to $(D-c_1I)\cdots(D-c_nI)$

$(D-c_jI)(D-c_iI)$

$$(D-c_iI)(Ce^{c_1t}) = Cc_1e^{c_1t} - Cc_ie^{c_1t} = C(c_1-c_i)e^{c_1t}$$

$$(D-c_iI)(t^{m_1-1}e^{c_1t}) = (m_1-1)t^{m_1-2}e^{c_1t} + c_1t^{m_1-1}e^{c_1t} - c_it^{m_1-1}e^{c_1t}$$

$$= [(m_1-1)t^{m_1-2} + (c_1-c_i)]e^{c_1t}$$

Self-Proof of Lemma 2

Idea: U not necessarily injective

When $TU(v) = 0$, either

$U(v) = 0$ or $U(v) \neq 0$

$v \in N(U)$ — $0 \neq U(v) \in N(T) \subseteq R(U) = V$

v linear comb of vect's v_i — $U(v)$ is a linear comb of vectors v_j

OR $\beta := \{v_\alpha \mid \alpha \in K\}$

$U(v_i) = 0 \quad \alpha \leq i \leq \alpha + n$

$T(v_j) = 0 \quad \beta \leq j \leq \beta + m$.

$\gamma := \{u_1, u_2, \dots, u_n\}$

$S := \{v \in V \mid TU(v) = 0\}$
$= N(TU)$

$U' := U_{N(TU)}$

$N(T) \subseteq R(U)$

$\text{rank}(U') + \text{nullity}(U') = \dim(U')$

$R(U') = \{U(v) \in V \mid TU(v) = 0\}$

$\text{nullity}(U') = \text{nullity}(U)$

~~$= U[N(TU)]$~~

$= N(T)$ by surjectivity of U

Proof

As usual, $U_{N(TU)}$ denotes the restriction of U to $N(TU)$: We notice $\text{rank}(U_{N(TU)}) + \text{nullity}(U_{N(TU)}) = \dim(N(TU))$. Where $R(U_{N(TU)}) := \{U(v) \in V \mid TU(v) = 0\} = N(T)$ by U's surjectivity so that $\text{rank}(U_{N(TU)}) = \text{nullity}(T)$, and $\text{nullity}(U_{N(TU)}) = \text{nullity}(U)$ is clear.

Hence, $\text{nullity}(TU) = \text{nullity}(T) + \text{nullity}(U)$ as expected.

Self-Proofs of Theorem 2.33 & 2.34 (including its lemma)

I accidentally proved both of these in my self-proof of Theorem 2.32 lol.

Self-Proof of Theorem 2.32

Let m_j be the number of times $D-c_jI$ is repeated in $p(D)$. We first claim that the set $S_n := \{t^{i_j}e^{c_jt} \in C^\infty \mid 1 \le j \le n \ \& \ 1 \le i_j \le m_j - 1\}$ is a basis for the nullspace of any nth order differential operator $p(D) = (D-c_1I)(D-c_2I)\cdots(D-c_nI)$. When $n=1$, this is just Theorem 2.30. So, assume that this is true for any differential operator $p(D)$ of a particular nth order. Then, for any differential operator $p(D)$ of order $n+1$, suppose that it has some $0 \le k \le n$ repeated roots. That is, $p(D) = (D-c_1I)^{m_1}(D-c_2I)^{m_2}\cdots(D-c_k)^{m_k}$ for some naturals m_j. For $p(D)(y)=0$, it simplifies to $z' - c_1z = 0$ by having $z := (D-c_1I)^{m_1-1}\cdots(D-c_k)^{m_k}(y)$. Therefore, by Theorem 2.30, $z := (D-c_1I)^{m_1-1}\cdots(D-c_k)^{m_k}(y) = Ae^{c_1t}$ —— $(\bigstar)$ for some $A \in \mathbb{C}$.

Notice $(D-c_1I)(t^{m_1-1}e^{c_1t}) = (m_1-1)t^{m_1-2}e^{c_1t}$. By repetition, $(D-c_1I)^{m_1-1}(t^{m_1-1}e^{c_1t}) = (m_1-1)!\,e^{c_1t}$. Continuing, $(D-c_jI)((m_1-1)!e^{c_1t}) = (m_1-1)!(c_1-c_j)e^{c_1t}$. Again repeating this, $(D-c_1I)^{m_1-1}(D-c_2I)^{m_2}\cdots(D-c_kI)^{m_k}(t^{m_1-1}e^{c_1t}) = Ce^{c_1t}$, eventually, where we define $C := (m_1-1)!\prod_{j=1}^{k}(c_1-c_j)^{m_j}$ for convenience. Hence, $t^{m_1-1}e^{c_1t}$ is a solution to $(\bigstar)$. Furthermore, since $Ce^{c_1 \cdot 0} = C \neq 0$ in the case that $t=0$, it is certainly not the zero function. As such, $t^{m_1-1}e^{c_1t}$ cannot be expressed as a linear combination of functions in S_n; implying the linear independence of S_{n+1}.

Presume f is a solution to $p(D)(y)=0$. Thence, f satisfies $(\bigstar)$ for some value of $A \in \mathbb{C}$, and so does $\frac{A}{C}t^{m_1-1}e^{c_1t}$ for the same value of A by the above result. Consequently, $(D-c_1I)^{m_1-1}(D-c_2I)^{m_2}\cdots(D-c_kI)^{m_k}(f - \frac{A}{C}t^{m_1-1}e^{c_1t}) = 0$. By our initial assumption / induction hypothesis, $f - \frac{A}{C}t^{m_1-1}e^{c_1t}$ is a linear combination of functions in S_n. Accordingly, f is a linear combination of functions in S_{n+1}. That is to say, $\operatorname{span}(S_{n+1}) = N(p(D))$. Now, S_{n+1} is a basis for $N(p(D))$. ~~by our initial assumption / induction hypothesis.~~

In other words, the initial claim is true of $n+1$ too. Wherefore, it is true for each $n \in \mathbb{N}$ by induction.

Exercises

1\. (a) True ✓

(b) True ✓

(c) False ✓

(d) ~~True~~ ✗ False "Any" solution!

(e) True ✓

(f) False ✓ (it is not given that $p(t)$ is of degree k)

(g) True ✓

$a = e$

$ae^c = e^e$ $\quad ae^{2c} = e^{e^2}$

$e^c = e^{e-1}$ $\quad e^{2c} = e^{e^2-1}$

ae^{ct} $\quad e^{e^t}$

2\. (a) The statement is false. Counterexample: We know $\mathrm{span}(\{e^{e^t}\})$ is a subspace of C^∞, yet for any homogeneous linear differential equation of order 1, i.e. one of the form $(D - cI)(y) = 0$ for some $c \in \mathbb{C}$, Theorem 2.30 says the solution space has basis e^{ct}. Since for any $a, c \in \mathbb{C}$: $ae^0 = e$, $ae^c = e^e$, and $ae^{2c} = e$ would imply $e^c = e^{e-1}$ and $e^{2c} = e^{e^2-1}$, a contradiction as $2e - 1 \neq e^2 - 1$, this means $\mathrm{span}(\{e^{e^t}\})$ is the solution space of no homogeneous linear differential equation of order 1, or in fact of any finite order as $\mathrm{span}(\{e^{e^t}\})$ is of dimension 1.

~~(b) This is true: simply take the differential equation $D^3(y) = 0$. Then Theorem 2.34 says it has basis {~~

(b) This is also false as Theorem 2.34 tells us that, for any homogeneous linear differential equation, if $\{t, t^2\}$ is included in its solution space, so must {1}

(c) Contrastingly, this is true. Consider any homogeneous linear differential equation $p(D)(y) = 0$. We notice $p(D)(y') = p(D)(D(y)) = D(p(D)(y)) = D(0) = 0$ which means y' is also a solution.

(d) Indeed, we see that $p(D)q(D)(x+y) = p(D)q(D)(x) + p(D)q(D)(y) = q(D)(p(D)(x)) + p(D)(q(D)(y)) = 0$. Hence, the statement must be t

(e) Ideas / Sketch

$(D-I)(D+I)(1) = (D^2 - I^2)(1) = D^2(1) - 1^2 = -1 \neq 0$

$e^t \quad e^{-t}$

$e^{t-t} = 1$

Proof

This is false: case in point: consider $p(D) := D - I$, $q(D) := D + I$, and naturally, $x := e^t$ and $y := e^{-t}$. Then, $p(D)q(D)(xy) =$ $(D^2 - I^2)(e^t e^{-t}) = D^2(1) - I^2(1) = -1 \neq 0$.

$\square$

3. (e) The auxillary polynomial here is $t^3 - t^2 + 3t + 5 = (t+1)(t-(1+2i))(t-(1-2i))$. Thus, it has roots $t = -1, 1+2i, 1-2i$.
Accordingly, our differential equation has basis $\{e^{-t}, e^{(1+2i)t}, e^{(1-2i)t}\}$. OR: $\{e^{-t}, e^{t}\cos(2t), e^{t}\sin(2t)\}$

4. (c) The corresponding differential equation has auxillary polynomial $t^3 + 6t^2 + 8t = t(t^2+6t+8) = t(t+4)(t+2)$, with roots $t = 0, 2, 4$. As such, $N(D^3 + 6D^2 + 8D)$ has basis $\{1, e^{-2t}, e^{-4t}\}$.

7. Ideas

$\frac{1}{2}e^{0i}$ $\quad \frac{1}{2}e^{\frac{i\pi}{2}}$ $\quad \frac{1}{2}(x+y)$ $\quad \frac{1}{2i}(x-y)$ $\quad x \cdot i = 1$

$x+y$ $\quad \frac{1}{i}(x-y) = -i(x-y)$
$= i(-x+y)$

$a+bi = \alpha(x+y) + i\beta(-x+y)$

$\alpha = \frac{a}{x+y}$ $\quad \beta =$

$(x+y \neq 0$ lest not basis$)$

Proof

Let $a+bi \in \mathbb{C}$. So, $\frac{2a}{x+y} \cdot \frac{1}{2}(x+y) + \frac{2b}{-x+y} \cdot \frac{1}{2i}(x-y) = a+bi$ because x and y are distinct and nonzero by the fact that $\{x, y\}$ is a basis.

8. Ideas

$e^{(a+ib)t}$
$= e^{at}[\cos(b) + i\sin(b)]$

$e^{at}\cos(b)$

$e^{(a-ib)t}$
$= e^{at}[\cos(-b) + i\sin(-b)]$
$= e^{at}[\cos(b) - i\sin(b)]$

Proof

This follows easily from 7.

10. & 11. See the self proofs of these Theorems.

Indep... means ... this is true ... Theorem 2.34 ... equation ... the solution ... ae^c ... differentials...

12. Ideas

$g(t)h(t) = h(t)g(t)$

$V := N(p(D))$

$y \in V: h(D)g(D)(y) = g(D)h(D)(y) = 0$

$\Rightarrow g(D)(y) \in N(h(D))$

$g(D)(V) \subseteq N(h(D))$

$h(D)(y) = 0$

$\Rightarrow h(D)g(D)(y) = g(D)h(D)(y) = 0$

$t^n e^{ct}$

$(D-cI)(u) = t^n e^{ct}$

$(D-cI)(t^{n+1}e^{ct}) = (n+1)t^n e^{ct} + ct^{n+1}e^{ct} - ct^{n+1}e^{ct}$

$(D-c_1I)(\alpha t^{n+1}e^{ct}) = t^n e^{ct}$

$\alpha(n+1)t^n e^{ct} + \alpha(c-c_1)t^{n+1}e^{ct} = t^n e^{ct}$

Proof

We see that $h(D)g(D)[V] = g(D)h(D)[V] := p(D)[V] = \{0\}$ so $g(D)[V] \subseteq N(h(D))$ is certain. Now, we claim that $N(h(D)) \subseteq g(D)[V]$. To show this, consider any basis vector $t^n e^{ct}$ of $N(h(D))$. As in the sort-proof of Theorem 2.32, we first notice $(D-cI)^m(t^{n+m}e^{ct}) = At^n e^{ct}$ for a constant $A \in \mathbb{C}$ (namely, $\frac{(n+m)!}{n!}$), and by setting $\alpha_{ki} := (-1)^{k-i}\frac{k!}{i!}(c-b_j)^{i-k-1}$, $(D-b_jI)\left(\sum_{i=1}^{k}\alpha_{ki}t^i e^{ct}\right) = t^k e^{ct}$ for $b \neq c$ and $k \in \mathbb{N}$. Therefore, there indeed exists a ~~positive~~ choice for $x \in V$ that meets the criteria of $g(D)(x) = t^n e^{ct}$, namely $x := A^{-1}\sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\cdots\sum_{i_r=0}^{i_{r-1}}\alpha_{ki_1}\alpha_{i_1i_2}\alpha_{i_2i_3}\cdots\alpha_{i_{r-1}i_r}t^{i_r}e^{ct}$, where $k := n+m$ here and r $(\leq \dim(V))$ is the number of (possibly repeated) roots $b_j \neq c$ in $p(t)$ and m the number of repeated roots $b = c$ in $p(t)$. As such, $N(h(D)) \subseteq g(D)[V]$ indeed holds.

Consequently, $N(h(D)) = g(D)[V]$.

$\square$

More ideation lol

consider $0 \leq i \leq n$

$(D-c_1I)(\alpha_i t^i e^{ct})$

$= i\alpha_i t^{i-1}e^{ct} + \alpha_i(c-c_1)t^i e^{ct}$

$\alpha_n(c-c_1)\cancel{t^n e^{ct}} = \cancel{t^n e^{ct}}$

$\alpha_n = (c-c_1)^{-1}$

$0 \leq i \leq n-1:$

$\alpha_{i+1}\cancel{t^i e^{ct}} + \alpha_i(c-c_1)\cancel{t^i e^{ct}} = 0$

$(i+1)\alpha_{i+1} + \alpha_i(c-c_1) = 0$

$n\alpha_n + \alpha_{n-1}(c-c_1) = 0$

$n + \alpha_{n-1}(c-c_1)^2 = 0$

$\alpha_{n-1} = -n(c-c_1)^{-2}$

$\alpha_i = \frac{-(i+1)\alpha_{i+1}}{c-c_1}$

$\alpha_{n-1} = \frac{-n(c-c_1)^{-1}}{c-c_1} = -n(c-c_1)^{-2}$

$\alpha_{n-2} = -n(n-1)(c-c_1)^{-3}$

Conjecture: $i = n-(n-i)$

$\alpha_i = (-1)^{n-i}\frac{n!}{i!}(c-c_1)^{i-n-1}$

Checked with Wolfram

$N(h(D)) \subseteq V \subseteq R(g(D))$

$\sum_{i_1=0}^{k}\alpha_{ki_1}\sum_{i_2=0}^{i_1}\alpha_{i_1i_2}\sum_{i_3=0}^{i_2}\alpha_{i_2i_3}$

$\sum_{i=0}^{k}\alpha_{ki}\sum_{j=0}^{i}\alpha_{ij}t^j e^j$

$A^{-1}\sum_{i_1=0}^{k}\sum_{i_2=0}^{i_1}\sum_{i_3=0}^{i_2}\cdots\sum_{i_r=0}^{i_{r-1}}\alpha_{ki_1}\alpha_{i_1i_2}\alpha_{i_2i_3}\cdots\alpha_{i_{r-1}i_r}t^{i_r}e^{ct}$

$t^i e^{ct}$

α_{ki}

$\sum \alpha_{ki}\alpha_{ij}$

13. (a) Idea:

$(D - cI)(y) = x$

$y' - cy = x$

$(D - cI)(cx + x') = cx' - c^2 x + x'' - cx'$

$(D - cI)(cx + cx' + x'')$
$= cx' - c^2 x$
$+ cx'' - c^2 x'$
$+ x''' - cx''$

$(D - cI)\left(\sum_{i=0}^{\infty} c^{-i-1} x^{(i)}\right) = \sum_{i=0}^{\infty} c^{-i-1} x^{(i+1)} - c^{-i} x^{(i)}$
$= -c^{-0} x^{(0)}$
$= x$

$y' - cy = x^2$

$2\alpha t + \beta - c\alpha t^2 - c\beta t - c\gamma = t^2$

$-c\alpha = 1$
$\alpha = -\frac{1}{c} = -c^{-1}$

$2\alpha - c\beta = 0$
$-2c^{-1} - c\beta = 0$
$-2 - c^2\beta = 0$
$\beta = -2c^{-2}$

$\beta - c\gamma = 0$
$\gamma = \beta c^{-1}$
$= -2c^{-3}$

$(D - cI)(c^{-1}x'') = c^{-1}x''' - x''$

For each $i \geq 0$, $c^{-i-1} x^{(i+1)} - c^{-[i+1]} x^{(i+1)} = 0$

$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y^{(1)} + a_0 y = x$

$(D - c_1 I)(D - c_2 I) \cdots (D - c_n I) = x$

Proof

We first focus on the case of a 1st order homogeneous linear differential equation: Notice that since $c^{-i-1} x^{(i+1)} - c^{-[i+1]} x^{(i+1)} = 0$ for $i \geq 0$ (where square brackets are used as normal brackets) $(D - cI)\left(\sum_{i=0}^{\infty} c^{-i-1} x^{(i)}\right) = \sum_{i=0}^{\infty} c^{-i-1} x^{(i+1)} - c^{-i} x^{(i)} = -c^{-0} x^{(0)} = x$. Indeed, (a) is true in this case. It is a simple inductive step to generalise this to all homogeneous linear differential equations of order any $n \in \mathbb{N}$.

(b) Suppose f is any solution to the nonhomogeneous linear differential equation $p(D)(y) = 0$. Then, taking the difference of the equations, we have that $p(D)(f - z) = 0$. So, $f = z + (f - z)$, where $f - z \in V$ follows from the previous line. That is, the set of all solutions to the nonhomogeneous equation is a subset of V.

Conversely, $p(D)(z + y) = x + 0 = x$ as expected. Therefore, equality holds. $\square$

(6) Suppo...

14. Ideas

ae^{ct} $\quad$ $ae^{ct} + bte^{ct}$ $\quad$ $ae^{c_1t} + be^{c_2t}$

$ae^0 = 0$ $\quad$ $ace^{ct} + be^{ct} + bcte^{ct}$ $\quad$ $ac_1e^{c_1t} + bc_2e^{c_2t}$

$a = c$ $\quad$ $(ac+b)e^{ct} + bcte^{ct}$ $\quad$ $ac_1 + bc_2 = 0,$

$a = 0$ $\quad$ $ac + b = 0$ $\quad$ $a + b = 0$

$b = 0$ $\quad$ $b = -a$ $\quad$ $a(c_1 - c_2) = 0$ $\quad$ $a = b = 0$

$p(D)(x) = p(D)(at^ke^{ct})$

$= Ce^{ct}$ for some $C \in \mathbb{C}$

$C = ak!(c-c_1)(c-c_2)\cdots(c-c_{n-k})$

$akt^{k-1}e^{ct} + a(c-c_1)t^ke^{ct}$

$akt^{k-1}e^{ct} + act^ke^{ct} - act^ke^{ct} = akt^{k-1}e^{ct}$

$n=1$ trivial

Assume true for $n \in \mathbb{N}$

$(D-cI)(at^ke^{ct}) = akt^{k-1}e^{ct} + act^ke^{ct}$ ...

$a(c-c_1)e^{ct}$ $\quad$ $(D-cI)(x) = x^{(1)} - cx =$

$0 = akt^{k-1}e^{ct}$, $t=0$

$at^ke^{ct} = \sum_{i=0}^{n} a_ib_i$, $k \le n$

$ak! = 0$ $\quad$ by assumption & $e^{c\cdot 0} = 1$

$a = 0$

Proof

~~When $n=1$, this follows from Theorem 2.30. So assume this is true for $n \in \mathbb{N}$ and $p(D)(y)=0$ is an $(n+1)$th order homogeneous differential equation in which any solution x is such that $x(t_0) = x'(t_0) = \cdots = x^{(n)}(t_0) = 0$. Then there exists $q(D)$ with $p(D) = (D-cI)q(D)$ for some $c \in \mathbb{C}$. Let $\{b_i \mid 1 \le i \le n\}$ be the basis, as specified in Theorem 2.34, for the solution space of $q(D)(y)=0$. For some $k \le n$, $\{b_i \mid 1 \le i \le n\} \cup \{t^ke^{ct}\}$ is a basis for the solution space of $p(D)(y)=0$. Given any solution $x = \sum_{i=1}^{n} a_ib_i + at^ke^{ct}$ as specified in our assumption, we notice $0 = [(D-cI)^k(x)](0) = [(D-cI)^k(\sum_{i=1}^{k} a_ib_i)](0) + ak!e^0 = ak!$ by assumption. As such, $a = 0$ by virtue of $k! \neq 0$. Therefore, the statement is true for every $p(D)(y)=0$ of order $n+1$ too. By induction, this is true for all $n \in \mathbb{N}$.~~

Since the case of $n=1$ follows from Theorem 2.30, assume this is true for $n \in \mathbb{N}$ and that $p(D)(y)=0$ is an $(n+1)$th order homogeneous linear differential equation, with $p(D) = (D-cI)q(D)$, such that $x(t_0) = x'(t_0) = \cdots = x^{(n)}(t_0) = 0$; $\{b_i \mid 1 \le i \le n\}$ and $\{b_i \mid 1 \le i \le n\} \cup \{t^ke^{ct}\}$ the bases for the solution spaces of $q(D)(y)=0$ and $p(D)(y)=0$, as specified by Thm 2.34. For any solution x of $p(D)(x)=0$, $q(D)(x) = q(D)\left(\sum_{i=1}^{n} a_ib_i + at^ke^{ct}\right) = Ce^{ct}$ where $C := ak!(c-c_1)(c-c_2)\cdots(c-c_{n-k})$ for the roots $c_i \neq c$ of $q(t)$. So, by construction, $Ce^{ct_0} = 0$ so $C = 0$. Hence, $a = 0$. Accordingly, each $a_i = 0$ by assumption, since $a = 0$ means x is a solution to $q(D)(x) = 0$ with $x(t_0) = x'(t_0) = \cdots = x^{(n-1)}(t_0) = 0$. Thus, the statement is true for $p(D)(y)=0$ of order $n+1$ too. By induction, this is true for all $n \in \mathbb{N}$. $\square$

5. (a) Linearity is clear from definition while the second claim follows from exercise 14. Since $N(\Phi)=0$, Φ is hence an isomorphism according to Theorem 2.5 because $\dim(V)=\dim(\mathbb{C}^n)=n$ is already known.

