Linear Algebra

By Friedberg, Insel, Spence

Shao Hong

Contents

Definitions and Theorems

1 Vector Spaces

1.1 Vector Spaces

Definition. A vector space (or linear space) V over a field F consists of a set on which two operations (called addition and multiplication respectively here) are defined so that;

(A1) (V is Closed Under Addition ["uniquely"]) For all $x, y \in V$, there exists a unique element $x + y \in V$.

(M1) (V is Closed Under Scalar Multiplication ["uniquely"]) For all elements $a \in F$ and elements $x \in V$, there exists a unique element $ax \in V$.

Such that the following properties hold:

(VS 1) (Commutativity of Addition) For all $x, y \in V$, $x + y = y + x$

(VS 2) (Associativity of Addition) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$

(VS 3) (Existance of The Zero/Null Vector) There exists an element in V denoted by $\vec{\mathbf{0}}$, such that $x + \vec{\mathbf{0}} = x$ for all $x \in V$.

(VS 4) (Existance of Additive Inverses) For all elements $x \in V$, there exists an element $y \in V$ such that $x + y = \vec{\mathbf{0}}$

(VS 5) (Multiplicative Identity) For all elements $x \in V$, $\mathbb{1}x = x$, where $\mathbb{1}$ denotes the multiplicative idenity in F.

(VS 6) (Compatibility of Scalar Multiplication with Field Multiplication) For all elements $a, b \in F$ and elements $x \in V$, $(ab)x = a(bx)$

(VS 7) (Distributivity of Scalar Multiplication over Vector Addition) For all elements $a \in F$ and elements $x, y \in V$, $a(x + y) = ax + ay$.

(VS 8) (Distributivity of Scalar Multiplication over Field Addition) For all elements $a, b \in F$, and elements $x \in V$, $(a + b)x = ax + bx$

- ☐ Example 2, (VS✓): $M_{m \times n}(F)$ is the set of all $m \times n$ matrices with entries from field F, which is a **vector space**.
  - Two $m \times n$ matrices, A and B, are equal iff $A_{i,j} = B_{i,j}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.
  - Matrix addition and scalar multiplication: $(A + B)_{i,j} = A_{i,j} + B_{i,j}$ and $(cA)_{i,j} = cA_{i,j}$.
    *Instead of $a_{i,j}$, the author uses $A_{i,j}$ btw.
- ☐ Example 3, (VS✓): Let S be any set and F be any field. $\mathcal{F}(S, F)$ denotes the set of all functions from S to F
  - $\mathcal{F}(S, F) = \{f \mid f \colon S \to F\}$.
  - It is a **vector space** with the operations with $(f + g)(s) = f(s) + g(s)$ and $(cf)(s) = c[f(s)]$.
- ☐ Example 3-4, (VS✓): $P(F)$ is the set of all polynomials with coefficients from F, which is a **vector space**.
  - The zero polynomial, $f(x) = 0$, has degree defined to be -1 for convenience.
  - Two polnomials, $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$, are equal iff $m = n$ and $a_i = b_i$ for all $0 \leq i \leq n = m$

- Is a VS with the operations of addition and multiplication defined as $f(x)+g(x)=\sum_{i=0}^{n}(a_i+b_i)x^i$ and $cf(x)=\sum_{i=0}^{n}ca_ix^i$ for $c\in F$.
- `"Wait, what even is` x `here?"`. See https://en.wikipedia.org/wiki/Polynomial_ring#:~:text=Over%20a%20field%2C%20every%20nonzero,r%20such%20that%20q%20%3D%20pr

☐ Example 5: A sequence is denoted as $\{a_n\}$.

Theorem 1.1. *(Cancallation Law for Vector Addition) If x, y, z are vectors in a vector space V such that $x+z=y+z$, then $x=y$.*

Corollary 1.1.1. *The vector $\vec{\mathbf{0}}$ described in (VS 3) is unique.*

Corollary 1.1.2. *The vector y described in (VS 4) is unique.*

Theorem 1.2. *In any vector space V, the follow statements are true.*

(a) $\mathbb{0}x=\vec{\mathbf{0}}$ *for all* $x\in V$.

(b) $(-a)x=-(ax)=a(-x)$ *for all* $a\in F$ *and* $x\in V$.

(c) $a\vec{\mathbf{0}}=\vec{\mathbf{0}}$ *for all* $a\in F$.

Interesting Tidbits

- $\mathbb{Z}_n$ is a quotient ring. Need to read more algebra before I can understand that lol

1.2 Subspaces

Definition. A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

In any vector space V, V and $\{\vec{\mathbf{0}}\}$ are subspaces. $\{\vec{\mathbf{0}}\}$ is called the zero subspace of V.

(VS 1-2), (VS 5-8) hold for all vectors in the vector space, and hence the vectors in any subsets.

So, $W \subseteq V$ is a subspace of V iff:

1. (A1) (W is closed under addition) $x + y \in W$ whenever $x \in W$ and $y \in W$.
2. (M1) (W is closed under scalar multiplication) $cx \in W$ whenever $c \in F$ and $x \in W$.
3. (VS 3) (Existance of the Zero/Null Vector) W has a zero vector.
4. (VS 4) (Actually Redundent) ~~(Existance of Additive Inverses) Each vector in W has an additive inverse in W.~~

Theorem 1.3. *Let V be a vector space and W a subset of V. Then W is a subspace of V iff the following 3 conditions hold for the operations defined in V.*

(a) $\vec{\mathbf{0}} \in W$

(b) $x + y \in W$ *whenever* $x \in W$ *and* $y \in W$.

(c) $cx \in W$ *whenever* $c \in F$ *and* $x \in W$.

Definition. Given a $m \times n$ matrix, A, its transpose A^t is a $n \times m$ matrix where $(A^t)_{i,j} = A_{j,i}$. Also, $(aA + bB)^t = aA^t + bB^t$

Definition. A symmetric matrix is one (square matrix) such that $A^t = A$.

E.g. 1. (SS✓): $P_n(F)$ is the set of polynomials in $P(F)$ having degree less than or equal to $n \in \mathbb{N}_0$. $P_n(F)$ is a subspace of $P(F)$.

E.g. 2. (SS✓): $C(\mathbb{R})$ is the set of all continuous real-valued functions defined on $\mathbb{R}$. $C(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

E.g. 3. (SS✓): An $n \times n$ (square) matrix is a diagonal matrix iff $M_{i,j} = 0$ whenever $i \neq j$. The set of $n \times n$ diagonal matrices is a subspace of $M_{n \times n}(F)$.

E.g. 4. (SS✓): $\text{tr}(M)$ is the trace of a $n \times n$ matrix M, the sum of the diagonal entries of M; $\text{tr}(M) = \sum_{k=1}^{n} M_{k,k}$. The set of $n \times n$ matrices of trace 0 is a subspace of $M_{n \times n}(F)$.
Also, $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$.

Theorem 1.4. *Any intersection of subspaces of a vector space V is a subspace of V*

Definition. (SS✓): A $m \times n$ matrix is called **upper triangular** iff all entries lying below the diagonal entries are zero, i.e. $A_{i,j} = 0$ for all $i > j$. The set of all upper triangular matrices is a subspace of $M_{m \times n}(F)$.

Ex 16. (SS✓): $C^n\mathbb{R}$ is the set of all real-valued functions that have a continuous nth derivative. $C^n\mathbb{R}$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Definition. If S_1 and S_2 are nonempty subsets of a vector space V, then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y \,|\, x \in S_1 \text{ and } y \in S_2\}$.

Definition. A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Ex 17. (SS✓): A (square) matrix M is called **skew-symmetric** iff $M^t = -M$. The set of all skew-symmetric $n \times n$ matrices (with entries from F) is a subspace of $M_{n\times n}(F)$.

Ex 31. Let W be a subspace of a vector space V over a field F. For any $v \in V$, the set $v + W = \{v\} + W = \{v + w \,|\, w \in W\}$ is called the **coset** of W **containing** v.
(SS✓): $v + W$ is a subspace of V if and only if $v \in W$.

*It is customary to write this coset as $v + W$ instead of $\{v\} + W$.

(VS✓): The set V/W is the **quotient space of** V **modulo** W. The set (vector space) $V/W = \{v+W \,|\, v \in V\}$ of all cosets W is a vector space with the operations of addition and scalar multiplication defined as: $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ for all $v_1, v_2 \in V$ and $a(v + W) = av + W$ for all $v \in V$ and $a \in F$.

Note. † — We use this symbol to identify an exercise that is cited and essential in some later section that is not optional.

1.3 Linear Combinations and Systems of Linear Equations

Definition. Let V be a vector space and S a nonempty subset of V. A vector $v \in V$ is called a **linear combination** of vectors of S if there exists a finite number of vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$ in F such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. In this case we also say that v is a linear combination of $u_1, u_2, \ldots, u_n$ and call $a_1, a_2 \ldots, a_n$ the **coefficients** of the linear combination.

In any vector space V, $\vec{\mathbf{0}}$ is a linear combination of any nonempty subset of V; as there indeed exists such a finite number of vectors and scalars: $\vec{\mathbf{0}} = 0v$ for all $v \in V$.