(b) By (a), the required unique $x \in V$ is given by $\Phi^{-1}\begin{pmatrix} x(t_0) \\ x'(t_0) \\ \vdots \\ x^{(n-1)}(t_0) \end{pmatrix}$. $\square$

9. Ideas / Thought Organisation

1. More generality = better, obviously isotrue:
2. The space of real solutions, $S_{\mathbb{R}}$, is a subspace of the solution space $V \subseteq C^\infty$ for any HLDE $p(D)(y)=0$, so it is generated by the same basis β for V as in Theorem 2.34. In fact, letting $c_1, c_2, \ldots, c_k$ be roots of $p(t)$ whose conjugates are also roots of $p(t)$, and $N(c_i)$ the number of times $t-c_i$ is repeated in $p(t)$, the set $\{t^{i_j}e^{t\,\mathrm{Re}(c_i)}\cos(t\,\mathrm{Im}(c_i)) + t^{i_j}e^{t\,\mathrm{Re}(c_i)}\sin(t\,\mathrm{Im}(c_i)) \mid 1 \le i \le k \;\&\; 0 \le i_j \le \min\{N(c_i), N(\overline{c_i})\}-1\}$ over $\mathbb{R}$ is a basis for $S_{\mathbb{R}}$.

So, we still know how to find the general real solution to any HLDE.

3. Moreover, if the coefficients of the HLDE are real, which is the case for physical motion, all conjugates of roots are also roots (of $p(t)$) and occur the same number of times.

Self-Proof of Theorem 3.13

Suppose x is a solution to $(CA)x = Cb$. Then $Ax = C^{-1}(CA)x = C^{-1}(Cb) = b$. Conversely, when x is a solution to $Ax = b$, $(CA)x = C(Ax) = Cb$ as expected. Thus, the biconditional holds.

Self-Proof of Corollary

Assume that $(A'|b')$ is obtained from $(A|b)$ by a finite number of elementary row operations. Let $E_1, E_2, \ldots, E_n$ be the associated elementary matrices and define $M := E_1E_2\cdots E_n$, an invertible matrix. So, Exercise 15 of Section 3.2 says $(A'|b') = (MA|Mb)$. That is, $A' = MA$ and $b' = Mb$. Therefore, since the system $(MA)x = Mb$ is equivalent to $Ax = b$ by the above Theorem, the system $A'x = b'$ must be equivalent to $Ax = b$.

Ideas

Self-Proof of Theorem 3.14

Let A be any $m\times n$ matrix. Exercise 12 of Section 3.1 says that ~~we can transform A into an upper triangular matrix A'' via elementary row operations~~. ~~without loss of generality~~, suppose A'' satisfies (a) and (c). Thus, we just need to focus on part 2 of Gaussian Elimination. If $n=1$, the matrix A is already in its reduced row echelon form, A''. So assume the result is true of $n \in \mathbb{N}$. Define

Apply induction similarly as in Exercise 12 of Section 3.1

Self-Proof of Theorem 3.15

(a) Let a_j be the jth column of A (the $m\times n$ matrix) of $(A|b)$ and ℓ_i be the least natural so $A_{i\ell_i} \neq 0$ for each $1 \le i \le r$. By condition (b) & (c) of the reduced row echelon form, $\{a_{\ell_i} \mid 1 \le i \le r\} = \{e_i \mid 1\le i\le r\}$ is linearly independent. In fact, as all other rows are zero by condition (a), $\operatorname{rank}(A) = \operatorname{rank}(A|b) = r$.

(b) Ideas / thought organization

For every $i \neq i'$, $A_{i'\ell_i} = 0$ / columns $a_{\ell_i} = e_j$ for some $1 \le j \le n$

$x_{\ell_i} = -\sum_{j=\ell_i+1}^{n} A_{ij}x_j - b_{i1}$ $\quad A_j = 0$ if $j \in \{\ell_i \mid 1\le i\le r\}$

$\sum_{i=1}^{n-r} t_i u_i = 0 \qquad b_{\ell_i} e_{\ell_i}$

Uh why did I tag this page again?

Proof

Let s_0 be a solution to $Ax=b$. For any $1 \le i \le n-r$, there exists $j_i \in \{1,2,\ldots,n\} - \{\ell_i \mid 1\le i\le r\}$ with $t_i = x_{j_i}$, so $(u_i)_{j_i 1} = 1$ and $(u_i)_{j_{i'} 1} = 0$ when $i \neq i'$. Hence, for any scalars $a_i \in F$, if $\sum_{i=1}^{n-r} a_i u_i = 0$, then every $a_i = 0$ (to have the j_ith entry be 0 for all i). Hence, since the solution set for $Ax=0$ is of rank $n-r$ from (a), this means $\{u_i \mid 1 \le i \le n-r\}$ is a basis for it as expected. Similarly, since we have that $(s_0)_{\ell_i 1} = b_{i1}$ if $1 \le i \le r$ and $(s_0)_{i1} = 0$ otherwise, therefore $As_0 = \sum_{i=1}^{r} (s_0)_{\ell_i 1} a_{\ell_i} = \sum_{i=1}^{r} b_{i1} e_{\ell_i} = b$ indeed.

Self-Proof of Theorem 3.1 $\underset{\text{1st row of}}{}\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} = \sum_{i=1}^{4} a_i r_i$

Exercise 7

Ideas

$p_j :=$ jth row of EA, $f_j :=$ jth row of E, $a_j :=$ jth row of A, i_j: the jth row of I_m : $\overset{I}{i_{j_1}} = \overset{E}{f_{j_2}}$ & $i_{j_2} = f_{j_1}$ for some $0 \le j_1, j_2 \le m$, $j_1 \ne j_2$

$q_j :=$ jth row of B

$\underset{A}{a_{j_1}} = \underset{B}{q_{j_2}}$ & $a_{j_2} = q_{j_1}$ $\qquad I_{j,k} = E_{j,k}$, $I_{j,k} \ne E_{j,k}$ otherwise

$A_{j,k} = B_{j,k}$, $A_{j,k} \ne B_{j,k}$

(1): $p_j = \sum_{k=1}^{m} (f_j)_{1k} a_k \overset{?}{=} q_j = \{$

$=$

$$(EA)_{ij} = \sum_{k=1}^{m} E_{ik} A_{kj}$$

$$\text{if } i \ne i_1, i_2 = \sum_{k \in M - \{i_1, i_2\}} I_{ik} B_{kj} + \overbrace{I_{i i_1}}^{0} B_{i_1 j} + \overbrace{I_{i i_2}}^{0} B_{i_2 j}$$

$E_{ii} A_{ij}$

$$= I_{ii} B_{ij}$$

$$= B_{ij}$$

if $i = i_1$, $\qquad E_{i_1 k} = I_{i_2 k}$

$$(EA)_{i_1 j} = \sum_{k \in M - \{i_1, i_2\}} I_{i_2 k} B_{kj} + I_{i_1 i_1} B_{i_1 j} + \overbrace{I_{i_2 i_1}}^{0} B_{i_2 j}$$

$$= B_{i_1 j}$$

Proof

Let E and B be the $m \times m$ and $m \times n$ matrices obtained from I and A respectively by means of a type 1 elementary operation. There exists distinct naturals $0 \le i_1, i_2 \le m$ with $I_{i_1 k} = E_{i_2 k}$, $I_{i_2 k} = E_{i_1 k}$, and $A_{i_1 k} = B_{i_2 k}$, $A_{i_2 k} = B_{i_1 k}$ for all $0 \le k \le m$. I agrees with E and A agrees with B everywhere else. Then, if $i \ne i_1$ and $i \ne i_2$, $(EA)_{ij} := \sum_{k=1}^{m} E_{ik} A_{kj} = \sum_{k=1}^{m} I_{ik} A_{kj} = A_{ij} = B_{ij}$. Even when $i = i_1$ (or $i = i_2$), $(EA)_{i_1 j} := \sum_{k=1}^{m} E_{i_1 k} A_{kj} = E_{i_1 i_2} A_{i_2 j} = I_{i_2 i_2} B_{i_1 j} = B_{i_1 j}$. Hence, $EA = B$. The rest follow similarly.

To get the same result for columns, just apply transposition. $\square$

Example 2 $\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\text{preserves}}{} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right)$

$$EA = \left(\overset{\text{row ex}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Self-Proof of Theorem 3.2

Ideas Elementary matrix E, $\qquad I = FE$

$I_m = EF$ (m×m)

$F = E^{-1}$ by Ex 10.(b) of section 2.4

Proof

Let E be an elementary m×m matrix. By Theorem 3.1, there exists some m×m matrix $\hat{F}$ of the same type with $I = FE$ since I can be obtained from E by means of an elementary row operation. Hence, $F = E^{-1}$ by exercise 10(b) of section 2.4. Same holds for elementary column operations. □

Exercises

1. (a) True
   (b) False, "scalar multiple". multiplication of any row/column by $\mathbb{I}$.
   (c) True
   (d) False: Consider two elementary matrices E and F obtained by interchanging rows 1 and 2, and rows 3 and 4 of I respectively. Then EF cannot be expressed with only one elementary row operation, since 4 rows differ from I, while any single elementary row operation can only fix at most two of those. Theorem 3.1 says
   (e) True, Theorem 3.2.
   (f) False, consider $I_2 + I_2$.
   (g) True
   (h) False
   (i) True

Nice

$$\begin{pmatrix} 0 & 3 & 2 \\ 0 & 2 & 2 \\ 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

2. An elementary operation that transforms A into B is one which adds -2 of the 1st column to the 2nd. For B to C, add -1 of the 1st row to the 2nd row. Finally, to convert C to I_3, the following elementary operations form one way:
   i. Multiplying the 2nd row by $-\frac{1}{2}$: $\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}$,
   ii. Adding -1 of the 3rd row to the 1st row: $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix}$,
   iii. Adding the 1st row to the 3rd row: $\begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}$,
   iv. Adding -2 of the 2nd row to the 1st row: $\begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}$,
   v. Add -1 of the 1st row to the 2nd row: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix}$,
   vi. Add -3 of the 2nd row to the 3rd row: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$,
   vii. Swap the 1st and 3rd rows: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$,
   viii. Swap the 2nd and 3rd rows: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Indeed, we have I_3 now.

3. (a) To obtain I_3, swap rows 1 and 3 of $\begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$. Hence, its inverse is given by swapping rows 1 and 3 of I_3, i.e. itself.

(b) To obtain I_3, multiply row 2 of $\begin{pmatrix} 1&0&0 \\ 0&3&0 \\ 0&0&1 \end{pmatrix}$ by $\frac{1}{3}$. Thus, its inverse is $\begin{pmatrix} 1&0&0 \\ 0&\frac{1}{3}&0 \\ 0&0&1 \end{pmatrix}$.

(c) To get I_3, add 2 of row 1 to row 3 of $\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ -2&0&1 \end{pmatrix}$. Therefore, its inverse is $\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 2&0&1 \end{pmatrix}$.

Let E be an elementary matrix.

4. We evaluate casewise by the type of E:

Type 1

Suppose row m is swapped with row n to obtain E from I. Then, swapping columns m and n of I, notice we get E still.

Type 2

Say, row m is multiplied by some scalar $c \in \mathbb{F}$ in I to get E. Now we also have E by multiplying column m by c in I.

Type 3

Consider a scalar multiple $c \in \mathbb{F}$ of a row m being added to row n to give E from I. It follows that E is also the result of adding c times of column m to column n of I.

Essentially, the arguments here boil down to $I_{ij} = 1$ if $i = j$, and $I_{ij} = 0$ otherwise. Indeed, E can be obtained in at least 2 ways.

5. Assume E is an elementary matrix. Evaluate casewise again by type. $E_{mk} = E_{km}$ $(E^t)_{mk} = E_{km} = I_{km}$ or I_{mm}

Type 1

Suppose row m is swapped with row n to obtain E from I. Then notice E is symmetric so $E^t = E$, because $(E^t)_{mk} = E_{km}$, which evaluates to $I_{km} = E_{km}$ provided $k \neq n$ and to $I_{mm} = E_{nm} = E_{km}$ if $k = n$.

Type 2

Trivial.

Type 3

Consider a scalar multiple $c \in \mathbb{F}$ of a row m being added to row n to give E from I. Now, by adding c times of row n to row m in I, we have E^t.

The converse is trivial since $(E^t)^t = E$

6. Similar to 5.

7. See self-proof

8. Notice that each elementary operation has an inverse of the same type.

9. Assume the elementary row operation $\overset{R}{\wedge}$ swaps rows m and n for any suitable matrix E. Then, the following operations done in order clearly results in swapping of rows m and n:

i. Add row n to row m; $E'_{mk} = E_{mk} + E_{nk}$ ($\&$ $E'_{nk} = E_{nk}$),

ii. Add $-\mathbb{1}$ of row m to row n; $E''_{nk} = E_{nk} - E'_{mk} = -E_{mk}$ ($\&$ $E''_{mk} = E'_{mk}$),

iii. Add row n to row m; $E'''_{mk} = E_{nk}$ ($\&$ $E'''_{nk} = E''_{nk}$),

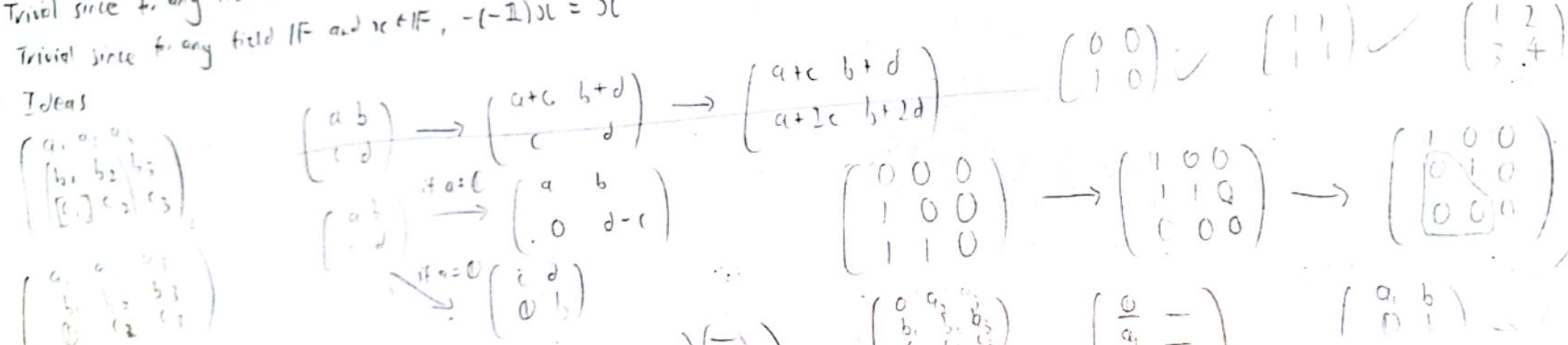
iv. Multiply row n by $-\mathbb{1}$; $E''''_{nk} = E_{nk} + E''''_{nk} = E_{mk}$ (for all k), so $E'''' = R(E)$ as expected.

(where the number of primes (′) p on E represents the matrix after applying the first p elementary row operations stated above, successively on E.) ~~www~~

10. Trivial since for any field $\mathbb{F}$ and $x \in \mathbb{F}$, $(x^{-1})^{-1} = x$.

11. Trivial since for any field $\mathbb{F}$ and $x \in \mathbb{F}$, $-(-\mathbb{1})x = x$

12. Ideas



~ Probably better to say: This sequence of elementary row ops keep $\bar{A}_n$ upper triangular as they only involve its multiple to other rows, which are all zero since $\bar{A}_n$ is upper triangular by construction. — 2nd ed.

→ Yeah the phrasing here isn't the best but I only did one iteration and no refinement.

Proof.

When $n = 1$, if ~~more than one entry of A is nonzero~~ $A_{a_1 1}, A_{a_2 1}, \dots, A_{a_k 1} \neq \mathbb{0}$ for some $k \in \mathbb{Z}^+_0$ and a_i's in $\mathbb{F}$, then apply the type 3 elementary row operations R_i, that adds $-\frac{A_{a_i 1}}{A_{a_1 1}}$ of row a_1 to row a_i, successively like this: $R_2 R_3 \cdots R_k(A) := \bar{A}$, swap rows a_1 and 1 of $\bar{A}$, thus forming an upper triangular matrix from A.

Assume this holds for $n \in \mathbb{N}$. For any $m \times (n+1)$ matrix A_{n+1}, let A_n be the $m \times n$ matrix defined by $(A_n)_{ij} := (A_{n+1})_{ij}$. If $n \geq m$, we are done. So, there is some k and sequence of elementary row operations O_i for $1 \leq i \leq k$ that transforms A_n into an upper triangular matrix $\bar{A}_n := O_1 O_2 \cdots O_k(A_n)$. Take the bottom right $(m-n+1) \times 1$ matrix A_1 defined by $(A_1)_{i1} := (\bar{A}_n)_{(i+n)(n+1)}$, for which from the $n=1$ case above, we know exists k' and a sequence of row operations R_i for $1 \leq i \leq k'$ with $R_1 R_2 \cdots R_{k'}(A_1)$ being upper triangular. Notice this sequence of transformations keep $\bar{A}_n$ upper triangular since $(\bar{A}_n)_{ij} = \mathbb{0}$ for $n+1 \leq i \leq m$ and $1 \leq j \leq n$.

Now, $R_1 R_2 \cdots R_{k'} O_1 O_2 \cdots O_k(A)$ must be an upper triangular matrix. Consequently, induction tells us this is true for all n (and m) in $\mathbb{N}$.

($\mathbb{1}$ by taking $O_1 O_2 \cdots O_k(A_{n+1})$)

Hence

... multiplied by some ... from I ... is ...

Self-Proof of Theorem 3.4

(a) By the invertibility of Q, $R(L_Q) = \mathbb{F}^n$ so $R(L_A L_Q) = R(L_A)$. Hence, $\operatorname{rank}(AQ) = \operatorname{rank}(A)$.

(b) Follows similarly.

(c) $\operatorname{rank}(PAQ) = \operatorname{rank}((PA)Q) \overset{(a)}{=} \operatorname{rank}(PA) \overset{(b)}{=} \operatorname{rank}(A)$ as expected.

Self-Proof of Corollary

Let $A \in M_{m \times n}(\mathbb{F})$ and B be obtained from A by an elementary row operation. By Theorem 3.1, there exists an elementary matrix E obtained from I_m by performing the same elementary row operation, such that $B = EA$. Since Theorem 3.2 says E is invertible, Theorem 3.4 tells us $\operatorname{rank}(B) = \operatorname{rank}(EA) = \operatorname{rank}(A)$.

Self Proof of Theorem 3.5

Ideas

Every Ax can be generated by $\{a_j \mid 1 \le j \le n\}$

$$Ax = \sum_{j=1}^{n} x_{j1} a_j$$

Proof

Let $A \in M_{m \times n}(\mathbb{F})$, a_j be the jth column of A and $x \in \mathbb{F}^n$; we know $Ax = \sum_{j=1}^{n} x_{j1} a_j$ so $R(L_A) = \operatorname{span}\{a_j \mid 1 \le j \le n\}$. And hence, $\operatorname{rank}(A) = \dim(\operatorname{span}\{a_j \mid 1 \le j \le n\})$ as expected.

Example 1

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

ups need column not row!

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Thus, $\operatorname{rank}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \dim\left(\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right\}\right) = 2$.

Self-Proof of Theorem 3.6

Similar to the proof for exercise 12 of Section 3.1.

Self-Proof of Corollary 1

From Theorems 3.1, 3.2, and 3.6.

$((C^t)^{-1}(B^{-1}D))^t$

$((C^t)^{-1}(D^t)\ (B^t)^{-1}$

Self-Proof of Corollary 2

(a) By Corollary 1, there exists a matrix D of the form $\begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$ and two invertible matrices B and C, with $D = BAC$ (or $A = B^{-1}DC^{-1}$). So, $\text{rank}(A^t) = \text{rank}(B^{-1}DC^{-1})^t = \text{rank}(((C^t)^{-1}(D^t)\ (B^t)^{-1}) = \text{rank}(D^t) = \text{rank}(D) = \text{rank}(A)$ by exercise 5 of section 2.4 and Theorem 3.4.

$R(A_n) = I_n$

$R^2(A_n) = R(I_n) = A^{-1}?$

(b) Follows from (a) and Theorem 3.5.

(c) By Theorem 3.5 and (b).

Self-Proof of Corollary 3

Let A be an invertible $m \times n$ matrix. So, it has rank m; by the proof of Corollary 1, there exists p, q and corresponding elementary matrices E_i and G_j for $1 \le i \le p$ and $1 \le j \le q$, with $(D =) I_n = \left(\prod_{i=1}^{p} E_i\right) A \left(\prod_{j=1}^{q} G_j\right)$. Hence, $A = \left(\prod_{i=1}^{p} E^{-1}_{p+1-i}\right)\left(\prod_{j=1}^{q} G^{-1}_{q+1-j}\right)$ where each E^{-1} and G^{-1} are elementary matrices by Theorem 3.2.

Self-Proof of Theorem 3.7:

(a) $R(UT) = U[R(T)] \subseteq U[W] = R(U)$. So, $\text{rank}(UT) \le \text{rank}(U)$.

(b) By the Dimension Theorem, $\text{rank}(UT) = \text{rank}(U_{R(T)}) = \text{rank}(T) - \text{nullity}(U_{R(T)}) \le \text{rank}(T)$.

(c) By (a).

(d) By (b).

$BI_n = A^{-1}$

$BBI_n = A^{-1}$

The Inverse of a Matrix

$I_n = BA$

$A^{-1} = B$

$R(A) = I_n$

$R(I_n) = A^{-1}$

$R^2(A) = A^{-1}$ ✓

$I_n = BAC$

$A = B^{-1}C^{-1}$

$A^{-1} = CB$

$O(A) = I_n$

$O(I_n) = BC + A^{-1}$

Exercises

1\. (a) False, its the number of linearly independent columns of the matrix.

(b) False, consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(e_1) = 0$ and $T(e_2) = T(e_3) = e_1$. Then, $\operatorname{rank}(T^2) = \operatorname{rank}(T_0) = 0 \neq 1 = \operatorname{rank}(T)$.

(c) True

(d) True

(e) False.

(f) True

(g) True

(h) True

(i) True

2\. (a) notice $\begin{pmatrix}1&1&0\\0&1&1\\1&1&0\end{pmatrix} \to \begin{pmatrix}0&1&0\\-1&1&1\\0&1&0\end{pmatrix} \to \begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}$ so $\operatorname{rank}\begin{pmatrix}1&1&0\\0&1&1\\1&1&0\end{pmatrix} = 2$.

(c) we see that $\begin{pmatrix}1&0&2\\1&1&4\end{pmatrix} \to \begin{pmatrix}1&0&2\\0&1&4\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$ thus $\operatorname{rank}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} = 2$ as well

(g) we note that $\begin{pmatrix}1&1&0&1\\2&2&0&2\\1&1&0&1\\1&1&0&1\end{pmatrix} \to \begin{pmatrix}1&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$ has rank 1.

3\. Trivial

4\. (a) $\begin{pmatrix}1&1&1&2\\2&0&-1&2\\1&1&1&2\end{pmatrix} \to \begin{pmatrix}0&1&0&0\\1&0&1&0\\0&1&0&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$ as required, so it has rank 2.

(b) $\begin{pmatrix}2&1\\-1&2\\2&1\end{pmatrix} \to \begin{pmatrix}0&1\\-5&2\\0&1\end{pmatrix} \to \begin{pmatrix}0&1\\1&2\\0&[illegible]\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$, again this has rank 2.

5\. (a) We compute $\left(\begin{array}{cc|cc}1&2&1&0\\1&1&0&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&2&1&0\\0&-1&-1&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&0&-1&2\\0&-1&-1&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&0&-1&2\\0&1&1&-1\end{array}\right)$. So, $\begin{pmatrix}1&2\\1&1\end{pmatrix}$ is of rank 2 and has the inverse $\begin{pmatrix}-1&2\\1&-1\end{pmatrix}$

(c) Again, $\left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\1&3&4&0&1&0\\2&3&-1&0&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&3&-1&1&0\\0&-1&-3&-2&0&0\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&3&-1&1&0\\0&0&0&-3&1&0\end{array}\right)$. Thus, $\begin{pmatrix}1&2&1\\0&1&3\\0&0&0\end{pmatrix}$ is of rank 2 and uninvertible.

(e) Once more, $\left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\-1&1&2&0&1&0\\1&0&1&0&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&3&3&1&1&0\\0&-2&0&-1&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&3&3&1&1&0\\0&-2&0&-1&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&3&3&1&1&0\\0&0&0&-\frac{1}{3}&\frac{2}{3}&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&1&1&1&0\\0&0&0&[illegible]&[illegible]&1\end{array}\right)$. Hence, $\begin{pmatrix}1&2&1\\-1&1&2\\1&0&1\end{pmatrix}$ has ran[k] [illegible]

2

$\to \left(\begin{array}{ccc|ccc}1&0&1&0&0&1\\0&1&1&\frac{1}{3}&\frac{1}{3}&0\\0&0&2&-\frac{1}{3}&\frac{2}{3}&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&0&[illegible]&[illegible]&[illegible]\\0&1&0&[illegible]&[illegible]&[illegible]\\0&0&1&[illegible]&[illegible]&[illegible]\end{array}\right)$

and cannot be inverted.

Hence it has rank 3 and inverse $\begin{pmatrix}[illegible]\end{pmatrix}$

6. (a) $T(1) = -1$, $T(x) = 2 - x$, $T(x^2) = 2 + 4x - x^2$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & -2 & 0 \\ 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \overset{X}{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -10 \\ 0 & 1 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)}$$

uh oops acci did column ops

T is invertible and is given by $T^{-1}(a + bx + cx^2) = -a - 2b - 10c - (b + 4c)x - cx^2$.

check: $f'(x) = -b - 4c - 2cx$, $f''(x) = -2c$, so $TT^{-1}(a + bx + cx^2) = -2c - 2b - 8c - 4cx + a + 2b + 10c + (b + 4c)x + cx^2$
$= a + bx + cx^2$

(c) $T(1,0,0) = (1,-1,1)$, $T(0,1,0) = (2,1,0)$, $T(0,0,1) = (1,2,1)$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 6 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

T is invertible and $T^{-1}(a,b,c) = \left(\frac{1}{6}a - \frac{1}{3}b + \frac{1}{2}c, \frac{1}{2}a - \frac{1}{2}c, -\frac{1}{6}a + \frac{1}{3}b + \frac{1}{2}c\right)$.

7. $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right]^{-1}.$$

And so $$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ Checked! with GC

8. By Theorem 2.15(c), $R(L_{cA}) = R(cL_A) = R(L_A)$. Thus $\text{rank}(cA) = \text{rank}(A)$.

9. See self-proof.

11. Let b_j be the jth column of B. We see that $b_1 \notin \text{Span}\{b_j \mid 2 \leq j \leq n\}$ so $\text{rank}(\bar{B}) = r - 1$ by Theorem 3.5, where $\bar{B}$ is the $m \times (n-1)$ matrix with $\bar{B}_{i,j} = B_{i,j+1}$.
Furthermore, since $\text{rank}(\bar{B})$ is clear isomorphic to to $\text{rank}(L_{B'})$, $\text{rank}(B') = \text{rank}(\bar{B}) = r - 1$.

(a) ... 1.6.2.

Self-Exercise

Prove that every inverse matrix A^{-1} can be computed exclusively by a finite sequence of elementary row operations on A.

Ideas: Let A_n be $n\times n$ invertible matrix.

$n=1$, A_1 trivial

Assume n, let a_j^c be A_{n+1}'s jth column, a_j^r its jth row

$a_1 \neq 0 \to e_1$ by $R_1 R_2 \cdots R_k$ $\quad$ $\{a_j^r \mid 1 \le j \le n\}$ is a basis for $M_{1\times n}(\mathbb{F})$

$\Rightarrow \sum_{j=1}^{n} c_j a_j^r = (1, 0, 0, \ldots, 0)$

Apply assumption / IH ✓

Proof. We first show that every $n\times n$ invertible matrix can be transformed into I_n by a finite sequence of row operations. When $n=1$, the result is trivial. So, assume this holds for $n\in\mathbb{N}$ and let A be a $(n+1)\times(n+1)$ invertible matrix, ~~e_j its jth column~~. As usual, the first column of A can be transformed into e_1 by a finite k_1 number of elementary row operations R_1 to R_{k_1}. Let $\bar{r}_j$ be the jth row of $R_1R_2\cdots R_{k_1}(A)$. Since $R_1R_2\cdots R_{k_1}(A)$ is of rank $n+1$, so by virtue of $\{\bar{r}_j \mid 1\le j\le n+1\}$ forming a basis for $M_{1\times(n+1)}(\mathbb{F})$, there exists some scalars $c_j\in\mathbb{F}$ for which $\sum_{j=1}^{n+1} c_j\bar{r}_j = (1,0,0,\ldots,0)$. Thus, there are some $k_2 \le n+1$ elementary row operations $\bar{R}_1$ to $\bar{R}_{k_2}$ with $\bar{R}_1\bar{R}_2\cdots\bar{R}_{k_2}(\bar{r}_1) = (1,0,0,\ldots,0)$. Now, the $n\times n$ matrix A' given by $A'_{i,j} := \bar{R}_1\bar{R}_2\cdots\bar{R}_{k_2}\bar{R}_1\bar{R}_2\cdots\bar{R}_{k_1}(A)_{i+1,j+1}$ has rank n by exercise 11 of this section, and can be transformed into I_n by a finite k_3 number of elementary row operations R'_1 to R'_{k_3}, as our assumption / induction hypothesis tells us. Therefore, the finite sequence $R'_1 R'_2\cdots R'_{k_3}\bar{R}_1\bar{R}_2\cdots\bar{R}_{k_2}R_1R_2\cdots R_{k_1}$ transforms A into I_{n+1}. ◇

Consequently, given any invertible $n\times n$ matrix A, there exists a finite k number of elementary row operations R_i with the composition $R := R_1R_2\cdots R_k$ that transforms A into I_n. Letting $E_i := R_i(I_n)$ be the corresponding elementary matrices and $B := E_1E_2\cdots E_k$ their product, we have $I_n = R(A) = BA$ so that $A^{-1} = BI_n = B^2A = R^2(A)$. Wherefore, the finite sequence of elementary row operations R^2 transforms A to A^{-1}. $\square$

13. See self-proof.

14. (a) Clear from the relevant definitions. ✓

(b) Trivial from (a) ✓

(c) Follows immediately from (b). ✓

15. Let x_j and y_j be the jth columns of M and $M(A|B)$, respectively. Since we know $y_j = \sum_{i=1}^{n} (A|B)_{ij} x_i$, it is clear that $M(A|B) = (MA \mid MB)$.

16. See self-proof.

17. We see that for some suitable scalars,

$\mathcal{B}$: $\operatorname{rank}(BC) \leq \operatorname{rank}(B) \leq 1$

$$BC = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} b_1c_1 & b_1c_2 & b_1c_3 \\ b_2c_1 & b_2c_2 & b_2c_3 \\ b_3c_1 & b_3c_2 & b_3c_3 \end{pmatrix} \to \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix}.$$ ✓

Hence, BC must have at most rank 1.

$\mathcal{B}$ → Ehh nah this kinda bad notation when you look at the following sentence.

Conversely, since A has rank 1, this suggests that its columns are scalar multiples of each other. That is, for some 3×1 matrix B, $A = \begin{pmatrix} B & c_2B & c_3B \end{pmatrix} = B\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}$. ✓

18. ~~Let $\bar{A}_j$ be the $m \times n$ matrix whose jth column is the jth column of A, and 0 everywhere else. Then, it is a clear extension of exercise 14, it Section 2.3 to say $AB = \sum_{j=1}^{n}\left(\sum_{i=1}^{p} B_{ji}\right.$~~

Ideas

(1) $(a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5)$

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1\alpha_1 + b_1\beta_1 & \cdots & \cdots \\ a_1\alpha_2 + b_1\beta_2 & & \\ a_1\alpha_3 + b_1\beta_3 & & \end{pmatrix}$$ rank ≤ 2 ?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 27 & 30 & 33 \\ 61 & 68 & 75 \\ 95 & 106 & 117 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & \frac{62}{61} & \frac{63}{61} \\ 61 & 68 & 75 \\ 95 & 106 & 117 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{62}{61} & \frac{63}{61} \\ 0 & 6 & 12 \\ 0 & -\frac{3890}{61} & -\frac{5985}{61} \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{61} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 & \frac{1}{61} \\ 0 & 6 & 12 \\ 0 & 0 & -\frac{95}{61} \end{pmatrix}$$

$\underbrace{v_1 \quad v_2}_{B}$ $\quad v_3 = av_1 + bv_2 \overset{?}{\to}$ $u_1 \quad u_2 \quad cu_1$

$v_1 - bv_2,\ (1-b)v_2,\ v_3 = av_1$

$u_1 + u_2 \quad 2u_2 \quad cu_1 + u_2$

$v_1 \quad v_2 \quad v_3$

$= c(u_1 + u_2) - \frac{(1-c)(2u_2)}{2}$

$= cv_1 - \frac{1-c}{2}v_2$

$b = \frac{c-1}{2}$

$2b + 1 = c$

$$AB = A\begin{pmatrix} I_{r_2} & O \\ O & O \end{pmatrix} = \begin{pmatrix} & O \\ & O \end{pmatrix}$$

$$\begin{pmatrix} | & | \\ | & | \\ | & | \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} | \\ | \\ | \end{pmatrix}$$

$$\left(\frac{A}{B}\right) M = \left(\frac{AM}{BM}\right)$$

18. Ideas

$AB \to \begin{pmatrix} I_{r_1} & O \\ O & O \end{pmatrix}\begin{pmatrix} I_{r_2} & O \\ O & O \end{pmatrix} \overset{\text{claim}}{=} \begin{pmatrix} I_{\min\{r_1,r_2\}} & O \\ O & O \end{pmatrix}$

1. $r_1 < i < r_2$
2. $r_1 > \quad > r_2$

Consider $1 \le i, j \le r$

$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{n} \delta_{ik}\delta_{kj} = \delta_{ij}$

$r_1 < i < r_2$: $(AB)_{ij} = \sum_{k=1}^{n} O B_{ij} = O$ as $A_{ik} = O$ if $r_1 < i$

$r_2 < i < r_1$: $(AB)_{ij} = O$

$(b_1 A \quad b_2 A \quad \cdots \quad b_p A)$ clearly rank ≤ 1

rank $(AET) \overset{\text{claim}}{=}$ rank (AT) if E is invertible

$R(AET) = L_A L_E [\![R(T)]\!]$

$= L_A \quad X$ since L_E is invertible

$\mathbb{F}^p \xrightarrow{L_B} \mathbb{F}^n \xrightarrow{L_A} \mathbb{F}^m$

rank$(L_B) \le n$ $\quad R(L_A L_B) = L_A[\![R(L_B)]\!]$

rank$(L_A L_B) =$ rank$(L_{A_{R(L_B)}})$

$\le n -$ nullity$(L_{A_{R(L_B)}})$

$\le n$.

Assume k

S_1 to S_n

$S_k :=$ the $m \times p$ matrix with $(S_k)_{ij} := A_{ij}$ if $i = j = k$, O otherwise.

Thm 3.7: rank$(AB) \le$ rank$(B) \le n$

$\sum_{k=1}^{n} S_k = \begin{pmatrix} I_{\text{rank}(AB)} & O \\ O & O \end{pmatrix} = P(AB)Q$ for invertible P, Q

$AB = \sum_{k=1}^{n} P^{-1} S_k Q^{-1}$

Proof

First, notice that by Theorem 3.7, rank$(AB) \le$ rank$(B) \le n$. Hence, for some invertible matrices P and Q, $\begin{pmatrix} I_{\text{rank}(AB)} & O \\ O & O \end{pmatrix} = P(AB)Q$ according to Corollary 1 to Theorem 3.6. Define the $m \times p$ matrices S_k by $(S_k)_{ij} = A_{ij}$ when $i = j = k$, and O otherwise. It is thus clear that rank$(S_k) \le 1$ and $\sum_{k=1}^{n} S_k = \begin{pmatrix} I_{\text{rank}(AB)} & O \\ O & O \end{pmatrix}$. Combining these results, we have that $AB = \sum_{k=1}^{n} P^{-1} S_k Q^{-1}$, where rank$(P^{-1} S_k Q^{-1}) =$ rank$(S_k) \le 1$ by Theorem 3.4(c). $\square$

Hence ... show that

19. Ideas $\mathbb{F}^p \xrightarrow{L_B} [\mathbb{F}^n \xrightarrow{L_A} [\mathbb{F}^m$

This suggests $p \geq n \geq m$, L_B is surjective so $\text{rank}(L_A L_B) = \text{rank}(L_A) = m$.

Proof

We see that $p \geq n \geq m$ for $\text{rank}(B) = n$ and $\text{rank}(A) = m$, where L_B must be surjective, so $\text{rank}(AB) = \text{rank}(L_A L_B) = \text{rank}(L_A) = m$.

20. (a) We want to find 2 possible linearly independent columns $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$ of M which are nonzero but will evaluate to zero in AM:

$$a_1\begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} + a_3\begin{pmatrix} -1 \\ 3 \\ 4 \\ -5 \end{pmatrix} + a_4\begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix} + a_5\begin{pmatrix} 1 \\ 0 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

By GC: $a_1 = a_3 + 3a_5$,
$a_2 = -2a_3 + a_5$,
$a_3 = a_3$,
$a_4 = -2a_5$,
$a_5 = a_5$.

So, two possible columns are $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -1 \\ 1 \\ -2 \\ 1 \end{pmatrix}$. Correspondingly, we have the matrix $M = \begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$. ✓

OR: $\begin{pmatrix} 1 & 3 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Indeed we can verify that $AM = O$.

(b) Ideas

$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 3 & -1 & 0 \\ -1 & 1 & 4 & -1 & 3 \\ 3 & -1 & 5 & 1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 3 & 1 & 4 \\ 0 & -1 & 8 & -5 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 11 & -6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & \frac{5}{2} & -1 & 0 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 8 & -6 & 3 \end{pmatrix}$

-5 ... -6 bruh

$\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 8 & -6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ X

$\mathbb{F}^5 \xrightarrow{L_B} \mathbb{F}^5 \xrightarrow{L_A} \mathbb{F}^4$

$\text{rank}(L_B) = 2$ $\quad \text{rank}(L_A) = 4$

$\text{nullity}(L_B) = 3$ $\quad \text{nullity}(L_A) = 1$

$\text{rank}(AB) + \text{nullity}(AB) = 5$

$0 \qquad \text{nullity}(AB) = 5$

$\text{nullity}(B) + \text{nullity}(A_{R(B)}) = 5$

$\text{nullity}(A_{R(B)}) = 2$

$L_{A_{R(B)}} : \underbrace{R(B)}_{\text{dim } 2} \longrightarrow \mathbb{F}^4$

$(a_1 \; a_2)\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

$\mathbb{R}^2 \to \mathbb{R}^1 \qquad \mathbb{R}^2 \to \mathbb{R}^2$

$(1, 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

20. (b) Notice A is of rank 3 so nullity$(L_A) = 2$ by the Dimension Theorem. As such, for $AB = O$, rank$(B) \leq 2$.

21 Since rank$(A) = m$, the columns a_i of A generate $\mathbb{F}^m$. So, for each $1 \leq j \leq m$, there exists scalars B_{ij} for $1 \leq i \leq n$, such that $e_j = \sum_{i=1}^{n} B_{ij} a_i$.
Defining B to be the $n \times m$ matrix with entries B_{ij}, it is clear that $AB = I_m$.

22. Similarly, by virtue of rank$(B) = m$, the rows b_j of B generate $M_{1 \times m}(\mathbb{F})$. As such, for any $1 \leq i \leq n$, there exists scalars A_{ij} for $1 \leq j \leq$ with $\sum_{j=1}^{n} A_{ij} b_j$ being the jth row of I_m. Hence, it is again straightforward to notice $AB = I_m$.

Self-Proof of Theorem 3.8

Trivial

Self-Proof of Corollary

If $m < n$, $\dim(K) = n - \text{rank}(A) \geq n - m > 0$.

Self-Proof of Theorem 3.9

Let s' be a solution to $Ax = b$. Then $A(s' - s) = 0$ so $s' = s + (s' - s) \in s + K_H$. Conversely, for $k \in K_H$, $A(s + k) = As + Ak = b$.
Hence, $K = s + K_H$

Self-Proof of Theorem 3.10

When A is invertible, for any solution x to $Ax = b$ we indeed have $x = A^{-1}b$. Which means $A^{-1}b$ is the only solution. So, consider $Ax = b$ having only 1 solution, say some s. Then, the solution space to $Ax = 0$ must be $\{0\}$, lest there exists some solution $s + k \neq s$ to $Ax = b$ by Theorem 3.9.
Accordingly, nullity$(A) = 0$ so A is invertible.

Self-Proof of Theorem 3.11

Ideas

If $\exists$ soln x, i.e. $Ax = b$, $\cancel{(A|b)\binom{x}{-1} = 0}$ b is a lin comb of columns of A.

If $b \neq 0$ & all soln y to $(A|b)y = 0$ are s.t. $y = \begin{pmatrix} \vdots \\ 0 \end{pmatrix}$, $\nexists$ no soln to $Ax = b$

rank$(A) \leq$ rank$(A|b)$

If rank$(A|b) =$ rank$(A)+1$, $\beta := \alpha \cup \{b\}$ is also lid (maximal) & $\sum_{a \in \alpha} c_a a \neq b$ $\forall c_a$

maximal α set of lid columns

$m \times (n+1)$

When rank$(A) =$ rank$(A|b)$, b lin comb of cols of A

$$b = \sum_{j=1}^{n} c_j a_j$$

Soln: $\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ -1 \end{pmatrix}$

to $(A|b)y = 0$

1 - Proof of Theorem 3.11

er a consistent system of linear equations $Ax = b$, with some solution x. Then, b can be expressed as a linear combination of columns in A.
efore, $\operatorname{rank}(A) = \operatorname{rank}(A|b)$ is clear by considering the maximal number of linearly independent columns. Conversely, when $\operatorname{rank}(A) = \operatorname{rank}(A|b)$,
s a linear combination of columns a_j of A. In other words, there are scalars $c_j \in \mathbb{F}$ with $b = \sum_{j=1}^{n} c_j a_j$. Hence, it is clear that $\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$ is a solution
$Ax = b$. Which implies the consistency of the system $Ax = b$. □

cises

lse ✓
alse ✓
True, considering any system $Ax = 0$, $x = 0$ is a solution.
False ✓
) False ✓ Consider $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$, then all $x \in \mathbb{F}^2$ is a solution.
) False ✓
g) True ✓ The only solution would be $x = A^{-1}0 = 0$.
) False ✓ consider $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the sole solution, thus the solution set does not contain $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and can't be a subspace of $\mathbb{F}^n$. ✓

Nice

(a) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ has rank 1. Thus, the solution space has dimension 1 and is spanned by the basis $\left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right\}$. ✓

(c) $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 3 & -1 \\ 3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ clearly has rank 2. So its solution space has dimension 1 and the basis $\left\{\begin{pmatrix} ? \end{pmatrix}\right\}$. ✓

(e) $(1\ 2\ {-3}\ 1)$ clearly is of rank 1, so its solution space is of dimension 3 and has the basis $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}\right\}$. Its linear independence is seen from the fact that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has rank 3.

theor- A is solution to Ax ... $s*(s=s)$ ... ere exist. $\sum a_{ij}x_j$.