Procedure for Solving System of Linear Equations. *(Gaussian Elimination but not in matrix form yet.)*
We use three types of operations to simplify the original system:

1. *interchanging the order of any two equations in the system;*
2. *multiplying any equation in the system by a nonzero constant;*
3. *adding a constant multiple of any equation to another equation in the system.*

Note that we employed these operations to obtain a system of equations that had the following properties:

1. *The first nonzero coefficient in each equation is one.*
2. *If an unknown is the first unknown with a nonzero coefficient in some equation, then that unknown occurs with a zero coefficient in each of the other equations.*
3. *The first unknown with a nonzero coefficient in any equation has a larger subscript than the first unknown with a nonzero coefficient in any preceding equation.*

In Section 3.4, we prove that these operations do not change the set of solutions to the original system.

Definition. Let S be a nonempty subset of a vector space V. The **span** of S, denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S. For convinence, we define $\text{span}(\varnothing) = \{\vec{\mathbf{0}}\}$.

Me: If we think a lil' about this, it should actually make sense as the additive identity is $\{\vec{\mathbf{0}}\}$.

Theorem 1.5. *The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V that contains S must also contatin the span of S.*
(i.e. For all subspaces W of V; if $S \subseteq W$, then $\text{span}(S) \subseteq W$)

Definition. A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$. In this case, we also say that the vectors of S generate (or span) V.

1.4 Linear Dependence and Linear Independence

Definition. A subset S of a vector space V is called **linearly dependent** if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = \vec{\mathbf{0}}.$$

In this case we also say that the vectors of S are linearly dependent.

For any vectors $u_1, u_2, \ldots, u_n$, we have $a_1u_1 + a_2u_2 + \cdots + a_nu_n = \vec{\mathbf{0}}$ if $a_1 = a_2 = \cdots = a_n = \vec{\mathbf{0}}$. We call this the trivial representation of $\vec{\mathbf{0}}$ as a linear combination of $u_1, u_2, \cdots, u_n$. Thus, for a set to be linearly dependent, there must exist a nontrivial representation of $\vec{\mathbf{0}}$ as a linear combination of vectors in the set. Consequently, any subset of a vector space that contains the zero vector is linearly dependent, because $\vec{\mathbf{0}} = \mathbb{1} \cdot \vec{\mathbf{0}}$ is a nontrivial representation of 0 as a linear combination of vectors in the set.

Definition. A subset S of a vector space that is not linearly dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.

Fact. The following facts about linearly independent sets are true in any vector space.

1. The empty set is linearly independent, for linearly dependent sets must be nonempty.
2. A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = \vec{\mathbf{0}}$ for some nonzero scalar a.
   Thus
   $$u = a^{-1}(au) = a^{-1}\vec{\mathbf{0}} = \vec{\mathbf{0}}.$$
3. A set is linearly independent if and only if the only representations of $\vec{\mathbf{0}}$ as linear combinations of its vectors are trivial representations.

E.g. 4. (LI✓): $p_k(x) = x^k + x^{k+1} + \cdots + x^n$ for all $k = 0, 1, \ldots, n$.
The set $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is linearly independent in $P_n(F)$.

Theorem 1.6. *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.*

Corollary. *Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.*

Theorem 1.7. *Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$.*

1.5 Bases and Dimension

Definition. A **basis** β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

E.g. 1. $\varnothing$ is a basis for the zero vector space.

E.g. 2. In F^n, let $e_1 = (\mathbf{1}, \mathbb{0}, \mathbb{0}, \ldots, \mathbb{0})$, $e_2 = (\mathbb{0}, \mathbb{1}, \mathbb{0}, \ldots, \mathbb{0})$, ..., $e_n = (\mathbb{0}, \mathbb{0}, \ldots, \mathbb{0}, \mathbb{1})$; $\{e_1, e_2, \ldots, e_n\}$ is the **standard basis** for F^n.

E.g. 4. In $P_n(F)$, the set $\{1, x, x^2, \ldots, x^n\}$ is a basis. We call this basis the standard basis for $P_n(F)$.

Note. A basis need not be finite. E.g.: In $P(F)$, the set $\{1, x, x^2, \ldots\}$ is a basis. In fact, no basis for $P(F)$ can be finite. Hence, not every vector space has a finite basis.

Theorem 1.8. *Let V be a vector space and $u_1, u_2, \ldots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \cdot + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Self-Proof.

2 Appendix

2.1 Appendix C Fields

Definition. A field F is a set on which two operations + and $\cdot$ (called addition and multiplication, respectively) are defined so that, for each pair of elements x, y in F, there are unique elements $x + y$ and $x \cdot y \in F$ for which the following conditions hold for all elements a, b, c in F.

(F 1) $a + b = b + a$ and $a \cdot b = b \cdot a$
(commutativity of addition and multiplication)

(F 2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(associativity of addition and multiplication)

(F 3) There exist distinct elements $\mathbb{0}$ and $\mathbb{1}$ in F such that

$$\mathbb{0} + a = a \quad \text{and} \quad \mathbb{1} \cdot a = a$$

(existence of identity elements for addition and multiplication)

(F 4) For each element a in F and each nonzero element b in F, there exist elements c and d in F such that

$$a + c = \mathbb{0} \quad \text{and} \quad b \cdot d = \mathbb{1}$$

(existence of inverses for addition and multiplication)

(F 5) $a \cdot (b + c) = a \cdot b + a \cdot c$
(distributivity of multiplication over addition)

The elements $x + y$ and $x \cdot y$ are called the **sum** and **product**, respectively, of x and y. The elements $\mathbb{0}$ (read "zero") and $\mathbb{1}$ (read "one") mentioned in (F 3) are called **identity elements** for addition and multiplication, respectively, and the elements c and d referred to in (F 4) are called an **additive inverse** for a and a **multiplicative inverse** for b, respectively.

Theorem C.1. ***(Cancellation Laws).*** *For arbitrary elements a, b, and c in a field, the following statements are true.*

(a) If $a + b = c + b$, then $a = c$.

(b) If $a \cdot b = c \cdot b$ and $b \neq \mathbb{0}$, then $a = c$

Corollary. *The elements $\mathbb{0}$ and $\mathbb{1}$ mentioned in (F 3), and the elements c and d mentioned in (F 4), are unique.*

Theorem C.2. *Let a and b be arbitrary elements of a field. Then, each of the following statements are true.*

(a) $a \cdot \mathbb{0} = \mathbb{0}$.

(b) $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.

(c) $(-a) \cdot (-b) = a \cdot b$.

Corollary. *The additive identity of a field has no multiplicative inverse.*

Definition. The characteristic of a field. The smallest positive integer p for which a sum of p $\mathbb{1}$'s equals $\mathbb{0}$, i.e. $\sum_{i=0}^{p} \mathbb{1} = \mathbb{0}$, is called the **characteristic** of F; if no such positive integer exists, then F is said to have **characteristic zero**. If F is a field of characteristic $p \neq 0$, then $\sum_{i=0}^{p} x = \mathbb{0}$ for all $x \in F$.

Exercises

1 Vector Spaces

1.1 Introduction

1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.
(a) $(3, 1, 2)$ and $(6, 4, 2)$
(b) $(-3, 1, 7)$ and $(9, -3, -27)$
(c) $(5, -6, 7)$ and $(-5, 6, -7)$
(d) $(2, 0, -5)$ and $(5, 0, -2)$

(a)

Since the coordinates of endpoints of both vectors are not all scaled by the same constant factor, i.e.:

$$\frac{3}{6} = \frac{1}{2}\ ,\ \frac{1}{4}\ ,\ \frac{2}{2} = 1$$

So, there does not exist a constant t such that $(3, 1, 2) = t(6, 4, 2)$. Therefore, the vectors emanating from the origin and terminating at these 2 points are not parallel.

(b)

Since the coordinates of endpoints of both vectors are not all scaled by the same constant factor, i.e.:

$$\frac{-3}{6} = -\frac{1}{2}\ ,\ \frac{1}{-3} = -\frac{1}{3}\ ,\ \frac{7}{-27} = -\frac{1}{3}$$

So, there does not exist a constant t such that $(-3, 1, 7) = t(9, -3, -27)$. Therefore, the vectors emanating from the origin and terminating at these 2 points are not parallel.

(c)

Since the coordinates of endpoints of both vectors are all scaled by the same constant factor, i.e.:

$$\frac{5}{-5} = -1\ ,\ \frac{-6}{6} = -1\ ,\ \frac{7}{-7} = -1$$

So, there does exist a constant $t = -1$ such that $(5, -6, 7) = -(-5, 6, -7)$. Therefore, the vectors emanating from the origin and terminating at these 2 points are indeed parallel!

(d)

Since the coordinates of endpoints of both vectors are all scaled by the same constant factor, i.e.:

$$\frac{2}{5}\ ,\ 0 = 0\ ,\ \frac{-5}{-2} = \frac{5}{2}$$

So, there does not exist a constant t such that $(2, 0, -5) = t(5, 0, -2)$. Therefore, the vectors emanating from the origin and terminating at these 2 points are not parallel.

2. Find the equations of the lines through the following pairs of points in space
(a) $(3, -2, 4)$ and $(-5, 7, 1)$
(b) $(2, 4, 0)$ and $(-3, -6, 0)$
(c) $(3, 7, 2)$ and $(3, 7, -8)$
(d) $(-2, -1, 5)$ and $(3, 9, 7)$

Let $x, t \in \mathbb{R}$ and x be an arbitrary point on the line

(a)

$(-5, 7, 1) - (3, -2, 4) = (-8, 9, -3)$
Therefore, the line passing through $(3, -2, 4)$ and $(-5, 7, 1)$ has an equation of
$x = (3, -2, 4) + t(-8, 9, -3)$

(b)

$(-3, -6, 0) - (2, 4, 0) = (-5, -10, 0)$
Therefore, the line passing through $(2, 4, 0)$ and $(-3, -6, 0)$ has an equation of
$x = (2, 4, 0) + t(-5, -10, 0)$

(c)

$(3, 7, -8) - (3, 7, 2) = (0, 0, -10)$
Therefore, the line passing through $(3, 7, 2)$ and $(3, 7, -8)$ has an equation of
$x = (3, 7, -8) + t(0, 0, -10)$

(d)

$(3, 9, 7) - (-2, -1, 5) = (5, 10, 2)$
Therefore, the line passing through $(-2, -1, 5)$ and $(3, 9, 7)$ has an equation of
$x = (-2, -1, 5) + t(5, 10, 2)$

3. Find the equations of the planes containing the following points in space.
(a) $(2,-5,-1)$, $(0,4,6)$, and $(-3,7,1)$
(b) $(3,-6,7)$, $(-2,0,-4)$ and $(5,-9,-2)$
(c) $(-8,2,0)$, $(1,3,0)$, and $(6,-5,0)$
(d) $(1,1,1)$, $(5,5,5)$, and $(-6,4,2)$

Let $x, s, t \in \mathbb{R}$ and x be an arbitrary point on the line

(a)

$(0,4,6)-(2,-5,-1)=(-2,-9,7)$ and $(-3,7,1)-(2,-5,-1)=(-5,12,2)$
Thus $x=(2,-5,-1)+s(-2,-9,7)+t(-5,12,2)$

(b)

$(-2,0,-4)-(3,-6,7)=(-5,6,-11)$ and $(5,-9,-2)-(3.-6.7)=(2,-3,-9)$
Hence $x=(3,-6,7)+s(5,-9,-11)+t(2,-3,-9)$

(c)

$(1,3,0)-(-8,2,0)=(9,1,0)$ and $(6,-5,0)-(-8,2,0)=(14,-7,0)$
Therefore $x=(-8,2,0)+s(9,1,0)+t(14,-7,0)$

(d)

$(5,5,5)-(1,1,1)=(4,4,4)$ and $(-6,4,2)-(1,1,1)=(-7,3,1)$
So, $x=(1,1,1)+s(4,4,4)+t(-7,3,1)$

4. What are the coordinates of the vector 0 in the Euclidean plane that satisfies property 3 on page 3? Justify your answer.
(Just gonna answer in matrix form vectors instead of coordinates/tuples cos its funner)

Let $a, b \in \mathbb{R}$ and $\vec{A}, \vec{B} \in \mathbb{R}^n$ Then, the $\vec{0}$ is:

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The reason is that this definition of $\vec{0}$ satisfies Properties 3 and 4, that:
3. There exists a vector denoted $\vec{0}$ such that $\vec{x} + \vec{0} = \vec{x}$ for each vector $\vec{x}$.
4. For each vector $\vec{x}$, there is a vector $\vec{y}$ such that $\vec{x} + \vec{y} = \vec{0}$.
i.e.:

Firstly, it satisfies property 3:

$$\vec{x} + \vec{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}$$

Secondly, it satisfies property 4:

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

5. Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

By the property of a vector space V over a field F that:
(VS 8) For $a, b \in F$ and $x = (a_1, a_2) \in V, (a + b)x = ax + bx$

$$\begin{aligned}(t+0)(a_1, a_2) &= t(a_1 + a_2) + 0(a_1, a_2)\\ t(a_1, a_2) &= t(a_1, a_2) + 0(a_1, a_2)\end{aligned}$$

Now suppose $t(a_1, a_2) \neq (ta_1, ta_2)$,

$$\begin{aligned}0(a_1, a_2) &\neq (0a_1, 0a_2) = (0, 0)\\ \Rightarrow t(a_1 + a_2) + 0(a_1 + a_2) &\neq t(a_1, a_2) + (0, 0)\\ \Rightarrow (t + 0)(a_1 + a_2) &\neq t(a_1 + a_2)\\ \Rightarrow t(a_1 + a_2) &\neq t(a_1 + a_2)\end{aligned}$$

However, this is not possible since $t(a_1 + a_2) = t(a_1 + a_2)$ trivially. So, by contradiction, $t(a_1 + a_2) = (ta_1 + ta_2)$.

Q.E.D. ■

6. Show that the midpoint of the line segment joining the points (a, b) and (c, d) is $((a + c)/2, (b + d)/2)$

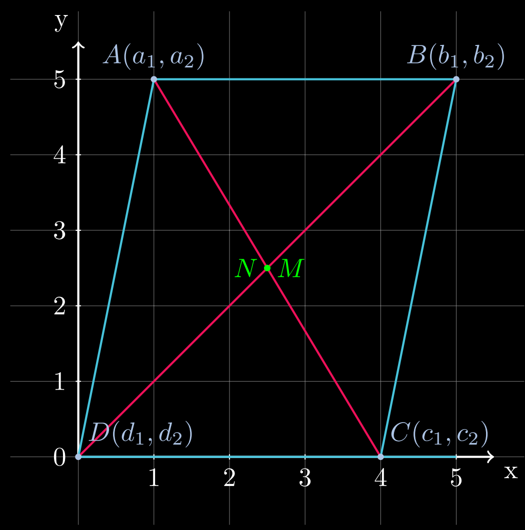
First, observe that $(c - a, d - b)$ is the full vector to be added to (a, b) to arrive at (c, d). Then, the midpoint of (a, b) and (c, d) just adding half of it, $\frac{1}{2}(c - a, d - b)$ to (a, b).
So,

$$\frac{1}{2}(c - a, d - b) + (a, b) = \left(\frac{c - a}{2} + a, \frac{d - b}{2} + b\right) = \left(\frac{a + c}{2}, \frac{b + d}{2}\right)$$

Q.E.D. ■

7. Prove that the diagonals of a parallelogram bisect each other.

Let our arbitrary parallelogram be $ABCD$, the coordinates of A, B, C, D be $(a_1, a_2), (b_1, b_2), (c_1, c_2)$, and (d_1, d_2) respectively.



Then, $\vec{AC} = (c_1 - a_1, c_2 - a_2)$ and $\vec{DB} = (b_1 - d_1, b_2 - d_2)$

Let the midpoint of the diagonal AC be M and that of the diagonal DB be N. M is $\left(\frac{a_1+c_1}{2}, \frac{a_2+c_2}{2}\right)$ and N is $\left(\frac{b_1+d_1}{2}, \frac{b_2+d_2}{2}\right)$.

Now note that by the definition of a parallelogram, $\vec{AB} = \vec{DC}$ and $\vec{DA} = \vec{CB}$, meaning:

$$\begin{aligned} \vec{AB} &= \vec{DC} \\ (b_1 - a_1, b_2 - a_2) &= (c_1 - d_1, c_2 - d_2) \\ b_1 - a_1 &= c_1 - d_1 \\ b_1 + d_1 &= a_1 + c_1 \end{aligned}$$

$$\begin{aligned} \vec{DA} &= \vec{CB} \\ (a_1 - d_1, a_2 - d_2) &= (b_1 - c_1, b_2 - c_2) \\ b_2 - a_2 &= c_2 - d_2 \\ b_2 + d_2 &= a_2 + c_2 \end{aligned}$$

So,

$$\left(\frac{a_1 + c_1}{2}, \frac{a_2 + c_2}{2}\right) = \left(\frac{b_1 + d_1}{2}, \frac{b_2 + d_2}{2}\right)$$
$$M = N$$

This means that the midpoint of the diagonals must be the point at which they intersect with each other. By definition, the midpoint of a line cuts the line into 2 equal segments, meaning that the diagonals cut each other into 2 equal segments too. Therefore, the diagonals of a parallelogram bisect each other.

1.2 Vector Spaces

1. Label the following statements as true or false.
(a) Every vector space contains a zero vector.
(b) A vector space may have more than one zero vector.
(c) In any vector space, $ax = bx$ implies that $a = b$.
(d) In any vector space, $ax = ay$ implies that $x = y$.
(e) A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$.
(f) An $m \times n$ matrix has m columns and n rows.
(g) In $P(F)$, only polynomials of the same degree may be added.
(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.
(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.
(j) A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero.
(k) Two functions in $F(S, F)$ are equal if and only if they have the same value at each element of S.

(a) True. ✓
(b) False. ✓
(c) True [in general (ignoring the case of $x = \vec{\mathbf{0}}$ of course) ✓].
(d) True [in general (ignoring the case of $a = 0$ of course) ✓].
(e) True. ✓
(f) False. ✓
(g) False. ✓
(h) True. × Careless mistake, *pain*. $f + (-f)$ is obviously not degree n.
(i) True. ✓
(j) True. ✓
(h) True. ✓

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Proof of Corollary 1 of Theorem 1.1:

Assume that there the zero vector described in (VS 3) is not unique, and let $\vec{\mathbf{0}}, \vec{\mathbf{0}}'$ be two distinct zero vectors of the vector space V.

By (VS 3), for all $x \in V$; $x + \vec{\mathbf{0}} = x$ and $x + \vec{\mathbf{0}}'$.
Thus, $x + \vec{\mathbf{0}} = x + \vec{\mathbf{0}}'$. By Theorem 1.1, $\vec{\mathbf{0}} = \vec{\mathbf{0}}'$.

This clearly condtradicts our assumption that $\vec{\mathbf{0}}$ and $\vec{\mathbf{0}}'$ are distinct. Therefore, it must be that the zero vector described in (VS 3) is unique. So, Corollary 1 is true. ■

Proof of Corollary 2 of Theorem 1.1:

Assume that the vector y described in (VS 4) is not unique, i.e. there exists at least 2 distinct vectors, $y, y' \in V$ that satisfy (VS 4).