2\. (g) $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$ has rank 2. Therefore, its solution space of dimension 2 has the basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

3\. (a) $x_1 = -3x_2$ NOT convert btw forms! 3. are nonhomogeneous! Oops

(c) $-x_1 = x_2 = x_3$

(e) Any solution can be represented as

$$a\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ -a-2b+3c \end{pmatrix},$$

which is equivalent to $x_4 = -x_1 - 2x_2 + 3x_3$.

(g) Similarly for (g), any solution can be written as $a\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2a-3b \\ b \\ a+b \\ a \end{pmatrix}$. Thence, any solution is given by the defining equations $x_1 = -2x_4 - 3x_2$ and $x_3 = x_4 + x_2$.

. (a)

(1) We first compute the inverse: $\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right)$.

(2) Then, the unique solution is given by $\begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}$, i.e. $-11x_1 = 5x_2$.

(b)

(1) Again, the inverse is computed by: $\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & -6 & 3 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & -9 & 4 & -6 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & 1 & -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{array}\right)$.

(2) Now, the unique solution is again $\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$. In other words, $3x_1 = -2x_3$.

One example is:

$$\begin{aligned} x_1 + x_2 + 0x_3 + 0x_4 + \cdots + 0x_n &= 1 \\ x_1 + x_2 + 0x_3 + 0x_4 + \cdots + 0x_n &= 1 \\ &\vdots \\ x_1 + x_2 + 0x_3 + 0x_4 + \cdots + 0x_n &= 1. \end{aligned}$$

It is represented by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

There are $2^{\aleph_0}$ solutions at least, because for each $x_1 \in \mathbb{R}$, $(x_1\ 1-x_1\ 0\ 0 \cdots 0)^t$ is a solution. As such, there indeed are infinitely many solutions.

6)

- We want to find $(a,b,c) \in \mathbb{R}^3$ such that $T(a,b,c) = (a+b, 2a-c) = (1, 11)$. We have the system of linear equations:

$$a+b=1 \quad \text{and} \quad 2a-c=11,$$

which translates to

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}.$$

To solve this, we first compute that $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has rank 2. The solution set of the corresponding homogeneous system hence has dimension 1 and basis $\left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\}$. Furthermore, $\begin{pmatrix} 11 \\ -10 \\ 22 \end{pmatrix}$ is a solution (to the non homogeneous system). This means that the required general solution is $\left\{\lambda\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 11 \\ -10 \\ 22 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{R}\right\}$.

7. (a) We compute the ranks:

i. $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 \\ 2 & 0 & 3 & 4 \end{pmatrix}$ has rank 2, while

ii. $\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right)$ has rank 3.

As such, no solutions exist.

(c) We again conduct rank computation:

$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$ tells us the coefficient matrix has identical rank as the augmented matrix.

Consequently, solutions certainly exist.

(e) Once more, compute ranks:

$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & -4 & 7 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & -3 & 4 & 1 \\ 1 & -6 & 8 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

The rank of the coefficient matrix is unequal to that of the augmented matrix. Thus, no solutions exist.

8. (a)

The condition $(a+b, b-2c, a+2c) = (1,3,-2)$ translates to the system of linear equations $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. We compute the ranks needed

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 1 & 0 & 2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 2 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 \end{array}\right).$$

By the rank equality, a solution exists

(b)

Again, we have a system of linear equations $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Computing the necessary ranks, $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right)$, we see that the rank is again equal. Hence, a solution exists.

9. Trivial.

10. When any $m \times n$ coefficient matrix A has rank m, the $m \times (n+1)$ augmented matrix $(A|b)$ must also be of equal rank. Hence, a solution definitely exists.

11. To solve $(I-A)x = 0$, we notice that Theorem 3.12 says it has a one-dimensional solution set spanned by a nonnegative vector, which can be $\begin{pmatrix} 1 \\ \frac{3}{4} \\ 1 \end{pmatrix}$ for example. Thus, $\frac{1}{1+\frac{3}{4}+1}\begin{pmatrix} 1 \\ \frac{3}{4} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{11} \\ \frac{3}{11} \\ \frac{4}{11} \end{pmatrix}$ is the required solution which tells us the ratio needed is $4:3$.

12. Simply solve $\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} x = 0$ into the required form.

13. We compute the inverse $(I-A)^{-1}$, if it exists:

$$\left(\begin{array}{cc|cc} \frac{1}{2} & -\frac{1}{5} & 1 & 0 \\ -\frac{1}{3} & \frac{4}{5} & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} \frac{1}{2} & -\frac{1}{5} & 1 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \frac{12}{5} & \frac{3}{5} \\ 0 & 1 & 1 & \frac{3}{2} \end{array}\right).$$

The inverse is thus $\begin{pmatrix} \frac{12}{5} & \frac{3}{5} \\ 1 & \frac{3}{2} \end{pmatrix}$. Accordingly, the solution is represented by $\begin{pmatrix} \frac{12}{5} & \frac{3}{5} \\ 1 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{29}{5} \\ \frac{19}{2} \end{pmatrix}$.

This means that $\frac{29}{5}$ units of commodity 1 and $\frac{19}{2}$ units of commodity 2 should be produced.

7. In the notation of the open model of Leontief, suppose that

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{3}{10} & \frac{3}{5} \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 90 \\ 20 \end{pmatrix}.$$

We notice that $(I-A)^{-1} = \begin{pmatrix} \frac{20}{7} & \frac{10}{7} \\ \frac{15}{7} & \frac{25}{7} \end{pmatrix}$. So, $(I-A)^{-1}d = \begin{pmatrix} \frac{2000}{7} \\ \frac{1850}{7} \end{pmatrix}$. Which means the total output of the economic system must be \$550 billion to support this defense system

Exercises

1. (a) False, its rows not columns.

(b) True

(c) True

(d) True

(e) False, we require the reduced row echelon form to be such that none of its rows has its first nonzero entry in the last column. [i.e. $\mathrm{rank}(A) = \mathrm{rank}(A|b) = \mathrm{rank}(A'|b')$

(f) True

(g) False X True: Oops *rows* not *cols*.

2. (a) We see that

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Hence, it is clear that $(4, -3, -1)^t$, or in other words; $x_1 = 4$, $x_2 = -3$, $x_3 = -1$ is the only solution.

(c) Notice:

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 3 & 5 & -1 & 6 & 17 \\ 2 & 4 & 1 & 2 & 12 \\ 2 & 0 & -7 & 11 & 7 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -4 & 0 & 7 & -5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{7} \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & \frac{44}{7} \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{7} \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{30}{7} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{7} \end{array}\right)$$

Thus, we again have only a singular solution, that is, $(\frac{30}{7}, 1, 0, -\frac{1}{7})^t$. X

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 6 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

The single solution is $(2, 3, -2, -1)^t$. ✓ 2nd try, at last I got it!! :)

(e) Once more, we compute the reduced row echelon form

$$\left(\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 2 & -8 & 1 & -4 & 9 \\ -1 & 4 & -2 & 5 & -6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & -3 & 6 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & -1 & 1 & 3 \\ 0 & 1 & -\frac{1}{8} & \frac{1}{2} & \frac{9}{8} \\ 0 & 0 & 1 & \frac{2}{3} & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & 0 & \frac{5}{3} & 4 \\ 0 & 1 & 0 & \frac{7}{12} & \frac{5}{4} \\ 0 & 0 & 1 & \frac{2}{3} & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 9 \\ 0 & 1 & 0 & \frac{7}{12} & \frac{5}{4} \\ 0 & 0 & 1 & \frac{2}{3} & 1 \end{array}\right)$$

$$\to \cancel{\left(\begin{array}{cccc|c} 1 & -4 & 0 & \frac{7}{3} & 4 \\ 0 & 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 0 & \frac{14}{3} & 10 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & 0 & \frac{5}{3} & 4 \\ 0 & 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & 0 & & \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right)}$$

Hence, the solution set is

$$\left\{ \begin{pmatrix} 9 \\ \frac{5}{4} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ -\frac{7}{12} \\ -\frac{2}{3} \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & -4 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left\{ \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \,\middle|\, r, s \in \mathbb{R} \right\}$$

2.(g)

$$\left(\begin{array}{ccccc|c} 2 & -2 & -1 & 6 & -2 & 1 \\ 1 & -1 & 1 & 2 & -1 & 2 \\ 4 & -4 & 5 & 7 & -1 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & -1 & -\frac{1}{2} & 3 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{3}{2} & -1 & 0 & \frac{3}{2} \\ 0 & 0 & 7 & -5 & 3 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 0 & \frac{8}{3} & -1 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & \cancel{-\frac{29}{3}}\ -\frac{1}{3} & 3 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & \cancel{-8} & 23 & -23 \\ 0 & 0 & 1 & 0 & & -6 & 7 \\ 0 & 0 & 0 & 1 & & -9 & 9 \end{array}\right)$$

The solution set is, as a result,

$$\left\{ \underbrace{\begin{pmatrix} 8 \\ 0 \\ \frac{33}{29} \\ \frac{9}{29} \\ 0 \end{pmatrix}}_{\times\ \begin{pmatrix} -23 \\ 0 \\ 7 \\ 9 \\ 0 \end{pmatrix}} + s \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\checkmark} + t \underbrace{\begin{pmatrix} -8 \\ 0 \\ -\frac{6}{29} \\ -\frac{9}{29} \\ 1 \end{pmatrix}}_{\times\ t\begin{pmatrix} -23 \\ 0 \\ 6 \\ 9 \\ 1 \end{pmatrix}} \;\middle|\; s, t \in \mathbb{R} \right\}$$

(f)

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

The solution set is hence

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; r \in \mathbb{R} \right\} \quad \times$$

$$\left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; r \in \mathbb{R} \right\}$$

ear that ...

2.(i) $\left(\begin{array}{ccccc|c} 3 & -1 & 2 & 4 & 1 & 2 \\ 1 & -1 & 2 & 3 & 1 & -1 \\ 2 & -3 & 6 & 9 & 4 & -5 \\ 7 & -2 & 4 & 8 & 1 & 6 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 1 & -1 \\ 0 & 2 & -4 & -5 & -2 & 5 \\ 0 & -1 & 2 & 3 & 2 & -3 \\ 0 & 5 & -10 & -13 & -6 & 13 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & -3 & -2 & 3 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 & 4 & -2 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Solution set : $\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + r\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -4 \\ 0 \\ -2 \\ 0 \end{pmatrix} \middle| \ r, s \in \mathbb{R} \right\}$

3.(a) Let rank $(A) = r$. When rank$(A') \neq$ rank$(A'|b')$, the $(r+1)$th row of A' is zero by Theorem 3.16 (a) and condition (a) but the same row of $(A'|b')$ must be nonzero by the difference in rank. Hence, this nonzero entry must lie in the last column. Conversely, consider the existence of a row k in which the only nonzero entry lies in the last column. So, the column b must be linearly independent of all other columns in A'. Consequently, rank$(A'|b') >$ rank(A').

(b) We see that $Ax = b$ is consistent iff rank(A) = rank$(A|b)$ iff rank(A') = rank$(A'|b')$ iff $(A'|b')$ contains no row in which the only nonzero entry lies in the last column by the Corollary to Theorem 3.13 and part (a) above.

4.(a) We compute the reduced row echelon form:

$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 2 & 1 & 1 & -1 & 3 \\ 1 & 2 & -3 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -3 & 3 & -3 & -1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & \frac{4}{3} \\ 0 & 1 & -1 & 1 & \frac{1}{3} \\ 0 & 0 & -2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & \cancel{\frac{8}{3}}\ \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right).$

So, the system is consistent by 3. and the solution set of the homogeneous system is $\left\{ r\begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix} \middle| \ r \in \mathbb{R} \right\}$ which has the basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix} \right\}$.

(c) We see that the reduced row echelon form has a row in which the only nonzero entry lies in the last column:

$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 1 \\ 0 & 0 & 4 & -2 & 1 \\ 0 & 0 & 2 & -1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & \frac{7}{4} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

Therefore, the system is inconsistent.

5. Let the matrix B be the reduced row echelon form of A. Then, using the notation of Theorem 3.16, $j_1 = 1$, $j_2 = 2$, $j_3 = 4$. Consequently, since $b_1 = 1 b_{j_1}$, $b_2 = 1 b_{j_2}$, and $b_4 = b_{j_3}$, $\{a_1, a_2, a_4\}$ should be linearly independent. To check for this, we row reduce:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & -1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Indeed, we see that the only solution is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. In other words, $\left\{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right\}$ is linearly independent and can be the first, second, and fourth columns of A, respectively.

7. As per normal, we row reduce:

$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -4 & 18 & -11 \\ 0 & -2 & 0 & 88 & -29 \\ 0 & -5 & 0 & -35 & 19 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 8 & -300 & 104 \\ 0 & 1 & -4 & 106 & -40 \\ 0 & 0 & -20 & 405 & -181 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -462 & \frac{193}{5} \\ 0 & 1 & 0 & 25 & -\frac{19}{5} \\ 0 & 0 & 1 & -\frac{81}{4} & \frac{181}{20} \end{pmatrix}$$

X

G.C. Check

$13 \overline{)\,-38}$ quotient -2, $-(-26)$, -12

$13 \overline{)\,91}$ quotient 7, remainder 0

$8 \cdot 13 = 80 + 24 = 104$

$7 \cdot 13 = 70 + 21 = 91$

So, $\{u_1, u_2, u_3\}$ is a basis for $\mathbb{R}^3$.

$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -4 & 18 & -11 \\ 0 & 13 & 0 & 41 & -38 \\ 0 & -5 & 0 & -35 & 19 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & -3 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Try 2

So, $\{u_1, u_2, u_5\}$ is a basis for $\mathbb{R}^3$.

11. (a) It suffices to notice that $1 - 2(2) + 3(1) - 0 + 2(0) = 1 - 4 + 3 = 0$, so $(1,2,1,0,0) \in V$. Hence $\{(1,2,1,0,0)\}$ is a linearly independent subset of V.

11b) Vectors in V have the form $(x_1, x_2, x_3, x_4, x_5) = (2t_1 - 3t_2 + t_3 - 2t_4, t_1, t_2, t_3, t_4) = t_1(2,1,0,0,0) + t_2(-3,0,1,0,0) + t_3(1,0,0,1,0) + t_4(-2,0,0,0,1)$. It is clear that $\{(2,1,0,0,0), (-3,0,1,0,0), (1,0,0,1,0), (-2,0,0,0,1)\}$ spans V by having $t_i := x_{i+1}$, and linear independence is trivial by noticing each vector in the above set has a nonzero entry that is zero in all other vectors. In other words, this set is a basis for V.

By row-reducing $\begin{pmatrix} 1 & 2 & -3 & 1 & -2 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, we got $\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Which means $\{(1,2,1,0,0), (2,1,0,0,0), (1,0,0,1,0), (-2,0,0,0,1)\}$ is a basis for V.

2. (a) First, notice that $S \subseteq V$ as expected since $0-(-1)+2-3+0=0$ and $2(0)-(-1)-0+3-4+4(0)=0$; $1-0+2-3+0=0$ and [illegible]. Furthermore, linear independence is ensured as $a(0,-1,0,1,1,0)+b(1,0,1,1,1,0)=(b,-a,b,a+b,a+b,0)=0$ means $-a=b=0$.

(b) As usual, we row reduce B: the coefficient matrix to find a basis for the solution space:

$$\begin{pmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 2 & -1 & -1 & 3 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 2 & -3 & 1 \\ 0 & 1 & -1 & -1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 3 \\ 0 & 1 & -1 & -1 & 2 & 2 \end{pmatrix}.$$

An equivalent system is hence

$$\begin{aligned} x_1 \quad - x_3 + x_4 - x_5 + 3x_6 &= 0 \\ x_2 - x_3 - x_4 + 2x_5 + 2x_6 &= 0. \end{aligned}$$

Let $t_i = x_{i+2}$, then any solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (t_1 - t_2 + t_3 - 3t_4,\ t_1 + t_2 - 2t_3 - 2t_4,\ t_1, t_2, t_3, t_4) = t_1(1,1,1,0,0,0) + t_2(-1,1,0,1,0,0) + t_3(1,-2,0,0,1,0) + t_4(-3,-2,0,0,0,1)$. By Theorem 3.15, the set $\{(1,1,1,0,0,0), (-1,1,0,1,0,0), (1,-2,0,0,1,0), (-3,-2,0,0,0,1)\}$ is a basis for V. Again, we row reduce to find that the reduced row echelon form of B: the new matrix with the first columns the vectors from S and the rest of the columns the basis vectors above

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{is} \quad \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

As such, it is clear from Theorem 3.16 (d) B: by looking at the pivot columns that an extension of S to a basis for V is the set

$\{(0,-1,0,1,1,0), (1,0,1,1,1,0), (-1,1,0,1,0,0), (-3,-2,0,0,0,1)\}$.

Ideas

Let B and C be the reduced row echelon form of A.

$MA = NA$

$M = N$

exactly r nonzero rows b_{j_i} of B and c_{k_i} of C

Proof

Let A be a $1 \times n$ matrix, then the row reduced echelon form ~~of A is either 0 or e_1, and there are~~ non overlapping cases. Hence, uniqueness holds [illegible]. Assume that this is true of any ~~$m \times n$ matrix~~ and let A now be a $(m+1) \times n$ matrix instead. Furthermore, let A_m be the $m \times n$ matrix with $(A_m)_{ij} = A_{ij}$, and [illegible] the $(m+1)$th row of ~~A. If~~ M and N are invertible matrices so that MA and NA are the reduced row echelon ~~forms of A, then~~ MA_m and NA_m are the reduced ~~echelon forms of A_m, by exercise 15 of section 3.2 and exercise 14 of this section~~. By our assumption / induction hypothesis, $MA_m = NA_m$. ~~Thus, M = N.~~

) vector ... ties to ... basis ... $(0\ 13\ -5\ 0\ 18 \ldots\ \sqrt{-38} \ldots\ 0\ -20\ 106\ 405\ 109\ -40)$ ... $(\ldots)$

15. Let A be any $m \times n$ matrix, and use the notation of Theorem 3.16 so the kth column of A is $\sum_{i=1}^{r} d_{ki} a_{j_i}$. Then, for any two (possibly distinct) reduced row echelon forms B and C of A, the kth columns of B and C are both just $\sum_{i=1}^{r} d_{ki} a_i$ by Theorem 3.16(d). Since this equality holds for all $1 \le k \le n$ columns of B and C, $B = C$ must consequently hold true. So, the reduced row echelon form of any matrix is unique. $\square$

X

Dude what was I thinking bruh.