Meaning, $x + y = \vec{\mathbf{0}}$ and $x + y' = \vec{\mathbf{0}}$. Hence, $x + y = x + y'$. By Theorem 1.1, $y = y'$.

Again, this contradicts our assumption that y, y' are distinct vectors of V. Wherefore, for each and every $x \in V$, the associated vector y described in (VS 4) is unique. ■

Proof of Theorem 1.2(c):

$$\begin{array}{rrcl} & a(x+\vec{\mathbf{0}}) \overset{\text{(VS 3)}}{=} ax & \text{and} & a(x+\vec{\mathbf{0}}) \overset{\text{(VS 7)}}{=} ax + a\vec{\mathbf{0}} \\ \implies & ax & = & ax + a\vec{\mathbf{0}} \\ & ax + (-ax) = a\vec{\mathbf{0}} & \text{and} & ax+(-ax) \overset{\text{(VS 8)}}{=} (a-a)x = 0x \overset{\text{T1.2(a)}}{=} \vec{\mathbf{0}} \\ \implies & a\vec{\mathbf{0}} & = & \vec{\mathbf{0}} \end{array}$$

Q.E.D. ■

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3. i.e.:

$$(f+g)(s) = f(s) + g(s) \text{ and } (cf)(s) = c[f(s)]$$

Let $s \in \mathbb{R}$ and $f, g, h \in V$. Then, f, g are differentiable by definition.

Indeed, V is closed under addition (A1), because $\frac{d}{dx}[(f+g)(s)] = \frac{d}{dx}[f(s)+g(s)] = \frac{d}{dx}f(s) + \frac{d}{dx}g(s)$ is differentiable, and hence an element of V. (for all $f, g \in V$)

Similarly, V is also closed under scalar multiplication (M1). $\frac{d}{dx}[(cf)(s)] = \frac{d}{dx}\big(c[f(s)]\big) = c \cdot \frac{d}{dx}f(s)$ which is again differentiable and in V by definition. (for all $f \in V$ and $c \in \mathbb{R}$)

(VS 1) Commutativity of Addition holds true; $(f+g)(s) = f(s) + g(s) = g(s) + f(s) = (g+f)(s)$. (for all $f, g \in V$)

(VS 2) Associativity of Addition also holds; $[(f+g)+h](s) = (f+g)(s) + h(s) = f(s) + g(s) + h(s) = f(s) + (g+h)(s) = [f + (g+h)](s)$. (for all $f, g, h \in V$ and $c \in \mathbb{R}$)

(VS 3) There exists a zero vector in V, specifically the (differentiable) function $\mathscr{O}\colon \mathbb{R} \to \mathbb{R}$ such that $\mathscr{O}\colon s \mapsto 0$, since;
$(f + \mathscr{O})(s) = f(s) + \mathscr{O}(s) = f(s) + 0 = f(s)$. (for all $f \in V$)

(VS 4) Additive inverses exist:
$[f + (-f)](s) = f(s) + (-f)(s) = f(s) - f(s) = 0 = \mathscr{O}(s)$. (for all $f \in V$)

(VS 5) A multiplicative idenity exists, $1 \in \mathbb{R}$: $(1f)(s) = 1[f(s)] = f(s)$. (for all $f \in V$)

(VS 6) is true in V: $(c\gamma f)(s) = c\gamma[f(s)] = c\big(\gamma[f(s)]\big) = c(\gamma f)(s)$. (for all $f, g \in V$ and $c, \gamma \in \mathbb{R}$)

(VS 7) holds in V: $[c(f+g)](s) = c[(f+g)(s)] = c[f(s)+g(s)] = c[f(s)] + c[g(s)]$. (for all $f, g \in V$ and $c \in \mathbb{R}$)

(VS 8) is also true; $[(c+\gamma)f](s) = (c+\gamma)[f(s)] = c[f(s)] + \gamma[f(s)]$. (for all $f, g \in V$ and $c, \gamma \in \mathbb{R}$)

Thence, since all properties of a vector space, i.e. (A1), (M1) and (VS 1) - (VS 8), are true in V, so V is a vector space.

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, a_2)$.

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

(VS 8) does not hold, because $(\alpha+\beta)(a_1,a_2) = ((\alpha+\beta)\cdot a_1, a_2) = (\alpha\cdot a_1 + \beta\cdot a_1, a_2)$. While $\alpha(a_1,a_2)+\beta(a_1,a_2) = (\alpha\cdot a_1, a_2) + (\beta\cdot a_1, a_2) = (\alpha\cdot a_1+\beta\cdot a_1, a_2^2)$. Therefore,

$$(\alpha+\beta)(a_1,a_2) = (\alpha\cdot a_1+\beta\cdot a_1, a_2) \neq (\alpha\cdot a_1+\beta\cdot a_1, a_2^2) = \alpha(a_1,a_2)+\beta(a_1,a_2)$$
$$(\alpha+\beta)(a_1,a_2) \neq \alpha(a_1,a_2)+\beta(a_1,a_2)$$

So, since not all the properties of vector spaces hold true in V, V is not a vector space.

Archive

V is closed under addition (A1), since $a_1+b_1\in\mathbb{R}$ and $a_2,b_2\in\mathbb{R}$. Thus, $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, a_2b_2) \in V = \mathbb{R}^2$.

V is also closed under scalar multiplication (M1), because $ca_1\in\mathbb{R}$ and $a_2\in\mathbb{R}$. Consequently, $c(a_1,a_2) = (ca_1,a_2)\in V=\mathbb{R}^2$.

(Of course, it goes without saying that the arguments are meant to hold for all $a_1,a_2,b_1,b_2,c\in\mathbb{R}$.)

(VS 2) Commutativity Of Addition holds, since $(a_1,a_2)+(b_1,b_2) = (a_1+b_1,a_2b_2) = (b_1,b_2)+(a_1,a_2)$.

(VS 3) There exists (0,1) as the zero vector, because $(a_1,a_2)+(0,1) = (a_1+0, a_2\cdot 1) = (a_1,a_2)$.

(VS 4) Existance of Additive Inverses holds, because $(a_1,a_2)+\left(-a_1+\frac{1}{a_2}\right) = \left(a_1-a_1, a_2\cdot\frac{1}{a_2}\right) = (0,1)$.

(VS 5) There exists a multiplicative identity that holds for all elements of V, namely 1; $1(a_1,a_2) = (1\cdot a_1,a_2) = (a_1,a_2)$.

(VS 6) is true, because $(\alpha\beta)(a_1,a_2) = (\alpha\beta\cdot a_1,a_2) = \alpha(\beta\cdot a_1,a_2) = \alpha(\beta(a_1,a_2))$.

(VS 7) holds, as $\alpha((a_1,a_2)+(b_1,b_2)) = \alpha(a_1+b_1,a_2b_2) = (\alpha\cdot(a_1+b_1),a_2,b_2) = (\alpha\cdot a_1+\alpha\cdot b_1,a_2,b_2) = (\alpha\cdot a_1,a_2)+(\alpha\cdot b_1,b_2) = \alpha(a_1,a_2)+\alpha(b_1,b_2)$.

19. Let $V = \{(a_1,a_2) | a_1,a_2 \in R\}$. Define addition of elements of V cooradinatewise, and for (a_1,a_2) in V and $c\in R$, define

$$c(a_2,a_2) = \begin{cases} (0,0) & \text{if } c=0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c\neq 0 \end{cases}$$

Is V a vector space over R with these operations? Justify your answer.

(VS 8) does not hold true in V; Let $\alpha,\beta\in\mathbb{R}$.
$(\alpha+\beta)(a_1,a_2) = \left((\alpha+\beta)a_1, \frac{a_2}{(\alpha+\beta)}\right) = \left((\alpha a_1+\beta a_1, \frac{a_2}{\alpha+\beta}\right)$.
While $\alpha(a_1,a_2)+\beta(a_1,a_2) = \left(\alpha a_1,\frac{a_2}{\alpha}\right) + \left(\beta a_1, \frac{a_2}{\beta}\right) = \left(\alpha a_1+\beta a_1, \frac{a_2}{\alpha}+\frac{a_2}{\beta}\right)$.
Thus,

$$(\alpha+\beta)(a_1,a_2) = \left((\alpha a_1+\beta a_1, \frac{a_2}{\alpha+\beta}\right) \neq \left(\alpha a_1+\beta a_1, \frac{a_2}{\alpha}+\frac{a_2}{\beta}\right) = \alpha(a_1,a_2)+\beta(a_1,a_2)$$
$$(\alpha+\beta)(a_1,a_2) \neq \alpha(a_1,a_2)+\beta(a_1,a_2)$$

Since not all properties of vector spaces hold in V, V is not a vector space.

22. How many matrices are there in the vector space $M_{m\times n}(\mathbb{Z}_2)$? (See Appendix C.) ✓

Since $\mathbb{Z}_2 = \{\mathbb{0}, \mathbb{1}\}$, there are only two choices to be made for each entry, $a_{i,j}$, of $A \in M_{m \times n}(\mathbb{Z}_2)$. There are a total of mn entries in A. So, $|M_{m \times n}(\mathbb{Z}_2)| = 2^{mn}$.

1.3 Subspaces

1. Label the following statements as true or false.

   (a) If V is a vector space and W a subset of V that is a vector space, then W is a subspace of V.

   (b) The empty set is a subspace of every vector space.

   (c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

   (d) The intersection of any two subsets of V is a subspace of V.