The indexing j_i may be different for B and C.

elf Proof of Theorem 4.1

et $u=(u_1,u_2)$, $v:=(v_1,v_2)$, and $w:=(w_1,w_2)$. Thus, it's clear that $\det\begin{pmatrix}u+kv\\w\end{pmatrix}=(u_1+kv_1)w_2-(u_2+kv_2)w_1 = u_1w_2-u_2w_1+k(v_1w_2-v_2w_1)=$ $\det\begin{pmatrix}u\\w\end{pmatrix}+k\det\begin{pmatrix}v\\w\end{pmatrix}$. It is similarly shown that $\det\begin{pmatrix}w\\u+kv\end{pmatrix}=\det\begin{pmatrix}w\\u\end{pmatrix}+k\det\begin{pmatrix}w\\v\end{pmatrix}$. $\square$

elf-Proof of Theorem 4.2

Tricks

$ad=bc \Rightarrow \begin{pmatrix}a&b\\c&d\end{pmatrix}\to\begin{pmatrix}ac&bc\\bc&bd\end{pmatrix}\to\begin{pmatrix}ad&bd\\0&0\end{pmatrix} \Rightarrow$ not invertible

f $\det(A)=0$

invertible $\Rightarrow \det \neq 0$

$c-a\left(\frac{c}{a}\right)$

If not invertible, $\begin{pmatrix}a&b\\c&d\end{pmatrix}\to\begin{pmatrix}a&b\\0&d-\frac{bc}{a}\end{pmatrix}\to\begin{pmatrix}a&0\\0&d-\frac{bc}{a}\end{pmatrix}$

$\begin{pmatrix}c&d\\a&b\end{pmatrix}$ $cb-ad$

$\left(\begin{matrix}a&b\\c&d\end{matrix}\middle|\begin{matrix}1&0\\0&1\end{matrix}\right)\to\left(\begin{matrix}a&b\\0&d-\frac{bc}{a}\end{matrix}\middle|\begin{matrix}1&0\\-\frac{c}{a}&1\end{matrix}\right)\to\left(\begin{matrix}a&b\\0&ad-bc\end{matrix}\middle|\begin{matrix}1&0\\-c&a\end{matrix}\right)$

$\to\left(\begin{matrix}a&0\\0&ad-bc\end{matrix}\middle|\begin{matrix}1+\frac{abc}{ad-bc}&-\frac{ab}{ad-bc}\\-c&a\end{matrix}\right)$

$\to\left(\begin{matrix}1&0\\0&1\end{matrix}\middle|\begin{matrix}\frac{1}{a}+\frac{bc}{a(ad-bc)}&-\frac{b}{ad-bc}\\-\frac{c}{ad-bc}&\frac{a}{ad-bc}\end{matrix}\right)$

(T1) zero doesn't matter

(T2) & (T3) $\Rightarrow$ $\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$

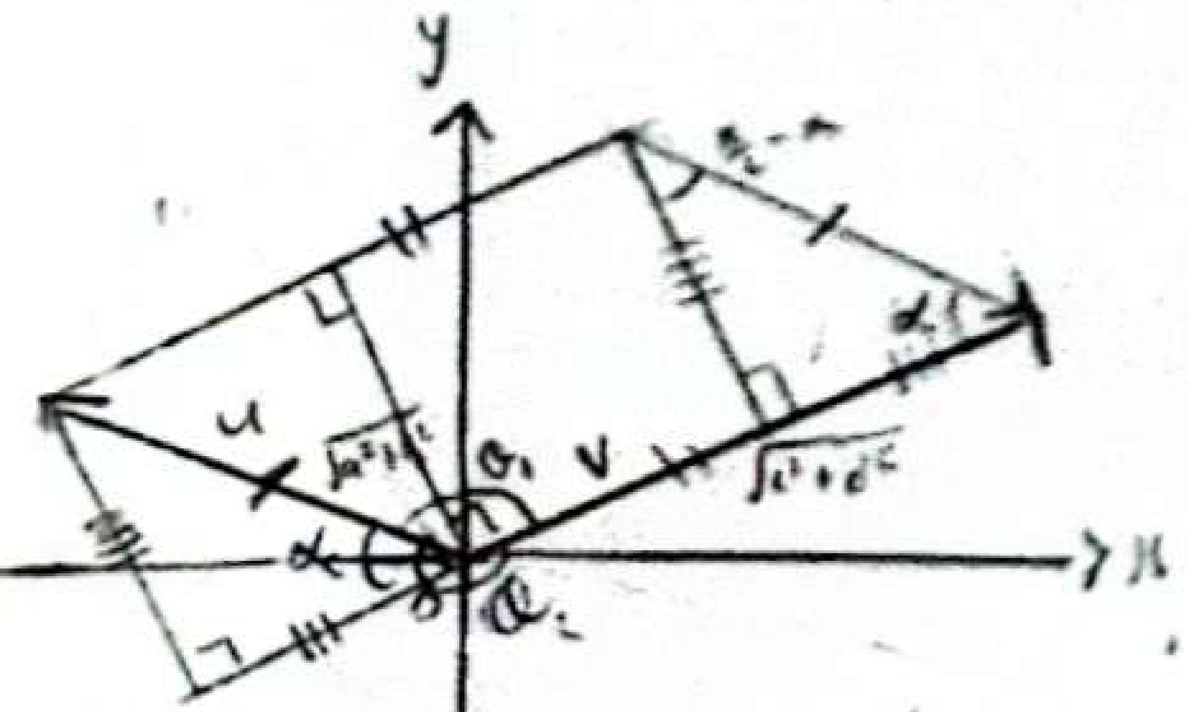
Proof

Let $A:=\begin{pmatrix}a&b\\c&d\end{pmatrix}$. If $a=c=0$, then $\det(A)=ad-bc=0$ and $\operatorname{rank}(A)\le 1$ means A cannot be invertible. Now consider a or c being nonzero. Without loss of generality, we can assume $a\neq 0$ as we can swap the rows of A otherwise anyways. Notice that by elementary row operations, $\begin{pmatrix}a&b\\c&d\end{pmatrix}\to\begin{pmatrix}a&b\\0&d-\frac{bc}{a}\end{pmatrix}\to\begin{pmatrix}a&0\\0&d-\frac{bc}{a}\end{pmatrix}$. As such, A is invertible iff $d-\frac{bc}{a}\neq 0$ iff $ad-bc\neq 0$ iff $\det(A)\neq 0$.

Furthermore, if A is invertible, then the augmented matrix $\left(\begin{matrix}a&b\\c&d\end{matrix}\middle|\begin{matrix}1&0\\0&1\end{matrix}\right)$ simplifies to $\left(\begin{matrix}1&0\\0&1\end{matrix}\middle|\begin{matrix}\frac{1}{a}+\frac{bc}{a(ad-bc)}&-\frac{b}{ad-bc}\\-\frac{c}{ad-bc}&\frac{a}{ad-bc}\end{matrix}\right)$. In other words,

$$A^{-1}=\begin{pmatrix}\frac{1}{a}+\frac{bc}{a(ad-bc)}&-\frac{b}{ad-bc}\\-\frac{c}{ad-bc}&\frac{a}{ad-bc}\end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix}\frac{ad-bc}{a}+\frac{bc}{a}&-b\\-c&a\end{pmatrix}=\frac{1}{\det(A)}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$$ as expected. $\square$

lf-Check:



$\theta_1=\pi-\alpha$

$\sin(\alpha)=\sin(\pi-\theta_1)$
$=\sin(\theta_1)$

$$A\begin{pmatrix}u\\v\end{pmatrix}=\sqrt{v_1^2+v_2^2}\cdot\sqrt{u_1^2+u_2^2}\sin(\alpha)$$
$$=\sqrt{(u_1\cos(\theta)-u_2\sin(\theta))^2+(u_1\sin(\theta)+u_2\cos(\theta))^2}\cdot\sqrt{u_1^2+u_2^2}\sin(\alpha)$$
$$=\sqrt{u_1^2\cos^2(\theta)-2u_1u_2\sin(\theta)\cos(\theta)+u_2^2\sin^2(\theta)+u_1^2\sin^2(\theta)+2u_1u_2\sin(\theta)\cos(\theta)+u_2^2\cos^2(\theta)}\cdot\sqrt{u_1^2+u_2^2}\sin(\alpha)$$
$$=\sqrt{u_1^2+u_2^2}\sqrt{u_1^2+u_2^2}\sin(\alpha)$$
$$=(u_1^2+u_2^2)\sin(\alpha)$$
$$=(u_1^2+u_2^2)\sin(\theta_1)=-(u_1^2+u_2^2)\sin(\theta_2)$$
$$=\left|\det\begin{pmatrix}u\\v\end{pmatrix}\right|$$

$\theta_2=2\pi-\theta_1$
$=2\pi-(\pi-\alpha)$
$=\pi+\alpha$
$\sin(\theta_2)=-\sin(\alpha)$

Anticlockwise rotation of u to coincide with v is either through θ_1 or θ_2.

is inver... ... $\mathbf{0}$, ... , fo ...

2)

Self-Proof of Lemma

Ideas

$$\det(B) := \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j})$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{e_1} \quad (-1)^{1+1} \cdot 1 = 1$$

$$\underbrace{(-1)^{1+1} \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{1} \;\Big|\; (-1)^{2+2} \underbrace{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{1} \;\Big|\; (-1)^{3+3} \underbrace{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{1}$$

$$(-1)^{1+1} \underbrace{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{1} + (-1)^{1+2} \underbrace{\det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{0} + (-1)^{1+3} \underbrace{\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{0}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix} \cdot \quad (-1)^{2+1} \det\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = (-1)(-4) = 4$$

$$1 \cdot (-1)^{1+1} \underbrace{\det\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}} + 2 \cdot (-1)^{1+2} \underbrace{\det\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}}_{1} + 3 \cdot (-1)^{1+3} \underbrace{\det\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}}_{2} = -2 + 6 = 4$$

$n = 2$ Simple casewise analysis

Assume n, when $i \geq 2$:

$$\sum_{j=1}^{n} (-1)^{2+j} B_{2j} \underbrace{\det(\tilde{B}_{2j})}_{(-1)^{i-1+k} \det((\tilde{\tilde{B}}_{2j})_{i-1,k})} = (-1)^{i+k} \sum_{j=1}^{n} (-1)^{j} \cancel{B_{2j}} \det\left((\tilde{\tilde{B}}_{2j})_{i-1,k}\right)$$

actually may be $k-1$ depending on n

$$\begin{pmatrix} \tilde{\tilde{b}}_2 \\ \vdots \\ \tilde{\tilde{b}}_{j-1} \\ \tilde{\tilde{b}}_{j+1} \\ \vdots \\ \tilde{\tilde{b}}_n \end{pmatrix}$$

$$\det(\tilde{B}_{ik}) = \cancel{\sum_{j=1}^{n} (-1)^{j} \det((\tilde{\tilde{B}}_{ik})_{2j})}$$

$i = 1$: there's only 1 nonzero summand

$$= \sum_{j=1}^{n} (-1)^{2+j} \underbrace{(\tilde{B}_{ik})_{2j}}_{(b'_2)_{2j}} \det\left((\tilde{\tilde{B}}_{ik})_{2j}\right)$$

Proof

When $n=2$, a simple casewise analysis suffices:

1. $i=1$ & $k=1$: $(-1)^{1+1} \det((d)) = d = \det\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$,
2. $i=1$ & $k=2$: $(-1)^{1+2} \det((c)) = -c = \det\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$,
3. $i=2$ & $k=1$: $(-1)^{2+1} \det((b)) = -b = \det\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$,
4. $i=2$ & $k=2$: $(-1)^{2+2} \det((a)) = a = \det\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$;

for any scalars $a, b, c, d \in \mathbb{F}$. Clearly, the result holds for $n=2$. So assume it holds for $n \geq 2$ ~~1~~ and let $B \in M_{(n+1)\times(n+1)}(\mathbb{F})$ so its ith row is e_k. Tackling the trivial case of ~~$i=1$~~ first, we see that $\det(B) := \sum_{j=1}^{n+1} (-1)^{2+j} B_{2j} \det(\tilde{B}_{2j}) = (-1)^{2+k} \det(\tilde{B}_{2j}) = (-1)^{i+k} \det(\tilde{B}_{ik})$. If $i \geq 2$, then for any row $b_j \in \mathbb{F}^{n+1}$ of B, let the row vectors $\tilde{b}_j \in \mathbb{F}^n$, $\tilde{b}'_j \in \mathbb{F}^n$, $\tilde{\tilde{b}}_j \in \mathbb{F}$ be b_j but with the jth entry, the kth entry, and the jth and kth entries, respectively removed. As such, using our assumption / induction hypothesis (I.H.) and the Corollary to Theorem 4.3 which tells us $\det\begin{pmatrix} \tilde{b}_2 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_{n+1} \end{pmatrix} = 0$ for $j = k$,

$$\det(B) := \sum_{j=1}^{n+1} (-1)^{2+j} B_{2j} \det\begin{pmatrix} \tilde{b}_2 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_{n+1} \end{pmatrix} \overset{\text{corollary}}{=} \sum_{\substack{j=0 \\ j \neq k}}^{n} (-1)^{2+j} B_{2j} \det\begin{pmatrix} \tilde{b}_2 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_{n+1} \end{pmatrix} \overset{\text{I.H.}}{=} (-1)^{i+k} \sum_{\substack{j=0 \\ j \neq k}}^{n} (-1)^{j} B_{2j} \det\begin{pmatrix} \tilde{\tilde{b}}_2 \\ \vdots \\ \tilde{\tilde{b}}_{i-1} \\ \tilde{\tilde{b}}_{i+1} \\ \vdots \\ \tilde{\tilde{b}}_{n+1} \end{pmatrix} = (-1)^{i+k} \det\begin{pmatrix} \tilde{b}'_1 \\ \vdots \\ \tilde{b}'_{i-1} \\ \tilde{b}'_{i+1} \\ \vdots \\ \tilde{\tilde{b}}_{n+1} \end{pmatrix} = (-1)^{i+k} \det(\tilde{B}_{ik}),$$

as expected. As a result, the claim is true for $n+1$. Consequently, it is true for all natural numbers $n \geq 2$.

Oops had a really dumb brain loading moment :')

Wrote some really dumb stuff ugh.

Self-Proof of Theorem 4.3

Ideas

When $n=1$, trivial

Assume n

$$\det\left(\begin{array}{c} \\ \end{array}\right) = \sum_{j=1}^{n+1} (-1)^{2+j} \widetilde{(a_i)_{2j}} \cdot \det\left(\tilde{A}_{2j}\right) \qquad \tilde{A}_{2j} = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{u}+k\tilde{v} \\ \vdots \\ \tilde{a}_{n+1} \end{pmatrix}$$

Proof

When $n=1$, the result is trivial as $\det((a+kb)) = a+kb = \det((a)) + k\det((b))$ for any $a, b, k \in \mathbb{F}$. Assume this holds for $n \in \mathbb{N}$. To show the same is true of $n+1$, first for any row vector $v \in \mathbb{F}^{n+1}$, let $\hat{v} \in \mathbb{F}^n$ be v but with the jth entry completely removed, so that $\widetilde{\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_{n+1} \end{pmatrix}}_{1j} = \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_{r-1} \\ \tilde{u}+k\tilde{v} \\ \tilde{a}_{r+1} \\ \vdots \\ \tilde{a}_{n+1} \end{pmatrix}$ by noticing $(\widetilde{u+kv}) = \tilde{u} + k\tilde{v}$.

Hence, when $r \neq 1$,

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_{n+1} \end{pmatrix} := \sum_{j=1}^{n+1} (-1)^{1+j} (a_1)_{1j} \det \widetilde{\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_{n+1} \end{pmatrix}}_{1j} = \sum_{j=1}^{n+1} (-1)^{1+j} (a_1)_{1j} \cdot \left[\det\begin{pmatrix} \tilde{a}_2 \\ \vdots \\ \tilde{a}_{r-1} \\ \tilde{u} \\ \tilde{a}_{r+1} \\ \vdots \\ \tilde{a}_{n+1} \end{pmatrix} + k\det\begin{pmatrix} \tilde{a}_2 \\ \vdots \\ \tilde{a}_{r-1} \\ \tilde{v} \\ \tilde{a}_{r+1} \\ \vdots \\ \tilde{a}_{n+1} \end{pmatrix}\right] = \sum_{j=1}^{n+1} (-1)^{1+j} (a_1)_{1j} \det\widetilde{\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_{n+1} \end{pmatrix}} + k\sum_{j=1}^{n+1} (-1)^{1+j} (a_1)_{1j} \det\widetilde{\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_{n+1} \end{pmatrix}} = \det\begin{pmatrix} a_1 \\ \vdots \\ u \\ \vdots \\ a_{n+1} \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ v \\ \vdots \\ a_{n+1} \end{pmatrix}$$

as expected. In other words, the claim holds for $n+1$. By induction, this is true of all natural n. $\square$

In the case that $r=1$, the result is immediate as $(a_1)_{1j} = (u+kv)_{1j} = u_{1j} + kv_{1j}$ and $\widetilde{\begin{pmatrix} u+kv \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix}}_{1j} = \widetilde{\begin{pmatrix} u \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix}}_{1j} = \widetilde{\begin{pmatrix} v \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix}}_{1j}$

Self-Proof of Corollary / Exercise 24

As in Theorem 4.3, let a_j denote the jth row of A and suppose $a_r = 0$ for some $1 \leq r \leq n$. Now, $\det(A) = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ 0-0 \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ 0 \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} - \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ 0 \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = 0$ as expected. $\square$

4.
(a) Area $= \left|\det\begin{pmatrix}3 & -2\\ 2 & 5\end{pmatrix}\right| = |15+4| = 19$ units^2 ✓

(b) Area $= \left|\det\begin{pmatrix}1 & 3\\ -3 & 1\end{pmatrix}\right| = |1+9| = 10$ units^2 ✓

(c) Area $= \left|\det\begin{pmatrix}4 & -1\\ -6 & -2\end{pmatrix}\right| = |-8 \mp 6| = 14$ units^2 x

(d) Area $= \left|\det\begin{pmatrix}3 & 4\\ 2 & -6\end{pmatrix}\right| = |-18-8| = 26$ units^2 ✓

5. Let $A := \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, so $B = \begin{pmatrix}c & d\\ a & b\end{pmatrix}$. Hence, $\det(B) = cb - da = -(ad-bc) = -\det(A)$.

6. In that case, A is of the form $\begin{pmatrix}a & a\\ b & b\end{pmatrix}$ so $\det(A) = ab - ab = 0$ as expected.

7. Once more, let $A := \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, so $A^t = \begin{pmatrix}a & c\\ b & d\end{pmatrix}$. As such, $\det(A^t) = ad - cb = ad - bc = \det(A)$.

8. Given A is diagonal, $A = \begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$ for some scalars a, b, d in $\mathbb{F}$. Accordingly, $\det(A) = ad - b\cdot 0 = ad$, as we wanted.

9. Method 1

Let $A := \begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}$ and $B = \begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}$. Thus, $AB = \begin{pmatrix}a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4\\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4\end{pmatrix}$. Thence, $\det(AB) = (a_1b_1+a_2b_3)(a_3b_2+a_4b_4) - (a_1b_2+a_2b_4)(a_3b_1+a_4b_3) =$
$a_1b_1a_3b_2 + a_1b_1a_4b_4 + a_2b_3a_3b_2 + a_2b_3a_4b_4 - (a_1b_2a_3b_1 + a_1b_2a_4b_3 + a_2b_4a_3b_1 + a_2b_4a_4b_3) = a_1b_1a_4b_4 - a_1b_2a_4b_3 + a_2b_3a_3b_2 - a_2b_4a_3b_1 =$
$a_1a_4(b_1b_4 - b_2b_3) - a_2a_3(b_1b_4 - b_2b_3) = (a_1a_4 - a_2a_3)(b_1b_4 - b_2b_3) = \det(A)\det(B)$.

Method 2 (Probably more easily generalizable + elegance)

Ideas

$p_i = \sum_{j=1}^{2} A_{ij} b_j$

$$\det\begin{pmatrix}p_1\\ p_2\end{pmatrix} = \sum_{j=1}^{2} A_{1j} \det\begin{pmatrix}b_j\\ p_2\end{pmatrix} = \sum_{j=1}^{2} A_{1j} \sum_{k=1}^{2} A_{2k} \det\begin{pmatrix}b_j\\ b_k\end{pmatrix}$$

$= A_{11}A_{22}\det\begin{pmatrix}b_1\\ b_2\end{pmatrix} + A_{12}A_{21}\det\begin{pmatrix}b_2\\ b_1\end{pmatrix}$ as $\det\begin{pmatrix}b_1\\ b_1\end{pmatrix} = \det\begin{pmatrix}b_2\\ b_2\end{pmatrix} = 0$

$= (A_{11}A_{22} - A_{12}A_{21})\det\begin{pmatrix}b_1\\ b_2\end{pmatrix}$ by exercise 5

Proof

Let p_i and b_i be the ith rows of AB and B respectively. We know $p_i = \sum_{j=1}^{2} A_{ij} b_j$, therefore

$\det(AB) = \det\begin{pmatrix}p_1\\ p_2\end{pmatrix} = \sum_{j=1}^{2} A_{1j} \sum_{k=1}^{2} A_{2k} \det\begin{pmatrix}b_j\\ b_k\end{pmatrix}$ by Theorem 4.1,

$= A_{11}A_{22}\det\begin{pmatrix}b_1\\ b_2\end{pmatrix} + A_{12}A_{21}\det\begin{pmatrix}b_2\\ b_1\end{pmatrix}$ as $\det\begin{pmatrix}b_1\\ b_1\end{pmatrix} = \det\begin{pmatrix}b_2\\ b_2\end{pmatrix} = 0$ by exercise 6 above,

$= (A_{11}A_{22} - A_{12}A_{21})\det\begin{pmatrix}b_1\\ b_2\end{pmatrix}$ since $\det\begin{pmatrix}b_1\\ b_2\end{pmatrix} = -\det\begin{pmatrix}b_2\\ b_1\end{pmatrix}$ by exercise 5 above,

$= \det(A)\det(B)$.

Exercises

$0 < \theta < \pi \Rightarrow 0 < \sin(\theta) \le 1$

13. Ideas

Example 2 of §2.1: $T_\theta(u) = v \Rightarrow (u_1 \cos(\theta) - u_2 \sin(\theta),\ u_1 \sin(\theta) + u_2 \cos(\theta)) = (v_1, v_2)$

$$\det\begin{pmatrix} u_1 & u_2 \\ u_1\cos(\theta) - u_2 \sin(\theta) & u_1 \sin(\theta) + u_2 \cos(\theta) \end{pmatrix} = u_1^2 \sin(\theta) + u_1 u_2 \cos(\theta) - u_1 u_2 \cos(\theta) + u_2^2 \sin(\theta)$$
$$= (u_1^2 + u_2^2)\sin(\theta)$$

sign 1 $\quad a^2 + b^2$

Proof

Let θ be the angle between u and v, not necessarily satisfying $0 < \theta < \pi$. By example 2 of section 2.1, we see that

$\det\begin{pmatrix} u \\ v \end{pmatrix} = \det\begin{pmatrix} u_1 & u_2 \\ u_1\cos(\theta) - u_2 \sin(\theta) & u_1 \sin(\theta) + u_2 \cos(\theta) \end{pmatrix} = u_1^2 \cos(\theta) + u_1 u_2 \cos(\theta) - u_1 u_2 \cos(\theta) + u_2^2 \sin(\theta) = (u_1^2 + u_2^2)\sin(\theta)$.

That is, $O\begin{pmatrix} u \\ v \end{pmatrix} = 1$ iff $0 < \theta < \pi$, to ensure $\sin(\theta) > 0$.

1. (a) False ✓
   (b) True ✓
   (c) False ✓
   (d) False ✓
   (e) True ✓

2. (a) $\det\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix} = 6\cdot 4 + 3 \cdot 2 = 24 + 6 = 18$. ✗ 30
   (b) $\det\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix} = -5 - 12 = -17$ ✓
   (c) $\det\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix} = -8 - 0 = -8$ ✓

3. (a) $\det\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix} = (-1+i)(2-3i) - (1-4i)(3+2i) = -2 + 3 + 3i + 2i - (3 + 8 - 12i + 2i) = -10 + 15i$. ✓
   (b) $\det\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix} = 35i + 14 + (3-i)(6+4i) = 14 + 35i + 18 + 4 - 6i + 12i = 36 + 41i$ ✓
   (c) $\det\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix} = -12 - 12 = -24$. ✓

1. (a) Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ once again, then $C = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Consequently, $AC = \begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{pmatrix} = CA$, which is clearly just $\det(A)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [\det(A)]I$.