   (e) An $n \times n$ diagonal matrix can never have more that n nonzero entries.

   (f) The trace of a square matrix is the product of its diagonal entries.

   (g) Let W be the xy-plane in $\mathbb{R}^3$; that is, $W = \{(a_1, a_2, 0) \mid a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$.

   (a) True × (If the assumptions I took to be the case; that W and V share the same operations and field, are taken as a priori, then I am correct. But if they are not, then I'd be wrong.)

   Thus, if we're looking at the info presented explicitly by the qns, then I am wrong.

   (b) False ✓

   (c) True ✓

   (d) False ✓

   (e) True ✓

   (f) False ✓

   (g) False ✓Interesting note; Apparently there is an isomorphism between the two.
   (well at least according to the nonofficial solution set I found online)*

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(F)$ and any $a, b \in F$.

   For all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant n$,
   $((aA + bB)^t)_{i,j} = (aA + bB)_{j,i} = (aA)_{j,i} + (bB)_{j,i} = aA_{j,i} + bB_{j,i} = a(A^t)_{i,j} + b(B^t)_{i,j}$.
   Therefore, by definition, $(aA + bB)^t = aA^t + bB^t$.

4. Prove that $(A^t)^t = A$ for each $A \in M_{m\times n}(F)$.

   For all $1 \leqslant i \leqslant m$ and all $1 \leqslant j \leqslant n$,
   $((A^t)^t)_{i,j} = (A^t)_{j,i} = A_{i,j}$.
   Once again, by (the) definition (of equality of matrices) , $(aA + bB)^t = aA^t + bB^t$.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

   Let A be an $n \times n$ (square) matrix: By exercise 2 & 3, $(A + A^t)^t = A^t + A = A + A^t$. Thus, by definition (of symmetric matrices), $A + A^t$ is symmetric.

6. Prove that $\operatorname{tr}(aA+bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B)$ for any $A, B \in M_{n\times n}(F)$.

$$\begin{aligned}
\operatorname{tr}(aA+bB) &= \sum_{i=0}^{n} (aA+bB)_{i,i} \\
&= \sum_{i=0}^{n} aA_{i,i} + bB_{i,i} \\
&= \sum_{i=0}^{n} aA_{i,i} + \sum_{i=0}^{n} bB_{i,i} \\
&= \left(a \cdot \sum_{i=0}^{n} A_{i,i}\right) + \left(b \cdot \sum_{i=0}^{n} B_{i,i}\right) \\
&= a\operatorname{tr}(A) + b\operatorname{tr}(B)
\end{aligned}$$

7. Prove that diagonal matrices are symmetric matrices.

Let A be a $n \times n$ diagonal matrix, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ be natural numbers. For all i, j;

$$(A^t)_{i,j} = A_{j,i} = \begin{cases} A_{j,i} & i = j \\ 0 & i \neq j \end{cases} = \begin{cases} A_{i,j} & i = j \\ 0 & i \neq j \end{cases} = A_{i,j}$$

Wherefore, A, a diagonal matrix, is also a symmetric matrix.

11. Is the set $W = \{f(x) \in P(F) \mid f(x) = 0 \text{ or } f(x) \text{ has degree n}\}$ a subspace of $P(F)$ if $n \geqslant 1$? Justify your answer.

No. It is not closed under addition nor multiplication. See counterexamples;

(A1) Given $f_1(x) = x^n + x^{n-1} \in W$ and $f_2(x) = -x^n \in W$, $f_1(x) + f_2(x) = x^n + x^{n-1} - x^n = x^{n-1} \notin W$. For $g(x) + h(x) \in W$, you actually need the extra condition that the sum of coefficients for x^n is nonzero, which does not necessarily hold true for all elements of the field F.

(M1) Given some $f_3(x) = x^n + x^{n-1} \in W$ and $\mathbb{I}(x) = x \in W$, $f_3(x) \cdot \mathbb{I}(x) = (x^n + x^{n-1}) \cdot x = x^{n+1} + x^n \notin W$. For the product of 2 elements in W to also be an element of W, the sum of their degrees must be at most n.

So, by Theorem 1.3, W is not a vector space.

12. An $m \times n$ matrix A is called upper triangular if all entries lying below the diagonal entries are zero, that is, if $A_{i,j} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m\times n}(F)$.

Let the set of $m \times n$ upper triangular matrices over F be $\triangle \subseteq M_{m\times n}(F)$.
The zero matrix, $\mathbf{O} \in \triangle$ since $\mathbf{O}_{i,j} = 0$ for all i, j; including when $i > j$.

Define $B, C \in \triangle$ and $c \in F$;

Whenever $i > j$, $B_{i,j} = 0$ and $C_{i,j} = 0$. Hence, $(B + C)_{i,j} = B_{i,j} + C_{i,j} = 0$ for all $i > j$, meaning $B + C \in \triangle$ and $\triangle$ is closed under addition (A1).

Similarly, $\triangle$ is indeed closed under multiplication too (M1): Whenever $i > j$ and for all $c \in F$, $(cB)_{i,j} = c(B_{i,j}) = c(0) = 0$. Wherefore, $cB \in \triangle$.

So, by Theorem 1.3, $\triangle$ is a subspace of $M_{m\times n}(F)$.

20. ✓× (TLDR: Be more careful + less careless)
The key idea to solve this qns is rather simple but it would have been better to add in some stuff.

Prove that if W is a subspace of a vector space V and $w_1, w_2, \cdots, w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalars $a_1, a_2, \cdots, a_n$.

By Theorem 1.3, W is closed under (A1) addition and (M1) multiplication, thence; if $w_1, w_2, \cdots, w_n \in W$, then (M1) $a_1w_1, a_2w_2, \cdots, a_nw_n \in W$. So, (A1) $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$.

Note: Closure Under Addition says that $u + v \in W$ for all $u, v \in W$. However, it does not directly state that for n summands. Therefore, to make it more rigorous, we should do it inductively. i.e.: For $n = 2$, $a_1w_1 + a_2w_2 \in W$ (A1). Suppose that $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for some $n \geq 2$, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n + a_{n+1}w_{n+1} \in W$ also. By induction, $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for all n (since we already showed the $n = 1$ case by talking about W being closed under multiplication) .

23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

(a) The zero vector is in $W_1 + W_2$:

By Theorem 1.3, $\vec{\mathbf{0}}$ is in both W_1 and W_2.
Thus, $\vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{0}} \in \{x + y \mid x \in W_1 \text{ and } y \in W_2\} = W_1 + W_2$.

Let $a, b \in W_1$ and $c, d \in W_2$. As well as $k \in F$ where F is the underlying field of V.

$W_1 + W_2$ is closed under addition (A1): $(a + c) + (b, d) = (a + b) + (c + d) \in W_1 + W_2$ where $(a + c), (c + d) \in W_1 + W_2$ individually.

$W_1 + W_2$ is closed under multiplication (M1): $k(a + c) = ka + kc \in W_1 + W_2$ because $ka \in W_1$ and $kc \in W_2$, as W_1 and W_2 are also closed under multiplication. (M1).

Wherefore, by Theorem 1.3, $W_1 + W_2$ is a subspace of V.

$W_1, W_2 \subseteq W_1 + W_2$: For all $a \in W_1$, $a + \vec{\mathbf{0}} \in W_1 + W_2$, thus $W_1 \subseteq W_1 + W_2$. For all $c \in W_2$, $\vec{\mathbf{0}} + c \in W_1 + W_2$, therefore $W_2 \subseteq W_1 + W_2$.

(b) Let S be a subspace of V containing both W_1 and W_2. i.e. $W_1, W_2 \subseteq S$.

For all $a \in W_1$ and $c \in W_2$; $a, c \in S$, and so, $a+c \in S$, because S is closed under addition (A1). Hence $W_1 + W_2$, which is the set of all such $a + c$, is a subset of S.

26. In $M_{m\times n}(F)$ define $W_1 = \{A \in M_{m\times n}(F) \mid A_{i,j} = 0 \text{ whenever } i > j\}$ and
$W_2 = \{A \in M_{m\times n}(F) \mid A_{i,j} = 0 \text{ whenever } i \leq j\}$. ($W_1$ is the set of all upper triangular matrices defined in Exercise 12.) Show that $M_{m\times n}(F) = W_1 \oplus W_2$.

$W_1 \cap W_2 = \{\mathbf{O}\}$:

The zero matrix, $\mathbf{O}$, is in W_1 and W_2 and hence in $W_1 \cap W_2$ too;
because all entries of $\mathbf{O}_{i,j}$ are 0, including whenever $i > j$ and also whenever $i \leq j$.

Assume there exists a $E \in W_1 \cap W_2$ such that $E \neq \mathbf{O}$. Then, for some i, j; $E_{i,j} \neq 0$. There are three cases to consider:

- There exists some some $i > j$, $E_{i,j} \neq 0$. However, by definition of W_1, $E_{i,j} = 0$ whenever $i > j$.
- There exists some $i \leq j$, $E_{i,j} \neq 0$. But, by definition of W_2, $E_{i,j} = 0$ whenever $i \leq j$.
- There exists some some $i > j$ and some $i \leq j$ such that $E_{i,j} \neq 0$. This case is true if the previous two are. Since we have shown them to be a contradiction, this case is not possible as well.

Wherefore, by contradiction, there exists no $E \in W_1 \cap W_2$ such that $E \neq \mathbf{O}$. Indeed $\mathbf{O}$ is the unique element of $W_1 \cap W_2$.

W_1 and W_2 are subspaces of $M_{m\times n}(F)$:

From their definitions, $W_1, W_2 \subseteq M_{m\times n}(F)$ is guarenteed to be true. By exercise 12, W_1 is a subspace of $M_{m\times n}(F)$.
Define $C, D \in W_2$ and $k \in F$; Whenever $i \geq j$, $C_{i,j} = D_{i,j} = 0$, which means for all such C, D, k

- W_2 is closed under addition (A1): $C + D \in W_2$ because $(C + D)_{i,j} = C_{i,j} + D_{i,j} = 0$ (if $i > j$), and
- W_2 is closed under multiplication (M1): $kC \in W_2$ as $(kC)_{i,j} = k(C_{i,j}) = k(0) = 0$ (if $i > j$)

From earlier, we also know that $\{\mathbf{O}\} \in W_2$. Wherefore, by Theorem 1.3, W_2 is also a subspace of $M_{m\times n}(F)$.

$M_{m\times n}(F) = W_1 + W_2$:

- $W_1 + W_2 \subseteq M_{n\times n}(F)$:

  Since W_1 and W_2 are subspaces of $M_{n\times n}(F)$, and $M_{n\times n}(F)$ is a vector space, hence closed under addition (A1); $A + C \in M_{n\times n}(F)$.
- $W_1 + W_2 \supseteq M_{n\times n}(F)$:

  Conversely, if $X \in M_{n\times n}(F)$, then we can construct a $Y \in W_1$ and $Z \in W_2$ such that $X = Y + Z$. Specifically, we define

$$Y_{i,j} = \begin{cases} X_{i,j} & i \leq j \\ 0 & i > j \end{cases} \qquad Z_{i,j} = \begin{cases} 0 & i \leq j \\ X_{i,j} & i > j \end{cases}$$

  So that;

$$(Y + Z)_{i,j} = Y_{i,j} + Z_{i,j} = \left\{ \begin{matrix} X_{i,j} & i \leq j \\ X_{i,j} & i > j \end{matrix} \right\} = X_{i,j}$$

  As a result, if $X \in M_{n\times n}(F)$, then $X \in W_1 + W_2$.

$M_{m\times n}(F) = W_1 \oplus W_2$, since W_1 and W_2 are subspaces of $M_{m\times n}(F)$ such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$.

28. A matrix M is called **skew-symmetric** if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n\times n}(F)$. Now assume that F is not of characteristic 2 (see Appendix C), and let W_2 be the subspace of $M_{n\times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(F) = W_1 \oplus W_2$.

W_1 is a subspace of $M_{n\times n}(F)$

Since $\mathbf{O}^t = \mathbf{O} = -\mathbf{O}$, thus the $n \times n$ matrix $\mathbf{O} \in W_1$ is skew-symmetric.

Let $A, B \in W_1$, $k \in F$:

- W_1 is Closed Under Additon (A1): $(A+B)^t = A^t + B^t = -A - B = -(A+B) \in W_1$.
- W_1 is Closed Under Multiplication (M1): $(kA)^t = k(-A) = -(kA) \in W_1$.

Thus, by Theorem 1.3, W_1 is a subspace of $M_{n\times n}(F)$.

Now assume F is of not characteristic 2, i.e. $1 + 1 \neq 0$; and $C, D \in W_2$:

$W_1 \cap W_2 = \{\mathbf{O}\}$:

Define $\omega \in W_1 \cap W_2$, then $\omega^t = -\omega$ and $\omega^t = \omega$. It follows that

$$\begin{aligned} \omega &= -\omega \\ \omega + \omega &= \mathbf{O} \\ (1+1)\omega &= \mathbf{O} \end{aligned}$$

So, by Theorem 1.2, $1 + 1 = 0$ or $\omega = \mathbf{O}$. Since we assumed $\text{char}(F) \neq 2$, $1 + 1 = 0$ is not possible and it can only be that $\omega = \mathbf{O}$. So, $W_1 \cap W_2 = \{\mathbf{O}\}$.

$M_{n\times n}(F) = W_1 + W_2$:

- $W_1 + W_2 \subseteq M_{n\times n}(F)$:

  Since W_1 and W_2 are subspaces of $M_{n\times n}(F)$, and $M_{n\times n}(F)$ is a vector space, hence closed under addition (A1); $A + C \in M_{n\times n}(F)$.
- $W_1 + W_2 \supseteq M_{n\times n}(F)$:

  Conversely, if $X \in M_{n\times n}(F)$, then we can construct a $A \in W_1$ and $C \in W_2$ such that $X = A + C$; by defining $A_{i,j} = \frac{X_{i,j} - X_{j,i}}{1+1}$ and $C_{i,j} = \frac{X_{i,j} + X_{j,i}}{1+1}$.

  (I) Indeed, $A_{i,j} + C_{i,j} = \frac{X_{i,j} - X_{j,i}}{1+1} + \frac{X_{i,j} + X_{j,i}}{1+1} = \frac{X_{i,j} - X_{j,i} + X_{i,j} + X_{j,i}}{1+1} = \frac{(1+1)X_{i,j}}{1+1} = X_{i,j}$.

  (II) $-(A_{i,j}) = -\frac{X_{i,j} - X_{j,i}}{1+1} = \frac{X_{j,i} - X_{i,j}}{1+1} = A_{j,i}$. Hence, A is indeed skew-symmetric and in W_1.

  (III) $C_{i,j} = \frac{X_{i,j} + X_{j,i}}{1+1} = \frac{X_{j,i} + X_{i,j}}{1+1} = C_{j,i}$. Consequently, C is indeed symmetric and in W_2.

  As a result, if $X \in M_{n\times n}(F)$, then $X \in W_1 + W_2$.

Accordingly, since $W_1 + W_2 \subseteq M_{n\times n}(F)$ and $W_1 + W_2 \supseteq M_{n\times n}(F)$, $M_{n\times n}(F) = W_1 + W_2$ is true.

Wherefore, $M_{n\times n}(F) = W_1 \oplus W_2$, since W_1 and W_2 are subspaces of $M_{n\times n}(F)$ such that $W_1 \cap W_2 = \{O\}$ and $W_1 + W_2 = M_{n\times n}(F)$.

30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

Let $y_1 \in W_1$ and $y_2 \in W_2$, then $y_1 + y_2 \in V$ also.

If $V = W_1 \oplus W_2$, then each vector in V can be uniquely written as $x_1 + x_2$:

Assume $V = W_1 \oplus W_2$. Which means $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$ and $W_1 + W_2 = V$. There are 6 cases to consider for which $x_1 + x_2 = y_1 + y_2$:

- $x_1 = y_1 + y_2 \in W_1$ (while $x_2 = \vec{\mathbf{0}}$): Since W_1 is a subspace of V, by (VS 4), there exists an additive inverse of y_1, i.e. $-y_1 \in W_1$. W_1 must also be closed under additon (A1), $x_1+(-y_1) = y_1+(-y_1)+y_2 = y_2 \in W_1$. This is impossible since $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$, except for the special case of $y_2 = \vec{\mathbf{0}}$.
- $x_2 = y_1 + y_2 \in W_2$ (while $x_1 = \vec{\mathbf{0}}$): By the same logic, this is not possible either. (other than $y_1 = \vec{\mathbf{0}}$)
- $y_1 = x_1 + x_2 \in W_1$ (while $y_2 = \vec{\mathbf{0}}$): By the same logic, this is not possible either. (other than $x_2 = \vec{\mathbf{0}}$)
- $y_2 = x_1 + x_2 \in W_2$ (while $y_1 = \vec{\mathbf{0}}$): By the same logic, this is not possible either. (other than $x_1 = \vec{\mathbf{0}}$)
- $x_1 = y_1$ and $x_2 = y_2$. This is possible since $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ by definition.
- $x_1 = y_2$ and $x_2 = y_1$. This is immediately impossible as it would mean $x_1 = y_2 \in W_1$, $y_2 = x_1 \in W_2$, $x_2 = y_1 \in W_2$, $y_1 = x_2 \in W_1$. i.e. $x_1, x_2, y_1, y_2 \in W_1 \cap W_2$. Hence contradicting our assumption that $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$. (unless $x_1 = x_2 = y_1 = y_2 = \vec{\mathbf{0}}$)

There is no loss of generality in spite of the execptions. (for further details see the following:)

There will always exist a $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ that such contradictions will occur (because of the universal quantifier on the elements of V), as long as $W_1 \neq \{\vec{\mathbf{0}}\}$ and $W_2 \neq \{\vec{\mathbf{0}}\}$. Even if one of them is equal to $\{\vec{\mathbf{0}}\}$, then as $W_1 + W_2 = V$, the other must be V. In which case the claim still holds (where for all $u, v \in V$, $\vec{\mathbf{0}} + u = \vec{\mathbf{0}} + v \in V$ directly implies $u = v \in V$). For $V \neq \{\vec{\mathbf{0}}\}$, it is impossible for $W_1 = W_2 = \{\vec{\mathbf{0}}\}$ as it would be that $W_1 + W_2 = \{\vec{\mathbf{0}}\}$, instantly failing to meet the criteria of $W_1 + W_2 = V$, and creating a contradiction with the assumption that $V = W_1 \oplus W_2$. While for $V = \{\vec{\mathbf{0}}\}$, $\vec{\mathbf{0}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{0}}$ implies $\vec{\mathbf{0}} = \vec{\mathbf{0}}$, making our claim true regardless.