~~(b) From A, $\det([\det(A)]I) = \det(A)\begin{pmatrix} 1 & 0 \\ 0 & \det(A) \end{pmatrix} = [\det(A)]^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(AC) = \det(A)\det(C)$. Hence, $\det(C) = \det(A)$ when $\det(A) \neq 0$.~~ Theorem 4.1 and exercise 9 of this section

~~Even if $\det(A) = 0$, $\det(C) = ad + bc$~~

(b) $\det(A) = ad - bc = da - (-b)(-c) = \det(C)$.

(c) The classical adjoint of $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, i.e. C^t.

(d) If A is invertible, Theorem 4.2 immediately says $A^{-1} = [\det(A)]^{-1}C$.

(a) If E is a type 2 elementary matrix, then we can suppose without loss of generality that $E = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ for some scalar $a \in \mathbb{F}$.
Hence, $\delta(E) \overset{(i)}{=} a\,\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{(iii)}{=} a = \det(E)$.

When E is a type 3 elementary matrix, then the existence of some scalar $b \in \mathbb{F}$ with $E = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is something we can assume without loss of generality.
Therefore, $\delta(E) \overset{(i)}{=} \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\,\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \overset{(ii)}{=} 1 = \det(E)$.

Finally, given E is a type 1 elementary matrix, it is certain that $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, $\delta(E) \overset{(i)}{=} \delta\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \overset{(i)\&(ii)}{=} \delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \delta\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \overset{(ii)}{=}$
$-\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -1 = \det(E)$ since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a type 3 elementary matrix.

So, as long as E is an elementary matrix, $\delta(E) = \det(E)$.

(b) Let us follow the notation and work done in exercise 9 of this section: We know that

$$\delta(EB) = E_{11}E_{22}\,\delta\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + E_{12}E_{21}\,\delta\begin{pmatrix} b_2 \\ b_1 \end{pmatrix}.$$

Now, notice that $\delta\begin{pmatrix} b_2 \\ b_1 \end{pmatrix} \overset{(i)}{=} \delta\begin{pmatrix} b_2 - b_1 \\ b_1 \end{pmatrix} + \delta\begin{pmatrix} b_1 \\ b_1 \end{pmatrix} \overset{(i)\&(ii)}{=} \delta\begin{pmatrix} b_2 - b_1 \\ b_2 \end{pmatrix} + \delta\begin{pmatrix} b_2 - b_1 \\ b_1 - b_2 \end{pmatrix} \overset{(i)}{=} \delta\begin{pmatrix} b_2 \\ b_2 \end{pmatrix} - \delta\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \delta\begin{pmatrix} b_1 - b_2 \\ b_1 - b_2 \end{pmatrix} \overset{(ii)}{=} -\delta\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Hence, $\delta(EB) = (E_{11}E_{22} - E_{12}E_{21})\,\delta\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$
$= \det(E)\,\delta\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \overset{(a)}{=} \delta(E)\delta(B)$.

First, if A is invertible, then $A = \prod_{i=1}^{k} E_i$ for some natural k and corresponding elementary matrices E_i, by Corollary 3 of Theorem 3.6. As such, $\delta(A) \overset{11(b)}{=} \prod_{i=1}^{k} \delta(E_i) \overset{11(a)}{=} \prod_{i=1}^{k} \det(E_i)$; which is just $\det(A)$ according to exercise 9. of this section. When A is not invertible, it reduces to the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix}$ by elementary row operations. Thus, $\delta(A) \overset{(ii)}{=} 0 = \det(A)$.

Self - Proof of Theorem 4.4

Ideas B_j be A but with ith row being e_j, $\Rightarrow a_i = \sum_{j=1}^{n} A_{ij} e_j$

$\det(A) = \sum_{j=1}^{n} A_{ij} \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} \det((\tilde{B}_j)_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} \det(\tilde{A}_{ij})$ as ith column is deleted so $(\tilde{B}_j)_{ij} = (\tilde{A})_{ij}$

Proof

Let B_j be A but with its ith row being e_j instead, so by virtue of the ith row of A being $\sum_{j=1}^{n} A_{ij} e_j$, we know $\det(A) = \sum_{j=1}^{n} A_{ij} \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det((\tilde{B}_j)_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$ because B_j and A only differ in the ith row, and since that is removed, we have $(\tilde{B}_j)_{ij} = \tilde{A}_{ij}$. $\square$

Self - Proof of Corollary

Ideas

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = (-1)^{1+1} (1) \det\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + (-1)^{1+2} (2) \det\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

Proof

When $n=2$, the result is clear from noticing that $\det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = ab - ba = 0$ for any $a, b \in \mathbb{F}$. And $n=1$ is vacuous. So, assume the statement is true for $n-1 \geq 2$, and that the ith and jth rows of $A \in M_{n\times n}(\mathbb{F})$ are identical though $i \neq j$. Let $1 \leq k \leq n$ so $k \neq i$ and $k \neq j$. Then, by Theorem 4.4, $\det(A)$ is just the cofactor expansion of A along the kth row: $\det(A) = \sum_{m=1}^{n} (-1)^{k+m} A_{km} \det(\tilde{A}_{km})$. Since the ith and jth rows of $\tilde{A}_{km}$ must be identical, $\det(\tilde{A}_{km}) = 0$ by assumption / the induction hypothesis, for each $1 \leq m \leq n$. Consequently, $\det(A)$ is a sum of zeros and must be 0 itself. In other words, the claim holds for n too. Wherefore, induction says that this is true for all $n \in \mathbb{N}$. $\square$

Self-Proof of Theorem 4.5 (Continuation)

As such, to represent $\tilde{B}_{kj}$ in a form suitable to apply our induction hypothesis, let $R_{pq}(A)$ be the matrix obtained from A by interchanging rows p and q. Note that $\tilde{A}_{ij} = R_{k-2,k-1} R_{k-3,k-2} \cdots R_{i,i+1}(\tilde{B}_{kj})$, a sequence of $k-1-i$ elementary row operations of type 1. As such, $\det(\tilde{A}_{ij}) = (-1)^{k-1-i} \det(\tilde{B}_{kj})$ by our assumption / induction hypothesis. Thus, $\det(A) := \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}) = -\sum_{j=1}^{n} (-1)^{k+j} B_{ij} \cdot \det(\tilde{B}_{ij}) := \det(B)$, as expected. So, the statements also holds for n. Consequently, it must hold true for all $n \in \mathbb{N}$.

Self-Proof of Theorem 4.6

Let a_j be the jth row of A. Say B is a matrix obtained by adding $c \in F$ times of the ith row to the jth row of A. Now, by Theorem 4.3 and the Corollary to Theorem 4.4,

$$\det(B) = \det\begin{pmatrix} a_1 \\ \vdots \\ a_j + c a_i \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} \overset{\text{Thm 4.3}}{=} c \det\begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} \overset{\text{Corollary}}{=} 0 + \det\begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} = \det(A)$$

as we thought.

(Here $i > j$ is shown but the display of a_i right above a_n is just for the sake of clarity. It is easy to notice that the above argument holds when $j > i$.)

Self-Proof of Corollary

As before, let a_j be the jth row of A. Given A has rank less than n, $a_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i a_i$ for some scalars $c \in F$ and $1 \leq j \leq n$. Hence, we notice that by Theorem 4.6 and the Corollary to Theorem 4.4

$$\det(A) \overset{\text{Thm 4.6}}{=} \det\begin{pmatrix} a_1 \\ \vdots \\ a_j - \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i a_i \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ 0 \\ \vdots \\ a_n \end{pmatrix} \overset{\text{Corollary}}{=} 0.$$

Exercises

1. (a) False ✓
(b) True ✓
(c) True ✓
(d) True ✓
(e) False ✓
(f) False ✓
(g) False ✓
(h) True ✓

~~Note: To save time, I'll avoid writing computations out in the same way I do with proofs, and instead just, well, compute~~

2. We notice that for row vectors $A, B, C \in \mathbb{R}^3$

$$\det\begin{pmatrix}3A\\3B\\3C\end{pmatrix} = 3\det\begin{pmatrix}3A\\B\\3C\end{pmatrix} = 3^3\det\begin{pmatrix}A\\B\\C\end{pmatrix}.$$

Hence, the desired k is 27.

3. By using the information about how row operations affect determinants mentioned in page 217,

$$\det\begin{pmatrix}2a_1 & 2a_2 & 2a_3\\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3\\ 7c_1 & 7c_2 & 7c_3\end{pmatrix} = (2)(3)(7)\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}.$$

So, $k = 42$. ✓

4. Again, repeating a similar procedure,

$$\det\begin{pmatrix}b_1+c_1 & b_2+c_2 & b_3+c_3\\ a_1+c_1 & a_2+c_2 & a_3+c_3\\ a_1+b_1 & a_2+b_2 & a_3+b_3\end{pmatrix} = \det\begin{pmatrix}b_1+c_1 & b_2+c_2 & b_3+c_3\\ a_1+c_1 & a_2+c_2 & a_3+c_3\\ b_1-c_1 & b_2-c_2 & b_3-c_3\end{pmatrix} = 2\det\begin{pmatrix}b_1 & b_2 & b_3\\ a_1+c_1 & a_2+c_2 & a_3+c_3\\ a_1 & a_2 & a_3\end{pmatrix} = 2\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}.$$

Thus, $k = 2$.

5. $$\det\begin{pmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ 2 & 3 & 0\end{pmatrix} = 0 + \overset{(-1)^3}{1}\cdot\det\begin{pmatrix}-1 & -3\\ 2 & 0\end{pmatrix} + \overset{(-1)^4}{2}\cdot\det\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix} = +(-6)+2(-3) = 6-6 = 0. \quad \overset{-6-6=-12}{} \quad X$$

7. $$\det\begin{pmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ 2 & 3 & 0\end{pmatrix} = \overset{(-1)^{2+1}}{-1}\cdot\det\begin{pmatrix}1 & 2\\ 3 & 0\end{pmatrix} + 0 - \overset{(-1)^{2+3}}{3}\cdot\det\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix} = -6-3(-2) = 0 \quad \overset{-6-6 \;=\; -12}{} \quad X$$

9. ~~$\det\begin{pmatrix}0 & 1+i & 2\\ -2i & 0 & 1-i\\ 3 & 4i & 0\end{pmatrix} = 0 + (-1)^{1+2}(1+i)\cdot\det\begin{pmatrix}-2i & 1-i\\ 3 & 0\end{pmatrix} + 2\cdot\det\begin{pmatrix}-2i & 0\\ 3 & 4i\end{pmatrix} = (1+i)(1-i)(-3) + \ldots$~~

$$\det\begin{pmatrix}0 & 1+i & 2\\ -2i & 0 & 1-i\\ 3 & 4i & 0\end{pmatrix} = (-1)^{3+1}\,3\cdot\det\begin{pmatrix}1+i & 2\\ 0 & 1-i\end{pmatrix} + (-1)^{3+2}\,4i\det\begin{pmatrix}0 & 2\\ -2i & 1-i\end{pmatrix} + 0 = 3(1+i)(1-i) + 4i(-4i) = 6 + 16 = 22 \checkmark$$

11.
$$\det\begin{pmatrix}0 & 2 & 1 & 3\\ 1 & 0 & -2 & 2\\ 3 & -1 & 0 & 1\\ -1 & 1 & 2 & 0\end{pmatrix} = (-1)^{4+1}(-1)\cdot\overset{-12}{\det\begin{pmatrix}2 & 1 & 3\\ 0 & -2 & 2\\ -1 & 0 & 1\end{pmatrix}} + (-1)^{4+2}(1)\cdot\overset{23}{\det\begin{pmatrix}0 & 1 & 3\\ 1 & -2 & 2\\ 3 & 0 & 1\end{pmatrix}} + (-1)^{4+3}(2)\overset{-14}{\det\begin{pmatrix}0 & 2 & 3\\ 1 & 0 & 2\\ 3 & -1 & 1\end{pmatrix}} + 0 \checkmark$$

$$= \left[0 - 2(-1)^{2+2}\det\begin{pmatrix}2 & 3\\ -1 & 1\end{pmatrix} + 2(-1)^{2+3}\det\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix}\right]^{-12} + \left[(-1)^{1+2}\det\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix} + 3(-1)^{1+3}\det\begin{pmatrix}1 & -2\\ 3 & 0\end{pmatrix}\right]^{23} \checkmark$$

$$-2\left[0 + 2(-1)^{1+2}\det\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix} + 3(-1)^{1+3}\det\begin{pmatrix}1 & 0\\ 3 & -1\end{pmatrix}\right]^{-14}$$

$$= \overset{-12}{-2(2+3) - 2(1)} \quad \overset{23}{-(1-6) + 3(6)} \quad + 2\left[-2(1-6) + 3(-1)\right]$$

$$= 25 \times -3$$

13. We transform the matrix into an upper triangular one via a type 1 row operation:
$\begin{pmatrix}0 & 0 & 1\\ 0 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} \xrightarrow{T1} \begin{pmatrix}4 & 5 & 6\\ 0 & 2 & 3\\ 0 & 0 & 1\end{pmatrix}$. Hence, the required determinant is just $-4\cdot 2\cdot 1 = -8$. ✓

15. As before, we transform the matrix with elementary row operations so it becomes upper triangular:
(Alternatively, we see that the matrix is of rank $2<3$ so its determinant can only be 0.)

$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} \xrightarrow{2\times T3} \begin{pmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & -6 & -12\end{pmatrix} \xrightarrow{T3} \begin{pmatrix}1 & 2 & 3\\ 0 & -3 & +6\\ 0 & 0 & 0\end{pmatrix}$. Thus, the determinant required is $1\cdot(-3)\cdot 0 = 0$ ✓

17. Again transforming the matrix to an upper triangular one,

$\begin{pmatrix}0 & 1 & 1\\ 1 & 2 & -5\\ 6 & -4 & 3\end{pmatrix} \xrightarrow{T1} \begin{pmatrix}1 & 2 & -5\\ 0 & 1 & 1\\ 6 & -4 & 3\end{pmatrix} \xrightarrow{T3} \begin{pmatrix}1 & 2 & -5\\ 0 & 1 & 1\\ 0 & -16 & 32\end{pmatrix} \xrightarrow{T3} \begin{pmatrix}1 & 2 & 5\\ 0 & 1 & 1\\ 0 & 0 & 16\end{pmatrix}$.

$33 + 16 = 49$

So the determinant we need has value $\overset{33}{} -1\cdot 1\cdot -16 = 16$. $\times -49$

$-1\cdot 1\cdot -49 = -49$

19. Transforming the matrix so it is upper triangular,

$$\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix} \xrightarrow{2\times T1} \begin{pmatrix} 3 & 1+i & 2 \\ -3i & 1 & 4+i \\ i & 2 & -1 \end{pmatrix} \xrightarrow{T3} \begin{pmatrix} 3 & 1+i & 2 \\ 0 & 5 & 2-i \\ i & 2 & -1 \end{pmatrix} \xrightarrow{T3} \begin{pmatrix} 3 & 1+i & 2 \\ 0 & 5 & 2-i \\ 0 & \frac{5}{3}-\frac{i}{3} & -1-\frac{2}{3}i \end{pmatrix} \xrightarrow{T3} \begin{pmatrix} 3 & 1+i & 2 \\ 0 & 5 & 2-i \\ 0 & 0 & -\frac{28}{15}-\frac{1}{15}i \end{pmatrix}.$$

As such, the determinant $\det\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$ is just $3(5)(-\frac{28}{15}-\frac{1}{15}i) = -28-i$.

21. Using elementary row operations to transform the given matrix into an upper triangular matrix,

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{3\times T3} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{pmatrix} \xrightarrow{2\times T3} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 19 & -38 \end{pmatrix} \xrightarrow{T3} \begin{pmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

As a result, the necessary determinant is $1\cdot 1\cdot 19\cdot 5 = 95$.

23. This is immediate for 1×1 matrices as $\det((a)) = a$ by definition (for any $a\in F$). Assume this is true for $n\times n$ matrices and let A be a $(n+1)\times(n+1)$ upper triangular matrix. We shall do cofactor expansion along the $(n+1)$th row of A. But first, notice that $A_{n+1,j} = \begin{cases} 1 & \text{if } j=n+1, \\ 0 & \text{otherwise.} \end{cases}$ and $\tilde{A}_{n+1,j}$ is an $n\times n$ upper triangular matrix since its (i_1,i_2)th entry is just $A_{i_1 i_2}$ for $1\le i_1, i_2 \le n+1$. So, our assumption / induction hypothesis says $\det(\tilde{A}_{n+1,j}) = \prod_{i=1}^{n}(\tilde{A}_{n+1,j})_{i,i} = \prod_{i=1}^{n} A_{i,i}$. Hence, $\det(A) = (-1)^{n+1+n+1} A_{n+1,n+1}\cdot\prod_{i=1}^{n} A_{i,i} = \prod_{i=1}^{n+1} A_{i,i}$. In otherwords, for $(n+1)\times(n+1)$ matrices, the statement holds. So by induction, it must be true of any square matrix.

24. See the corresponding self-proof.

25. Let a_j represent the jth row of A, then

$$\det(kA) := \det\begin{pmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix} = k\det\begin{pmatrix} a_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix} = k^n \det\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = k^n \det(A)$$

by using Theorem 4.3 n times.

26. By exercise 25, $\det(-A) = (-1)^n\det(A)$. So, the desired equality $\det(-A) = \det(A)$ holds iff n is even.

27. Suppose $A \in M_{n\times n}(F)$ has two identical columns, so $\operatorname{rank}(A) < n$. ~~We notice that elementary row operations keep identical columns identical.~~ Hence, $\det(A) = 0$ by the Corollary to Theorem 4.6.

28. By example 4 of this section, we know $\det(I) = 1$. So, by using the information on how elementary row operations affect determinants, as provided on page 217, we see that $\det(E_1) = -1$, $\det(E_2) = c$ where E_2 is obtained from I by multiplying one row by the scalar $c \in F$, and $\det(E_3) = 1$.

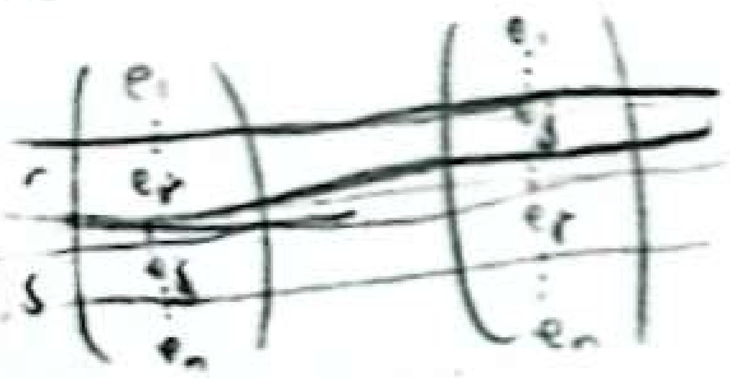
29. Ideas

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & 1,3 & \\ & & s,r & \end{array}$$

$$\begin{pmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to I_4$$

T3: $E_{ij} = \begin{cases} c & \text{if } i=r \text{ \& } j=s, \\ (I_n)_{ij} & \text{otherwise} \end{cases}$



T2: Equality holds by symmetry.

T1: $E_{ij} = \left\{\begin{array}{l} 1 \text{ if } i=j \text{ is not } r \text{ or } s, \\ 1 \text{ if } i=r \text{ \& } j=s \text{ or } i=s \text{ \& } j=r, \\ 0 \text{ otherwise.} \end{array}\right\} = \left\{\begin{array}{l} 1 \text{ if } j=i \text{ is not } r \text{ or } s, \\ 1 \text{ if } j=r \text{ \& } i=s \text{ or } j=s \text{ \& } i=r, \\ 0 \text{ otherwise.} \end{array}\right\} = E_{ji}$

Proof

We first see that for $n\times n$ elementary matrices E of types 1 and 2, they are symmetric. If E is of type 1, that is, obtained by interchanging rows r and s of I for some $1 \le r, s \le n$, then $E_{ij} = \left\{\begin{array}{l} 1 \text{ if } i=r \text{ \& } j=s \text{ or } i=s \text{ \& } j=r, \\ I_{ij} \text{ otherwise.} \end{array}\right\} = \left\{\begin{array}{l} 1 \text{ if } j=r \text{ \& } i=s \text{ or } j=s \text{ \& } i=r, \\ I_{ji} \text{ otherwise.} \end{array}\right\} = E_{ji}$. Similarly, when E is of type 2, then it is obtained by multiplying some row r of I by a nonzero scalar $c \in F$, so E is only nonzero on its diagonal entries; it is symmetric. Hence, $E = E^t$ in both cases. As such, $\det(E) = \det(E^t)$ holds. Now consider E being of type 3, thus it is obtained by adding $c \in F$ times of row r to row s of I. Then, E^t is obtained by adding $c \in F$ times of row s to row r of I because $(E^t)_{ij} = E_{ji} = \begin{cases} c \text{ if } j=s \text{ \& } i=r, \\ I_{ji} \text{ otherwise.} \end{cases} = \begin{cases} c \text{ if } i=r \text{ \& } j=s, \\ I_{ij} \text{ otherwise.} \end{cases}$ In other words, E^t is also a type 3 elementary matrix.

By exercise 28 of this section, $\det(E) = \det(E^t) = 1$.