Therefore, it follows that for all vector spaces V; subspaces $W_1, W_2 \in V$; $x_1, y_1 \in W_1$; $x_2, y_2 \in W_2$, $x_1 + x_2 = y_1 + y_2 \in V$ implies $x_1 = y_1$ and $x_2 = y_2$. In other words, if $V = W_1 \oplus W_2$, then each vector in V can be uniquely written as $x_1 \in W_1$ and $x_2 \in W_2$.

If each vector in V can be uniquely written as $x_1 + x_2$, then $V = W_1 \oplus W_2$:

Consider the case that each vector in V can be uniquely written as $x_1 + x_2$, then: for all $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$, $x_1 + x_2 = y_1 + y_2 \in V$ implies $x_1 = y_1$ and $x_2 = y_2$. This directly means $W_1 + W_2 = V$.

Suppose that $W_1 \cap W_2 \neq \{\vec{\mathbf{0}}\}$, meaning there exists some $z \neq \vec{\mathbf{0}}$ such that it is in W_1 and W_2 simultaenously. Then, by (VS 4), the inverse of z, $-z$ exists. Consequently, as W_1 and W_2 are closed under additon (A1),

$$\begin{aligned} \text{if} \quad & z + x_2 = y_1 + z \\ \text{then} \quad & z + x_2 + (-z) = y_1 + z + (-z) \\ & x_2 = y_1 \end{aligned}$$

However, this contradicts with our assumption that for all $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$, $x_1 + x_2 = y_1 + y_2 \in V$ implies $x_1 = y_1$ and $x_2 = y_2$. So, it must be that $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$. The only case that it would not result in such a contradiction is if $W_1 = \{\vec{\mathbf{0}}\}$ and or $W_2 = \{\vec{\mathbf{0}}\}$. But in that case, $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$ is immediately true anyways.

So, $V = W_1 \oplus W_2$, because the subspaces of V, W_1 and W_2, are such that $W_1 \cap W_2 = \{\vec{\mathbf{0}}\}$ and $W_1 + W_2 = V$.

It is proven that if each vector in V can be uniquely written as $x_1 + x_2$, then $V = W_1 \oplus W_2$.

Conclusion: Wherefore, now we can conclude that, indeed, V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$.

31. Let W be a subspace of a vector space V over a field F. For any $v \in V$. the set $\{v\} + W = \{v + w \,|\, w \in W\}$ is called the coset of W containing v. It is customary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.

(b) Prove that $v_1 + W = v_2 +$ W if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{v + W \,|\, v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = av + W$$

for all $v \in V$ and $a \in F$.

(c) Prove that the preceding operations are well defined; that is, show that if $v_1+W = v_1'+W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + w)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in F$.

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the quotient space of V modulo W and is denoted V/W.

(a) If $v + W$ is a subspace of V;

Then, $\vec{\mathbf{0}} \in W$ by Theorem 1.3. In the case that $v = \vec{\mathbf{0}}$, $v \in W$ already holds. Now, consider $v \neq \vec{\mathbf{0}}$. From (VS 3), $v + \vec{\mathbf{0}} = v \in v + W$. Using (VS 4), there exists an additive inverse for v, i.e. $-v$, such that $v + (-v) = \vec{\mathbf{0}} \in v + W$. In order for this to be true, for $v \neq \vec{\mathbf{0}}$, $-v \in W$ by the definition of $v + W$. Again, with (VS 4), we know that the additive inverse of $-v$, that is $-(-v)$ must exist such that

$$\begin{aligned} -v + (-(-v)) &= \vec{\mathbf{0}} \\ v + (-v) + (-(-v)) &= v && \text{since } v + W \text{ is closed under addition (A1)} \\ \vec{\mathbf{0}} + (-(-v)) &= v && \text{by (VS 4)} \\ v &= -(-v) && \text{by (VS 3)} \end{aligned}$$

We have hence proven that if $v + W$ is a subspace of V, then $v \in W$.

presume $v \in W$: Now, by (VS 4), the additive inverse of v, $-v$ exists. Which means $v + (-v) = \vec{\mathbf{0}} \in v + W$. Given any two elements of $v + W$, i.e. $v + w_1$ and $v + w_2$ where $w_1, w_2 \in W$; $w_1 + w_2 \in W$ and $v + (w_1 + w_2) \in W$ because W is closed under addition (A1). Thence $(v + w_1) + (v + w_2) = v + (v + w_1 + w_2) \in v + W$. This means $v + W$ is indeed closed under addition as well (A1). Let $a \in F$. Accordingly,

$$\begin{aligned} a(v + w) &= av + aw && \text{by (VS 7)} \\ &= (a - 1 + 1)v + aw \\ &= v + [(a - 1)v + aw] && \text{by (VS 8)} \end{aligned}$$

Once more, due to W being closed under multiplication (M1), $(a - 1)v, aw \in W$. As it is also closed under additon (A1), $(a - 1)v + aw \in W$. Now, it is clear that $v + [(a - 1)v + aw] \in v + W$. $v + W$ is closed udner multiplication (M1).

With all 3 criterion of Theorem 1.3 met, $v+W$ is indeed a subspace of V. In othe words, $v \in W$ implies $v+W$ is a subspace of V.

Combining our two true conditional statements, the statement $v+W$ is a subspace of V if and only if $v \in W$ holds.

(b) Assume $v_1 + W = v_2 + W$:

As a consequence, for all $v_1 + w_1 \in v_1 + W$, there exists $w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$. Using our result from part (a), $v_1, v_2 \in W$. Then by (VS 4), the additive inverses $-w_1, -v_2 \in W$ exists. W is closed under addition (A1):

$$\begin{aligned} v_1 + (-v_2) + w_1 + (-w_1) &= v_2 + (-v_2) + w_2 + (-w_1) \\ v_1 + (-v_2) + \vec{\mathbf{0}} &= \vec{\mathbf{0}} + w_2 + (-w_1) && \text{by (VS 4)} \\ v_1 + (-v_2) &= w_2 + (-w_1) && \text{by (VS 3)} \end{aligned}$$

$w_2 + (-w_1) \in W$ because, again, it is closed under addition (A1). Following this, $v_1 + (-v_2) = w_2 + (-w_1) \in W$.

Therefore, if $v_1 + W = v_2 + W$, then $v_1 + (-v_2) \in W$.

Suppose $v_1 + (-v_2) \in W$:

For all $w \in W$, we can define $\Omega = w - (v_1 + (-v_2))$, since W is closed under addition (A1) and $-(v_1 + (-v_2)) \in W$ by (VS 4).
By (VS 3), $w - (v_1 + (-v_2)) = w + \vec{\mathbf{0}} = w = \Omega + v_1 + (-v_2)$. i.e. $w = \Omega + v_1 + (-v_2)$.

$$\begin{aligned} v_2 + W &= \{v_2 + w \mid v_2 \in V \text{ and } w \in W\} \\ &= \{v_1 + v_2 + (-v_2) + \Omega \mid v_1, v_2 \in V \text{ and } \Omega \in W\} \\ &= \{v_1 + \vec{\mathbf{0}} + \Omega \mid v_1 \in V \text{ and } \Omega \in W\} && \text{by (VS 4)} \\ &= \{v_1 + \Omega \mid v_1 \in V \text{ and } \Omega \in W\} && \text{by (VS 3)} \\ &= v_1 + W \end{aligned}$$

Thence, $v_1 + (-v_2) \in W$ implies $v_1 + W = v_2 + W$.

The result is that we have proven that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

(c) If $v_1 + W = v_1' + W$ and $v_2 + W - v_2' + W$, then:

By part (b), $v_1' - v_1 \in W$ and $v_2' - v_2 \in W$. Using the same logic we used to show that we can define $w = \Omega + v_1 + (-v_2)$ previosuly, now let $w_1 = \Omega_1 + v_1' + (-v_1)$ and $w_2 = \Omega_2 + v_2' + (-v_2)$ such that $w = w_1 + w_2$ and $\Omega = \Omega_1 + \Omega_2$:

$$\begin{aligned} v_1 + v_2 + w &= v_1 + v_2 + w_1 + w_2 \\ &= v_1 + v_2 + [\Omega_1 + v_1' + (-v_1)] + [\Omega_2 + v_2' + (-v_2)] \\ &= \vec{\mathbf{0}} + \vec{\mathbf{0}} + \Omega_1 + v_1' + \Omega_2 + v_2' && \text{by (VS 4)} \\ &= v_1' + v_2' + \Omega_1 + \Omega_2 && \text{by (VS 3)} \\ &= v_1' + v_2' + \Omega \\ &= (v_1' + v_2') + \Omega \end{aligned}$$

As a result, for all vectors $v_1 + v_2 + w$, $v_1 + v_2 + w \in (v_1 + v_2) + W$ iff $v_1 + v_2 + w \in (v_1' + v_2') + W$. In other words, $(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$.

Applying the same principle again with the fact that W is closed under multiplication (M1) and Theorem 1.2, now let $w = a\Omega + av_1' + (-(av_1))$ and $\omega = a\Omega$:

$$\begin{aligned} av_1 + w &= av_1' + av_1 + (-(av_1)) + a\Omega \\ &= av_1' + \vec{\mathbf{0}} + \omega && \text{by (VS 4)} \\ &= av_1' + \omega && \text{by (VS 3)} \end{aligned}$$

Thus, for all $av_1 + w$, $av_1 + w \in av_1 + W$ iff $av_1' + w \in av_1' + W$.
i.e, $a(v_1 + W) = a(v_1' + W)$.