~~Or, for T3, we could use a cofactor expansion along~~

30. If $n = 2m+1$ for some $m \in \mathbb{N}$, then we see that:

$$B = \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{m+1} \\ a_m \\ a_{m-1} \\ \vdots \\ a_2 \\ a_1 \end{pmatrix} \xrightarrow{1st} \begin{pmatrix} a_1 \\ a_{n-1} \\ \vdots \\ a_{m+1} \\ a_m \\ a_{m-1} \\ \vdots \\ a_2 \\ a_n \end{pmatrix} \xrightarrow{2nd} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m+1} \\ a_m \\ a_{m-1} \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} \xrightarrow{3rd} \cdots \xrightarrow{mth} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-1} \\ a_m \\ a_{m+1} \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = A.$$

Even if $n = 2m$ for some $m \in \mathbb{N}$ instead, simply remove the 'a_m' entry from above and the same argument still applies. In any case, m type 1 row operations suffice to transform B to A. So, $\det(B) = (-1)^m \det(A) = \begin{cases} (-1)^{\frac{n-1}{2}} \det(A) & \text{if } n \text{ odd}, \\ (-1)^{\frac{n}{2}} \det(A) & \text{if } n \text{ even}. \end{cases}$

Self-Proof of Theorem 4.9 (Cramer's Rule)

Ideas

If $\det(A)\neq 0$, $\operatorname{rank}(A)=n$ and since $\operatorname{rank}(A)\le\operatorname{rank}(A|b)\le n$, $\operatorname{rank}(A|b)=n$ also. So, $\operatorname{ref}(A|b)=\overset{(A^{-1}A\,|\,A^{-1}b)}{(I_n|b')}$ for some column vector $b'\in\mathbb{F}^n$.

Indeed, system has exactly one solution, namely $b'=(b'_1, b'_2, \dots, b'_n)$.

$$b'_k=\frac{\det(N_k)}{\det(I_n)} \text{ because } \det(N_k)=\det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & b'_1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & b'_2 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & & b'_3 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \ddots & & & & & & 0 & 0\\ 0 & 0 & 0 & & \ddots & & & & & 0 & 0\\ \vdots & \vdots & \vdots & & & & \ddots & & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & & b'_n & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}=\det\begin{pmatrix} & 0 & \\ & 0 & \\ & \vdots & \\ & b'_k & \\ & \vdots & \\ & 0 & \end{pmatrix}=b'_k\det(I_n)$$

$$=\frac{\det(M_k)}{\det(A)\det(A^{-1})}$$

$$\overset{?}{=}\frac{\det(A^{-1}M_k)}{\det(A)\det(A^{-1})}$$

$$N_k\overset{?}{=}A^{-1}M_k$$

$\det(e_1\ e_2\ \cdots\ b'\ \cdots\ e_n)=\det(e_1\ e_2\ \cdots\ b'_k e_k\ \cdots\ e_n)$ where $\{e_i \mid 1\le i\le n\}$ is the standard ordered basis for $\mathbb{F}^n$ in column form

$$M_k=(a_1\ a_2\ \cdots\ b\ \cdots\ a_n)$$

$$A^{-1}M_k=(A^{-1}a_1\ A^{-1}a_2\ \cdots\ A^{-1}b\ \cdots\ A^{-1}a_n)=(e_1\ e_2\ \cdots\ b'\ \cdots\ e_n)$$

(Exercise 15 of Section 3-2)
$\hookrightarrow M(A|B)=(MA|MB)$

Proof

Given $\det(A)\neq 0$, $\operatorname{rank}(A)=n$ by the Corollary to Theorem 4.6, (or: Corollary to Thm 4.7 & Thm 3.10) and since $\operatorname{rank}(A)\le\operatorname{rank}(A|b)\le n$, $\operatorname{rank}(A|b)=n$ too. Accordingly, the solution space of the homogeneous system corresponding to $Ax=b$ has dimension 0. Hence, the nonhomogeneous system $Ax=b$ has exactly one solution, namely $b':=(b'_1\ b'_2\ \cdots\ b'_n):=A^{-1}b$ because $(A^{-1}A\,|\,A^{-1}b)=(I_n|b')$ is the reduced row echelon form of $(A|b)$. Let N_k be the $n\times n$ matrix obtained from I_n by replacing column k of I_n ~~by~~ b'. First, notice that $\det(N_k)=\det(e_1\ e_2\ \cdots\ e_{k-1}\ b'\ e_{k+1}\ \cdots\ e_{n-1}\ e_n)\overset{\text{Thm 4.6}}{=}\det(e_1\ e_2\ \cdots\ e_{k-1}\ b'_k e_k\ e_{k+1}\ \cdots\ e_{n-1}\ e_n)\overset{\text{Thm 4.3}}{=}b'_k\det(I_n)=b'_k$ from Theorems 4.6 and 4.3, where $\{e_i \mid 1\le i\le n\}$ is the standard ordered basis of $\mathbb{F}^n$ in column form. Secondly, by having a_j be the jth column of A, we see that $A^{-1}M_k=A^{-1}(a_1\ a_2\ \cdots\ a_{k-1}\ b\ a_{k+1}\ \cdots\ a_{n-1}\ a_n)\overset{\text{Ex 15}}{=}(A^{-1}a_1\ A^{-1}a_2\ \cdots\ A^{-1}a_{k-1}\ A^{-1}b\ A^{-1}a_{k+1}\ \cdots\ A^{-1}a_{n-1}\ A^{-1}a_n)=(e_1\ e_2\ \cdots\ e_{k-1}\ b'\ e_{k+1}\ \cdots\ e_{n-1}\ e_n)=N_k$ from exercise 15 of section 3-2.

As such, $$b'_k=\frac{\det(N_k)}{\det(I_n)}=\frac{\det(A^{-1}M_k)}{\det(A^{-1}A)}\overset{\text{Thm 4.7}}{=}\frac{\det(A^{-1})\cdot\det(M_k)}{\det(A^{-1})\cdot\det(A)}=\frac{\det(M_k)}{\det(A)}.$$

$\square$

Self-Proof of Corollary

Ideas

If A invertible, $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$ by Theorem 4.7 so $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$.

When $\det(A) \neq 0$, $\operatorname{rank}(A) = n$ by the Corollary of Theorem 4.6.

Proof

When A is invertible, by Theorem 4.7 we see that $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$. So, $\det(A) \neq 0$ and $\det(A^{-1}) = \frac{1}{\det(A)}$ as expected.

Conversely, given $\det(A) \neq 0$, the Corollary to Theorem 4.6 says $\operatorname{rank}(A) = n$. Consequently, A is invertible. Therefore, the biconditional holds. □

Self-Proof of Theorem 4.8

$m \times n \quad n \times p$

T en $n \times m$

Ideas

A not invertible $\Rightarrow$ A^t not invertible $\Rightarrow$ trivial

A invertible $\Rightarrow$

Proof

If A is not invertible, then A^t — being of equal rank — must not be invertible. Thus, $\det(A^t) = \det(A) = 0$. Now consider A being invertible.

Then for some nonnegative integer n and elementary matrices E_i, $A = E_n E_{n-1} \cdots E_1$ so $\det(A^t) = \det(E_1^t E_2^t \cdots E_n^t) \overset{\text{Thm 4.7}}{=} \det(E_1^t) \cdot \det(E_2^t) \cdot \cdots \cdot \det(E_n^t)$

$\overset{\text{Ex 29}}{=} \det(E_1) \cdot \det(E_2) \cdot \cdots \cdot \det(E_n) = \det(E_n) \cdot \det(E_{n-1}) \cdot \cdots \cdot \det(E_1) \overset{\text{Thm 4.7}}{=} \det(E_n E_{n-1} \cdots E_1) = \det(A)$ by Theorem 4.7 and exercise 29 of section 4.2. □

1. (a) False, $\det\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix} = 2$

(b) True

(c) False, it should be $\det(M) \neq 0$ instead

(d) True

(e) False, $\det(A^t) = \det(A)$

(f) True

(g) False

(h) False

Nice! :)

3. Since $\det\begin{pmatrix}2&1&-3\\1&-2&1\\3&4&-2\end{pmatrix} = -\det\begin{pmatrix}1&-2&1\\0&5&-5\\0&10&-5\end{pmatrix} = -\det\begin{pmatrix}1&-2&1\\0&5&-5\\0&0&5\end{pmatrix} = -1\cdot5\cdot5 = -25$, Cramer's Rule indeed applies. Using the notation of Theorem 4.9, we have $x_1 = \dfrac{\det\begin{pmatrix}5&1&-3\\10&-2&1\\0&4&-2\end{pmatrix}}{-25} = \dfrac{\det\begin{pmatrix}5&1&-3\\0&-4&7\\0&4&-2\end{pmatrix}}{-25} = \dfrac{\det\begin{pmatrix}5&1&-3\\0&-4&7\\0&0&5\end{pmatrix}}{-25} = \dfrac{(5)(-4)(5)}{-25} = 4$,

$x_2 = \dfrac{\det\begin{pmatrix}2&5&-3\\1&10&1\\3&0&-2\end{pmatrix}}{-25} = \dfrac{-\det\begin{pmatrix}1&10&1\\0&-15&-5\\0&-30&-5\end{pmatrix}}{-25} = \dfrac{\det\begin{pmatrix}1&10&1\\0&-15&-5\\0&0&5\end{pmatrix}}{25} = \dfrac{\overbrace{1\cdot-15\cdot5}^{1\cdot-3\cdot25}}{25} = -3$, and $x_3 = \dfrac{\det\begin{pmatrix}2&1&5\\1&-2&10\\3&4&0\end{pmatrix}}{-25} = \dfrac{-\det\begin{pmatrix}1&-2&10\\0&5&-15\\0&10&-30\end{pmatrix}}{25} = \dfrac{\det\begin{pmatrix}1&-2&10\\0&5&-15\\0&0&0\end{pmatrix}}{25} = 0$

Therefore, the unique solution to this system of linear equations is $(x_1, x_2, x_3) = (4, -3, 0)$.

To verify,
$2(4)+(-3)-3(0) = 8-3 = 5$ ✓
$4-2(-3)+0 = 4+6 = 10$ ✓
$3(4)+4(-3)-2(0) = 12-12 = 0$ ✓

5. Since $\det\begin{pmatrix}1&-1&4\\-8&3&1\\2&-1&1\end{pmatrix} = \det\begin{pmatrix}1&-1&4\\0&-5&33\\0&1&-7\end{pmatrix} = -\det\begin{pmatrix}1&-1&4\\0&1&-7\\0&0&-2\end{pmatrix} = -1(1)(-2) = 2$, using the notation of Theorem 4.9, Cramer's Rule says

$x_1 = \dfrac{\det\begin{pmatrix}-4&-1&4\\8&3&1\\0&-1&1\end{pmatrix}}{2} = \dfrac{\det\begin{pmatrix}-4&-1&4\\0&1&9\\0&0&10\end{pmatrix}}{2} = \dfrac{-4(1)(10)}{2} = -20$, $x_2 = \dfrac{\det\begin{pmatrix}1&-4&4\\-8&8&1\\2&0&1\end{pmatrix}}{2} = \dfrac{\det\begin{pmatrix}1&-4&4\\0&-24&33\\0&8&-7\end{pmatrix}}{2} = \dfrac{-\det\begin{pmatrix}1&-4&4\\0&8&-7\\0&0&12\end{pmatrix}}{2} = \dfrac{-1(8)(12)}{2} = -48$, and

$x_3 = \dfrac{\det\begin{pmatrix}1&-1&-4\\-8&3&8\\2&-1&0\end{pmatrix}}{2} = \dfrac{\det\begin{pmatrix}1&-1&-4\\0&-5&-24\\0&1&8\end{pmatrix}}{2} = \dfrac{-\det\begin{pmatrix}1&-1&-4\\0&1&8\\0&0&16\end{pmatrix}}{2} = \dfrac{-1(1)(16)}{2} = -8$. Thus, the unique solution to this system is $(x_1, x_2, x_3) = (-20$

Checking:
$-20-(-48)+4(-8) = -20+48-32 = -4$ ✓
$-8(-20)+3(-48)+(-8) = 160-144-8 = 8$ ✓
$2(-20)-(-48)+(-8) = -40+48-8 = 0$ ✓

7. As $\det\begin{pmatrix} 3 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & -5 & -2 \end{pmatrix} = -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} = -1(3)(\frac{4}{3}) = -4$, Cramer's Rule can be used. Hence, using Theorem 4.9's ...

$x_1 = \dfrac{\det\begin{pmatrix} 4 & 1 & 1 \\ 12 & -1 & 0 \\ -8 & 2 & 1 \end{pmatrix}}{-4} = \dfrac{\det\begin{pmatrix} 4 & 1 & 1 \\ 0 & -4 & -3 \\ 0 & 4 & 3 \end{pmatrix}}{-4} = \dfrac{\det\begin{pmatrix} 4 & 1 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & 0 \end{pmatrix}}{-4} = 0$; $x_2 = \dfrac{\det\begin{pmatrix} 3 & 4 & 1 \\ -2 & 12 & 0 \\ 1 & -8 & 1 \end{pmatrix}}{-4} = \dfrac{-\det\begin{pmatrix} 1 & -8 & 1 \\ 0 & -4 & 2 \\ 0 & 28 & -2 \end{pmatrix}}{-4} = \dfrac{-\det\begin{pmatrix} 1 & -8 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 12 \end{pmatrix}}{-4} = \dfrac{1\cdot -4 \cdot 12}{-4} = -12$,

and lastly, $x_3 = \dfrac{\det\begin{pmatrix} 3 & 1 & 4 \\ -2 & -1 & 12 \\ 1 & 2 & -8 \end{pmatrix}}{-4} = \dfrac{-\det\begin{pmatrix} 1 & 2 & -8 \\ 0 & 3 & -4 \\ 0 & -5 & 28 \end{pmatrix}}{-4} = \dfrac{-\det\begin{pmatrix} 1 & 2 & -8 \\ 0 & 3 & -4 \\ 0 & 0 & \frac{64}{3} \end{pmatrix}}{-4} = \dfrac{-1(3)(\frac{64}{3})}{-4} = 16$. The unique solution to the system is hence

$(x_1, x_2, x_3) = (0, -12, 16)$

To check:
$3(0) - 12 + 16 = 4$ ✓
$-2(0) - (-12) + 0 = 12$ ✓
$1(0) + 2(-12) + 16 = -8$ ✓

8. Let u, v, a_i be column vectors in $\mathbb{F}^n$ for $1 \le i \le n$ and k be a scalar in $\mathbb{F}$. Then by Theorems 4.3 and 4.8, given any $1 \le r \le n$, $\det(a_1 \cdots a_{r-1}\ u+kv\ a_{r+1} \cdots a_n) \overset{\text{Thm 4.8}}{=} \det(a_1 \cdots a_{r-1}\ u+kv\ a_{r+1} \cdots a_n)^t = \det\begin{pmatrix} a_1^t \\ \vdots \\ a_{r-1}^t \\ u^t+kv^t \\ a_{r+1}^t \\ \vdots \\ a_n^t \end{pmatrix} \overset{\text{Thm 4.3}}{=} \det\begin{pmatrix} a_1^t \\ \vdots \\ a_{r-1}^t \\ u^t \\ a_{r+1}^t \\ \vdots \\ a_n^t \end{pmatrix} + k\det\begin{pmatrix} a_1^t \\ \vdots \\ a_{r-1}^t \\ v^t \\ a_{r+1}^t \\ \vdots \\ a_n^t \end{pmatrix} \overset{\text{Thm 4.8}}{=} \det\begin{pmatrix} a_1^t \\ \vdots \\ a_{r-1}^t \\ u^t \\ a_{r+1}^t \\ \vdots \\ a_n^t \end{pmatrix}^t + k\det\begin{pmatrix} a_1^t \\ \vdots \\ a_{r-1}^t \\ v^t \\ a_{r+1}^t \\ \vdots \\ a_n^t \end{pmatrix}^t$

$\overset{\text{Thm 4.8}}{=} \det(a_1 \cdots a_{r-1}\ u\ a_{r+1} \cdots a_n) + k\det(a_1 \cdots a_{r-1}\ v\ a_{r+1} \cdots a_n)$. So, n-linearity holds column-wise.

An upper triangular matrix is invertible iff its determinant is nonzero, by the Corollary to Theorem 4.7, iff the product of its diagonal entries is nonzero, by Exercise 23 of section 4.2, iff all of its diagonal entries are nonzero, since fields have no zero divisors.

When $\det(M) \ne 0$, the Corollary to Theorem 4.7 says M is invertible. So, M can be expressed as a product of some n elementary matrices E_i $(1 \le i \le n)$. That is, $M = E_n E_{n-1} \cdots E_1$. Since multiplication with elementary matrices is rank preserving, $\text{rank}(M^k) = \text{rank}(E_n^k E_{n-1}^k \cdots E_1^k) = \text{rank}(E_n E_{n-1} \cdots E_1) = \text{rank}(M) = n$ for any positive integer k. Since $\text{rank}(O) = 0$, $M^k \ne O$ for each positive integer k.

Given that M is skew symmetric and n is odd, Theorem 4.8 together with exercise 25 of section 4.2 says $\det(M) \overset{\text{Thm 4.8}}{=} \det(M^t) = \det(-M) \overset{\text{ex 25}}{=} (-1)^n \det(M) \overset{n \text{ odd}}{=} -\det(M)$. So, $2\det(M) = 0$. Thus, $\det(M) = 0$ means M is not invertible by the Corollary to Theorem 4.7. When n is even, let M be the matrix with its jth row, $m_j := \begin{cases} e_{j+1} & \text{if } j \text{ odd}, \\ -e_{j-1} & \text{if } j \text{ even}. \end{cases}$ so $\text{rank}(M) = n$ is certain as $j+1 \ne j-1$ for any integer j. Moreover, $M_{ij} = \begin{cases} 1 & \text{if } j \text{ odd } \& \ i = j+1, \\ -1 & \text{if } j \text{ even } \& \ i = j-1, \\ 0 & \text{otherwise}. \end{cases} = \begin{cases} -1 & \text{if } i \text{ even } \& \ j = i-1, \\ 1 & \text{if } i \text{ odd } \& \ j = i+1, \\ 0 & \text{otherwise}. \end{cases} = -M_{ji}$ shows that M is skew-symmetric. Therefore, for any even n, a $n \times n$ skew symmetric matrix can possibly have nonzero determinant, for example the skew-symmetric matrix above has $\det(M) \ne 0$ as $\text{rank}(M) = n$.

11. Ideas

$$\cancel{M_{ij} := \begin{cases} 0 \text{ if } i=j, \\ -1 \text{ if } i>j, \\ 1 \text{ if } i<j. \end{cases} \quad m_j \text{ being row } j \text{ of } M \quad m_j = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 & \underset{j\text{th}}{0} & 1 & 1 & \cdots & \underset{n\text{th}}{1} \end{pmatrix}}$$

$$M_{ij} := \begin{cases} 1 \text{ if } j=i+1, \\ -1 \text{ if } i=j+1, \\ 0 \text{ otherwise.} \end{cases}$$

let m_j be row j,

$m_1 = e_2$

$m_n = -e_{n-1}$

$m'_3 := m_1 + m_3 \qquad m'_{n-3} := m_n + m_{n-3}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

12. Assume Q is orthogonal. From Theorems 4.7 and 4.8, $\det(Q)^2 \overset{\text{Thm 4.8}}{=} \det(Q)\det(Q^t) \overset{\text{Thm 4.7}}{=} \det(QQ^t) = \det(I) = 1$. Hence, $\det(Q) = \pm 1$.

13. (a) When $n=1$, $\det(\overline{(a+bi)}) = \det((a-bi)) = a-bi = \overline{a+bi} = \overline{\det((a+bi))}$ for any $a, b \in \mathbb{R}$. Hence, $\det(\overline{M}) = \overline{\det(M)}$ for $M \in M_{1\times1}(\mathbb{C})$. Suppose this is true of all matrices $M \in M_{n\times n}(\mathbb{C})$. Then, letting $M \in M_{(n+1)\times(n+1)}(\mathbb{C})$ instead, $\det(\overline{M}) := \sum_{j=1}^{n+1} (-1)^{1+j} (\overline{M}_{1j}) \cdot \det((\overline{M})_{1j}) = \sum_{j=1}^{n+1} (-1)^{1+j} \overline{M_{1j}} \det(\overline{(M)_{1j}}) \overset{\text{I.H.}}{=} \sum_{j=1}^{n+1} (-1)^{1+j} \overline{M_{1j}}\, \overline{\det(\tilde{M}_{1j})}$ by our assumption / induction hypothesis (I.H.) $= \overline{\sum_{j=1}^{n+1} (-1)^{1+j} M_{1j} \det(\tilde{M}_{1j})} = \overline{\det(M)}$. So, the claim is still true for every $(n+1)\times(n+1)$ matrix. By induction, the claim holds for all $n \times n$ matrices and any natural number n.

(b) Assume Q is a unitary matrix. Then by Theorems 4.7 and 4.8, as well as Exercises 12 and 13(a) above, we have that $1 = \det(I) = \det(QQ^*) \overset{\text{Thm 4.7}}{=} \det(Q)\det(\overline{Q^t}) \overset{13(a)}{=} \det(Q)\overline{\det(Q^t)} \overset{\text{Thm 4.8}}{=} \det(Q)\overline{\det(Q)} = |\det(Q)|^2$. Hence, $|\det(Q)| = 1$.

14. Clearly, $\det(B) \neq 0$ iff $\operatorname{rank}(B) = n$, by the Corollary to Theorem 4.6, iff $\dim(\operatorname{span}(\beta)) = n$ iff β is a basis for $\mathbb{F}^n$, since $|\beta| = n$.

15. Recall that matrices $A, B \in M_{n\times n}(\mathbb{F})$ are similar iff there exists another (invertible) matrix $Q \in M_{n\times n}(\mathbb{F})$ so $B = Q^{-1}AQ$. Consequently, using the definitions above and supposing $A, B \in M_{n\times n}(\mathbb{F})$ are indeed similar, we have that $\det(B) \overset{\text{Thm 4.7}}{=} \det(Q^{-1})\det(A)\det(Q) = \det(A)\det(Q^{-1})\det(Q) \overset{\text{Thm 4.7}}{=} \det(A)\det(Q^{-1}Q) = \det(A)\cdot 1 = \det(A)$ by Theorem 4.7.

16. By Theorem 4.7 we have $\det(A)\cdot\det(B) \overset{\text{Thm 4.7}}{=} \det(AB) = \det(I) = 1$. Hence, $\det(A)$ and $\det(B)$ must be nonzero, implying that A and B are invertible. That is, $B = A^{-1}(AB) = A^{-1}I = A^{-1}$.

17. Suppose n is odd and $\mathbb{F}$ is not a field of characteristic two. Then by Theorems 4.3 and 4.7, $\det(A)\cdot\det(B) = \det(AB) = \det(-BA) \overset{\text{Thm 4.7}}{=} \det(-B)\cdot\det(A) \overset{\text{Thm 4.3}}{=} (-1)^n \cdot \det(B)\cdot\det(A) \overset{n \text{ odd}}{=} -\det(A)\cdot\det(B)$. So, $\det(A)\cdot\det(B) = 0$. Since fields have no zero divisors, either $\det(A)$ or $\det(B)$ must be 0. This corresponds to A or B being uninvertible by the Corollary to Theorem 4.7.

19. Since Theorem 4.8 informs us that $\det(A) = \det(A^t)$, $\det(A)$ is the product of the diagonal entries of itself; as $(A^t)_{ii} = A_{ii}$

20. Let $A \in M_{m\times n}(\mathbb{F})$, for some $m \leq n$. ~~There are some p elementary $m\times m$ matrices E_i of types 1 and 3, of which k are type 1, so we have $E_p \cdots E_1 A$ being upper triangular. Hence, letting $G := E_p \cdots E_1$, we see that~~ High (good) phrasing can be so annoying to achieve

20. There exists some m_1 and m_3 elementary row operations of types 1 and 3 respectively, so A is transformed into an upper triangular matrix A'. By applying the same sequence of elementary row operations on M, it becomes some $\begin{pmatrix} A' & B' \\ O & I \end{pmatrix}$. Thus, $\det(M) = (-1)^{m_1} \det \begin{pmatrix} A' & B' \\ O & I \end{pmatrix}$ $= (-1)^{m_1} \prod_{i=1}^{\dim(A)} A'_{ii} = \det(A)$. $\square$

(finally a satisfactory phrasing!) :)

21. Almost the same as 20 except that we need to apply 2 sequence of eros ($A \xrightarrow{1st} A'$ & $C \xrightarrow{2nd} C'$) to M. (offset by $\dim(A)$) $\square$

22. (a) Notice $T(x^j) = (c_0^j, c_1^j, \cdots, c_n^j) = \sum_{i=1}^{n+1} c_i^j e_i$. So, $M_{ij} = c_i^j$ as we thought. $\square$

(b) Since T is an isomorphism according to exercise 22 of section 2.4, L_M is also an isomorphism (as Figure 2.2 commutes). As such, $\operatorname{rank}(M) = n+1$ so the Corollary to Theorem 4.7 says $\det(M) \neq 0$.

(c) Idea

$\exists P_{n\times n} : P\begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_0^k \\ \vdots \\ c_n^k \end{pmatrix}$?

$n=1$: $\det(M) = 1$, vacuously true

Assume true for $n=k$. For $M \in M_{(n+1)\times(n+1)}(\mathbb{F})$,

$(i, k+1)$th entry at the $(k+1)$th iteration: $i_{k+1} = i_k +$

$\frac{c_0 - c_i}{c_0} = 1 - \frac{c_i}{c_0}$ $\quad$ $c_2^k \cancel{- \ldots + \ldots} - c_2 c_0^{k-1}$

$- \frac{c_2 - c_0}{1 \cdot c_0}$

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 0 & c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ \vdots & \vdots & & & \\ 0 & c_n - c_0 & c_n^2 - c_0^2 & \cdots & c_n^n - c_0^n \end{pmatrix} \to \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 0 & c_1 - c_0 & c_1^2 - c_0^2 & \cdots & c_1^n - c_0^n \\ 0 & 0 & c_2^2 - c_2 c_0 & \cdots & c_2^n - c_2 c_0^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & c_n^2 - c_n c_0 & \cdots & c_n^n - c_n c_0^{n-1} \end{pmatrix}$$

$c_j^k - c_j c$

$(x_{ij})_{k+1}$

$(x_{ij})_k - \frac{x_{ij}}{c_0}$ $\quad$ $c_i^j \sim \frac{c_i^k}{c_0^k}$, x_{ij}

$(x_{ij})_k - (x_{i,k+1})_k c_0^{j-k-1}$ $\quad$ $(x_{ij})_{k+1} = (x_{ij})_k - \frac{(x_{i,k+1})_k}{c_0^{k+1}}$.

$(x_i)_{k+1} = (x_i)_k - \frac{(x_k)_{i,k+1}}{c_0^{k+1}} x_1$

$c_i^2 - c_i c_0$

$c_i^j \cancel{- c_i^2 c_0^{j-1}} + \cancel{c_i c_0^{j-1}} - c_i^2 c_0^{j-2}$

$c_j^i \to c_j^i - c_i^{i-1} c_0^{j+1-i}$ ☆ $c_i^i \to c_i^i - c_i^{i-1} c_0$ ☆

0th $\quad c_i^j - c_0^j$

1st $\quad c_i^j - c_0^j + \frac{c_0 - c_i}{c_0} \cdot c_0^j = c_i^j - c_i c_0^{j-1}$

2nd $\quad c_i^j - c_i^2 c_0^{j-1} + \frac{c_i c_0 - c_i^2}{c_0^2} \cdot c_0^j = c_i^j - c_i^2 c_0^{j-2}$

From 0: $c_i^j - c_i^k c_0^{j-k}$ $\quad$ From 1: $c_i^j - c_i^{k+1} c_0^{j-k-1}$

22. Ideas

$m_j - c_0 m_{j-1}$

$c_i^j - c_0 c_i^{j-1} = c_i^{j-1}(c_i - c_0)$

Proof

Clearly, if $M \in M_{1\times 1}(\mathbb{F})$, $\det(M) = 1 = \prod_{0 \le i < j \le 0} (c_j - c_i)$, the empty product. So assume this is true of all $(n+1)\times(n+1)$ matrices, and let $M \in M_{(n+2)\times(n+2)}(\mathbb{F})$ instead, with m_j being its jth column. Starting from $j = n+2$, we subtract c_0 times of m_{j-1} from m_j till we reach $j=1$, hence transforming each c_i^j into $c_i^{j-1}(c_i - c_0)$. Thus, we have that by assumption / our induction hypothesis [I.H.]:

$$\det(M) = \left(\prod_{i=1}^{n+1} c_i - c_0\right) \cdot \det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ (c_1 - c_0)^{-1} & 1 & c_1 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ (c_{n+1} - c_0)^{-1} & 1 & c_{n+1} & \cdots & c_{n+1}^n \end{pmatrix} = (-1)^{1+1} \cdot 1 \cdot \left(\prod_{i=1}^{n+1} c_i - c_0\right) \cdot \det\begin{pmatrix} 1 & c_1 & \cdots & c_1^n \\ \vdots & \vdots & & \vdots \\ 1 & c_{n+1} & \cdots & c_{n+1}^n \end{pmatrix}$$

$$\overset{\text{I.H.}}{=} \left(\prod_{i=1}^{n+1} c_i - c_0\right) \cdot \left(\prod_{1 \le i < j \le n+1} c_j - c_i\right)$$

$$= \prod_{0 \le i < j \le n+1} c_j - c_i$$

Therefore, the statement is true of all $M \in M_{(n+2)\times(n+2)}(\mathbb{F})$. Consequently, it is true by induction for all matrices in $\mathbb{F}$. $\square$

22. (a) Ideas

Goal: $\underbrace{[(c_1-c_0)(c_2-c_0)\cdots(c_n-c_0)]}_{n}\underbrace{[(c_2-c_1)(c_3-c_1)\cdots(c_n-c_1)]}_{n-1}\cdots\underbrace{[(c_n-c_{n-1})]}_{1}$

$c_3^2-c_3c_0 \quad c_3^3-c_3c_0^2+\frac{c_3c_0-c_3^2}{c_0^2}\cdot c_0^3 = c_3^3-c_3c_0^2+c_3c_0^2-c_3^2c_0 = c_3^3-c_3^2c_0$

$1(c_1-c_0)(c_2^2-c_2c_0)(c_3^3-c_3^2c_0)\cdots(c_n^n-c_n^{n-1}c_0)$

$c_2c_3^2c_4^3\cdots c_n^{n-1}(c_1-c_0)(c_2-c_0)\cdots(c_n-c_0)$

$n=2: (c_2-c_1)\cancel{(c_2-c_0)}\cancel{(c_1-c_0)} \stackrel{?}{=} c_2\cancel{(c_1-c_0)}\cancel{(c_2-c_0)}$

When $n=0$: $1=1$

$n=1$: $c_1-c_0=c_1-c_0$

$1(c_1-c_0)(c_2^2-c_2c_0)$

$c_2(c_1-c_0)(c_2-c_0)$

Assume true for $n-1$, $\Pi_n=(c_n-c_0)(c_n-c_1)\cdots(c_n-c_{n-1})\,\Pi_{n-1}$

Show $(c_n-c_0)(c_n-c_1)\cdots(c_n-c_{n-1}) = c_n^{n-1}(c_n-c_0)$

$\frac{c_0-c_2}{c_1-c_0}$ $\quad c_2^2-c_0^2-(c_2-c_0)(c_1+c_0)$

$=(c_2-c_0)(c_2+c_0-c_1-c_0)$

$=(c_2-c_0)(c_2-c_1)$

$$\det\begin{pmatrix}1 & c_0 & c_0^2\\ 1 & c_1 & c_1^2\\ 1 & c_2 & c_2^2\end{pmatrix} = \det\begin{pmatrix}1 & c_0 & c_0^2\\ 0 & c_1-c_0 & c_1^2-c_0^2\\ 0 & c_2-c_0 & c_2^2-c_0^2\end{pmatrix} = \det\begin{pmatrix}1 & c_0 & c_0^2\\ 0 & c_1-c_0 & c_1^2-c_0^2\\ 0 & 0 & c_2^2-c_2c_0\end{pmatrix} = (c_1-c_0)(c_2^2-c_2c_0)$$

$= c_2(c_2-c_0)(c_1-c_0)$

since $(c_2^2-c_0^2)+\frac{c_0-c_2}{c_0}\cdot c_0^2$

$= c_2^2-c_0^2+c_0^2-c_2c_0$

$= c_2^2-c_2c_0$

$\begin{pmatrix}1 & c_0 & c_0^2\\ 0 & c_1-c_0 & c_1^2-c_0^2\\ 0 & 0 & \end{pmatrix}$

Qn suggests this is $(c_2-c_1)\cancel{(c_2-c_0)}\cancel{(c_1-c_0)} = c_2(c_2-c_0)(c_1-c_0)$

$\Rightarrow c_2-c_1=c_2$

$c_1=0$???

25. Ideas

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -15 \end{pmatrix} = \begin{pmatrix} -40 \\ 0 \\ 0 \end{pmatrix} = -40 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$c_{11} = 5 \quad c_{12} = 0 \quad c_{13} = -15$

$\det = -40$

$c_{21} = 4 \quad c_{22} = -8 \quad c_{23} = 4$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -40 \\ 0 \end{pmatrix} = -40 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

When $\det(A) = 0$, $(A|0) \rightarrow$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} = 0$$

$c_{11} = -3 \quad c_{12} = 6 \quad c_{13} = -3$

$Ax = \det(A) \cdot e_j$

Given $\det(A) \neq 0$,

$$x_k = \frac{\det(M_k)}{\det(A)} = \det(M_k) = \det($$

$$= (-1)^{j+k} \cdot 1 \cdot \det(\bar{A}_{jk})$$

$$= c_{jk}$$

(a)

b) Proof

23. (a) Clearly, for any $m\times m$ submatrix A_m of A with nonzero determinant, $\operatorname{rank}(A_m) = m \le \operatorname{rank}(A) =: r$. Let $\{a_{k_1}, a_{k_2}, \ldots, a_{k_r}\}$ be an extension of a_1 to a basis for $\mathbb{F}^n$, where a_j is the jth row of A and $1 = k_1 < k_2 < \cdots < k_r \le n$. Then for each $j \notin \{k_i \mid 1 \le i \le k\}$, $a_j = \sum_{i=1}^{k} c_{ij} a_{k_i}$ for some scalars c_{ij}. By type 3 row operations, all these $n-r$ rows a_j are reduced to 0 in the transformed matrix A'. Repeating this process on the columns of A' now, we get a matrix A'' with $n-r$ zero rows and $n-r$ zero columns. Removing them gives a submatrix A_r of A that has rank r, so $\det(A_r) \ne 0$. As such, $r = k$.

(b)
Proven in (a).

Oops, A contains entries '-1', not '1'!

24. Let $R_{i,j}(A+tI)$ denote the matrix $A+tI$ after t^{-1} times of its ith row is added to its jth row. We see that

$$R_{n-1,n} R_{n-2,n-1} \cdots R_{1,2}(A+tI) = \begin{pmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ 0 & t & 0 & \cdots & 0 & a_1 + t^{-1}a_0 \\ 0 & 0 & t & \cdots & 0 & a_2 + t^{-1}a_1 + t^{-2}a_0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} + \sum_{i=0}^{n-2} t^{i-n+1} a_i + t \end{pmatrix}.$$

Oops forgot the tI on the entry!

So, $\det(A) = t^{n-1}\left(a_{n-1} + \sum_{i=0}^{n-2} t^{i-(n-1)} a_i\right) =$

$$t + t^{n-1}a_{n-1} + \sum_{i=0}^{n-2} t^i a_i = \left(\sum_{i=0}^{n-1} t^i a_i\right) + t^{n-1}$$

25. (a) Ideas

$\det(B) \overset{4.8}{=} \det(B^t) = (-1)^{k+j} \det(\widetilde{(B^t)}_{kj}) = (-1)^{k+j} \det(\tilde{B}_{jk})$

Proof

By Theorem 4.8, $\det(B) \overset{\text{Thm 4.8}}{=} \det(B^t) = (-1)^{k+j} \det(\widetilde{(B^t)}_{kj}) = (-1)^{k+j} \det(\tilde{B}_{jk}) = (-1)^{k+j} \det(\tilde{A}_{jk}) = c_{jk}$, by doing cofactor expansion along the kth row of B^t.

(b) Ideas

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$c_{11} = 1 \quad c_{21} = 0$

$c_{12} = 0 \quad c_{22} = 1$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} - \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot e_1$$

$c_{11} = 4 \quad c_{21} = -2$

$c_{12} = -3 \quad c_{22} = 1$

$\det = -2 \quad \det = -2$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$\prod_{0 \le i < j \le n+1} (c_j - c_i)$ ... $\left(\prod_{0 \le i < j \le n} (c_j - c_i) \right)$ ... C^n_{n+1} / $(-1)^{2+2} \cdot \prod$ ...

25. Ideas

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -15 \end{pmatrix} = \begin{pmatrix} -40 \\ 0 \\ 0 \end{pmatrix} = -40 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$c_{11} = 5 \quad c_{12} = 0 \quad c_{13} = -15$

$\det = -40$

$c_{21} = 4 \quad c_{22} = -8 \quad c_{23} = 4$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -40 \\ 0 \end{pmatrix} = -40 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

When $\det(A) = 0$, $(A \mid 0) \to$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} = 0$$

$c_{11} = -3 \quad c_{12} = 6 \quad c_{13} = -3$

$Ax = \det(A) \cdot e_j$

Given $\det(A) \neq 0$,

$$x_k = \frac{\det(M_k)}{\det(A)} = \det(M_k) = \det(M_k)^t$$

$$= (-1)^{j+k} \cdot 1 \cdot \det(\bar{A}_{jk})$$

$$= c_{jk}$$

(a)

(b) Proof

·R. ... matrix ...

26. (a) $\begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix}^t \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$

(e) $\begin{pmatrix} -3i & 4 & 4+10i \\ 0 & -1+i & -5-3i \\ 0 & 0 & 3+3i \end{pmatrix}^t \begin{pmatrix} -3i & 0 & 0 \\ 4 & -1+i & 0 \\ 10+16i & -5-3i & 3+3i \end{pmatrix}$

Oops been thinking of 25.

26·(e) $\det = (-3i)(-1+i)(3+3i) = (3-3i)(-3+3i) = -(9-9-18i) = 18i$ $\quad A\det = -3-3i \quad (-3-3i)^2 = 18i$

27. (a) Ideas

$\det(C) = \sum_{j=1}^{n+1} (-1)^{1+j} C_{1j} \det(\tilde{C}_{1j})$ $\qquad [\det(A)]^n = \left[\sum_{j=1}^{n+1} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j})\right]^n = \left[\sum_{j=1}^{n+1} A_{1j} C_{1j}\right]^n$

Find a $n \times n$ matrix B so the cofactor of row i, column k of B is $(\tilde{C}_{1j})_{ik}$? $\quad 1 \le i \le n,\ 1 \le k \le n$

$$(-1)^{i+k}\det(\tilde{B}_{ik}) = (-1)^{i+k}\det(\tilde{A}_{i+1,k}) \quad \text{if } k<j$$
$$= (-1)^{i+k}\det(\tilde{A}_{i+1,k+1}) \quad \text{if } k \ge j$$

(a) Proof I

From exercise 25·(c), it is straightforward to notice that $\det(A)\det(C) = [\det(A)]^n$ so $\det(C) = [\det(A)]^{n-1}$.

(b) We see that $(C^t)_{ij} = c_{ji} = (-1)^{j+i}\det(\tilde{A}_{ij}) = (-1)^{j+i}\det(\tilde{A}_{ij})^t = (-1)^{i+j}\det((\widetilde{A^t})_{ji})$, the ji-cofactor of A^t, since removing row i and column j then transposing is the same as removing row j and column i after transposing.

(c) Ideas $\quad A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ $\quad adf \neq 0 \quad a,d,f \neq 0 \implies$ $c_{21} = -bf$, $c_{32} = be-cd$, $c_{32} = -ae$ $\implies C = \begin{pmatrix} df & 0 & 0 \\ -bf & af & 0 \\ be-cd & -ae & ad \end{pmatrix}^t$

$\det(\tilde{A}_{ij})$

$$\left(\begin{array}{c|c} N & I_n \\ ae_{n+1} & e_{n+1} \end{array}\right) \to \left(\begin{array}{c|c} I_n & N^{-1} \\ ae_{n+1} & e_{n+1} \end{array}\right)$$

Proof I

When $n=1$, there is nothing to prove since all 1×1 matrices are upper triangular. So assume B^{-1} is upper triangular if $B \in M_{n\times n}$ is, and let $A \in M_{n\times n}(\mathbb{F})$. Thus, by some elementary row operations, $(A \mid I_n) \to \left(\begin{array}{cc|cc} I_{n-1} & \vdots \\ 0\cdots0 & A_{nn} \end{array}\ \middle|\ \begin{array}{cc} \tilde{A}_{nn}^{-1} & 0 \\ 0\cdots0 & 1 \end{array}\right) \to \left(I_n \ \middle|\ \begin{array}{cc} \tilde{A}_{nn}^{-1} & -c_{i}A_{nn}^{-1} \\ 0\cdots0 & A_{nn}^{-1} \end{array}\right)$

Since $\tilde{A}_{nn}^{-1}$ is upper triangular by assumption / our induction hypothesis, $A^{-1} = \begin{pmatrix} \tilde{A}_{nn}^{-1} & -c_iA_{nn}^{-1} \\ 0\cdots0 & A_{nn}^{-1} \end{pmatrix}$ is clearly also upper triangular, as expected.

Hence, by induction, for any upper triangular matrix A, A^{-1} is also upper triangular. By exercise 25(d), $C = \det(A)\cdot A^{-1}$ is also upper triangular.

$_{ij}(A+tI)$ denote the n.

in $R_{n-2,n-1} \dots R$

Oops, A contains an

$\det(A_r) \neq 0$. A_r.

A^{-1} with $n-r$

a_j are red..

$\dots, k$

rank(A)

Think about alt. proof of 27.(4)

Alternate Proofs

27. (4) Ideas

$\det(C) = [\det(A)]^{n-1} = [\det(A)]^{-1} \det(\det(A) I) \Rightarrow \det(AC) = \det(A) I$

$\det(C) = \det(A^{n-1})$

$=$

$i < s \leq r < j$

$$\begin{cases} A_{rs} & \text{if } r \geq j,\ s < i \\ A_{r+1,s} & \text{if } r < j,\ s \geq i \\ A_{r,s+1} & \text{if } r \geq j,\ s \geq i \\ A_{r+1,s+1} & \text{if} \end{cases}$$

$r < j < i \leq s$

(c) Ideas

A^{-1} (?

$i=1$ trivial

Assume $i-1$

$$\begin{pmatrix} a_1 & b_1 & b_3 \\ 0 & 0 & b_4 \\ 0 & 0 & a_4 \end{pmatrix}$$

21

Proof 2

In Proof 1, we showed A^{-1} is upper triangular without using exercise 25. So, we shall now show a proof for C being upper triangular, without using exercise 25. We notice that

$$(\tilde{A}_{ji})_{rs} = \begin{cases} A_{rs} & \text{if } r < j \ \&\ s < i, \\ A_{r+1,s} & \text{if } r \geq j \ \&\ s < i, \\ A_{r,s+1} & \text{if } r < j \ \&\ s \geq i, \\ A_{r+1,s+1} & \text{if } r \geq j \ \&\ s \geq i. \end{cases}$$

Thus, when $n \geq i > j \geq 1$ and $r > s$, the only possibility for $(\tilde{A}_{ji})_{rs} \neq 0$ is if $r < j$ & $s \geq i$, such that $(\tilde{A}_{ji})_{rs} = A_{r,s+1}$. But then $r < j < i \leq s$, a contradiction. Therefore, $\tilde{A}_{ji}$ must be upper triangular. Furthermore, $(\tilde{A}_{ji})_{jj} = A_{j+1,j} = 0$ means $\det(\tilde{A}_{ji})_{jj} = 0$. Hence, $C_{ij} = 0$. Consequently, C is certainly upper triangular.