(d)

S(A1) For all $v_1 + W_1 \in S$ and $v_2 + W_2 \in S$, $(v_1 + W_1) + (v_2 + W) = (v_1 + v_2) + W \in S$ since $v_1 + v_2 \in V$ as V is closed under addition itself (A1). This element of S is unique as addition is well-defined, as proven in part (c).

S(M1) For all $a \in F$ and $v+W \in S$, $a(v+W) = av+W \in S$ because $av \in V$ resulting from V being closed under multiplication (M1). Once more, this element of S is unique as scalar multiplication is well-defined from part (c).

S(VS 1) For all elements $v_1 + W$ and $v_2 + W$ in S;

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ &= (v_2 + v_1) + W && \text{by (VS 1) of } V \\ &= (v_2 + W) + (v_1 + W)\end{aligned}$$

S(VS 2) For all elements $v_1 + W$, $v_2 + W$, and $v_3 + W$ in S;

$$\begin{aligned}[(v_1 + W) + (v_2 + W)] + (v_3 + W) &= [(v_1 + v_2) + W] + (v_3 + W) \\ &= [(v_1 + v_2) + v_3] + W \\ &= [v_1 + (v_2 + v_3)] + W && \text{by (VS 2) of } V \\ &= (v_1 + W) + [(v_2 + v_3) + W] \\ &= (v_1 + W) + [(v_2 + W) + (v_3 + W)]\end{aligned}$$

S(VS 3) By (VS 3) of V, $\vec{\mathbf{0}} \in V$. $\vec{\mathbf{0}} + W \in S$ is then the zero vector of S, as for all $v + W \in S$,

$$\begin{aligned}(\vec{\mathbf{0}} + W) + (v + W) &= (\vec{\mathbf{0}} + v) + W \\ &= v + W && \text{by (VS 3) of } V\end{aligned}$$

S(VS 4) By (VS 4) of V; for all $v \in V$, there exists $-v \in V$ such that $v + (-v) = \vec{\mathbf{0}}$. Thus, for all $v + W \in S$, there exists a $-v + W \in S$, such that $(v + W) + (-v + W) = [v + (-v)] + W = \vec{\mathbf{0}} + W$, which is the zero vector of S as shown in *S(VS 3)*

S(VS 5) For all elements $v + W \in S$,

$$\begin{aligned}\mathbb{1}(v + W) &= \mathbb{1}v + W \\ &= v + W && \text{by (VS 5) of } V\end{aligned}$$

S(VS 6) For all $a, b \in F$ and elements $v + W \in S$;

$$\begin{aligned}(ab)(v + W) &= (ab)v + W \\ &= a(bv) + W && \text{by (VS 6) of } V \\ &= a(bv + W) \\ &= a[b(v + W)]\end{aligned}$$

S(VS 7) For all $a \in F$, elements $v_1 + W$ and $v_2 + W$ in S;

$$\begin{aligned}a[(v_1 + W) + (v_2 + W)] &= a[(v_1 + v_2) + W] \\ &= a(v_1 + v_2) + W \\ &= (av_1 + av_2) + W && \text{by (VS 7) of } V \\ &= (av_1 + W) + (av_2 + W) \\ &= a(v_1 + W) + a(v_2 + W)\end{aligned}$$

S(VS 8) For all $a, b \in F$ and $v + W \in S$,

$$\begin{aligned}(a+b)(v+W) &= (a+b)v + W \\ &= (av+bv)+W && \text{by (VS 8) of } V \\ &= (av+W)+(bv+W) \\ &= a(v+W)+b(v+W)\end{aligned}$$

Wherefore, since (A1), (M1), and (VS 1) to (VS 8) are satisfied, S is indeed a vector space.

1.4 Linear Combinations and Systems of Linear Equations

Self-Exercise 1: Let V be a vector space and the set $S \subseteq V$. span(S) is a subspace of V.

Proof:

If $S = \varnothing$, by definition, span(S) $= \{\vec{\mathbf{0}}\}$ which the zero subspace of V.

Now, consider the case of $S \neq \varnothing$: Define $\sum_{k=0}^{n} a_k i_k, \sum_{k=0}^{m} b_k j_k \in \text{span}(S)$; where $n, m \in \mathbb{N}$, for all natural k $a_k, b_k \in F$ and $i_k, j_k \in S$.

(a) By Theorem 1.2, $0u = \vec{\mathbf{0}}$, for all $u \in S$. Thus, by definition, $\vec{\mathbf{0}} \in \text{span}(S)$.

(b) $\sum_{k=0}^{n} a_k i_k + \sum_{k=0}^{m} b_k j_k$ is a finite sum of multiples of vectors in S, and hence, in span(S). (Specfically, this is a sum of $n+m$ terms) Which means that it is closed under addition (A1).

(c) $c \sum_{k=0}^{n} a_k i_k = \sum_{k=0}^{n} c(a_k i_k) = \sum_{k=0}^{n} (ca_k) i_k$ by (VS 6) of V. As a result, this is once again a finite sum of multiples of vectors in S and an element of span(S). i.e. span(S) is closed under multiplication (M1).

Wherefore, with Theorem 1.3 we know that span(S) is a subspace of V.

Q.E.D. ■

1. Label the following statements as true or false.

   (a) The zero vector is a linear combination of any nonempty set of vectors.

   (b) The span of $\varnothing$ is $\varnothing$.

   (c) If S is a subset of a vector space V, then span(S) equals the intersection of all subspaces of V that contain S.

   (d) In solving a system of linaer equations, it is permissible to multiply an equation by any constant.

   (e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.

   (f) Every system of linear equations has a solution.

   (a) True. Given any vector space V and nonempty subset S of V, $0u = \vec{\mathbf{0}}$ for all $u \in S$. ✓

   (b) False. By definition, span($\varnothing$) $= \{\vec{\mathbf{0}}\}$. ✓

   (c) True. By Theorem 1.5 span(S) is itself a subspace that contains S and all subspaces containing S must contain span(S) as well. Hence, their intersection must be span(S). ✓

   (d) True. False — × `any constant.` We missed out one special case where it does not hold: Of course if we multiply an equation by the zero constant, the set of possible solutions for a system of linear equations may change.

   (e) True. ✓

   (f) False. Counterexample: The system of linear equations

$$b = 1$$
$$b = 2$$

   has no solutions. ✓

2. Solve the following systems of linear equations by the method introuduced in this section.

(c)
$$\begin{aligned} x_1 + 2x_2 - \ x_3 + \ x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - \ x_3 + 4x_4 &= 8 \end{aligned}$$

Assume there exists a set of solutions for the unknowns, x_1, x_2, x_3, x_4, in the afforementioned system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - \ x_3 + \ x_4 &= 5 \\ 2x_2 - 2x_3 - 4x_4 &= 1 \\ -\ x_2 + \ x_3 + 2x_4 &= -2 \end{aligned}$$

If $2z_2 - 2x_3 - 4x_4 = 1$, then $x_2 - x_3 - 2x_4 = \frac{1}{2}$. Thus:

$$\begin{aligned} x_1 + 2x_2 - \ x_3 + \ x_4 &= 5 \\ 2x_2 - 2x_3 - 4x_4 &= 1 \\ 0 &= -\frac{3}{2} \end{aligned}$$

However, we know that $0 \neq -\frac{3}{2}$. Consequently, by contradiction, there exists no set of solutions for the unknowns, x_1, x_2, x_3, x_4, in this system of linear equations. ✓

(d)
$$\begin{aligned} x_1 + 2x_2 + \ 6x_3 &= -1 \\ 2x_1 + \ x_2 + \ x_3 &= 8 \\ 3x_1 + \ x_2 - \ x_3 &= 15 \\ x_1 + 3x_2 + 10x_3 &= -5 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + \ 6x_3 &= -1 \\ -3x_2 - 11x_3 &= 10 \\ -5x_2 - 19x_3 &= 18 \\ x_2 + \ 4x_3 &= -4 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + \ 6x_3 &= -1 \\ x_2 + \ 4x_3 &= -4 \\ -3x_2 - 11x_3 &= 10 \\ -5x_2 - 19x_3 &= 18 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + \ 6x_3 &= -1 \\ x_2 + \ 4x_3 &= -4 \\ x_3 &= -2 \\ -5x_2 - 19x_3 &= 18 \end{aligned}$$

With $x_3 = -2$, $x_2 + 4(-2) = -4$ and $x_2 = 4$. Hence, $x_1 = 3$ too:

$$\begin{aligned} x_1 \qquad\qquad &= 3 \\ x_2 \qquad &= 4 \\ x_3 &= -2 \\ -5(4) - 19(-2) &= 18 \end{aligned}$$

$$\begin{aligned} x_1 \qquad &= 3 \\ x_2 \quad &= 4 \\ x_3 &= -2 \\ 18 &= 18 \end{aligned}$$

Therefore, the solutions are $x_1 = 3, x_2 = 4, x_3 = -2$. ✓

5. In each part, determine whether the given vector is in the span of S.

(f) $2x^3 - x^2 + x + 3$, $S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\}$

Assume $2x^3 - x^2 + x + 3 \in \text{span}(S)$; Then there exists the (real-valued) constants a, b such that:

$$\begin{aligned} 2x^3 - x^2 + x + 3 &= a(x^3 + x^2 + x + 1) + b(x^2 + x + 1) + c(x + 1) \\ &= ax^3 + (a+b)x^2 + (a+b+c)x + (a+b+c) \end{aligned}$$

By comparing coefficients of x^3, x^2, x, as well as comparing constants, we get the system of linear equations

$$\begin{aligned} a \qquad &= 2 \\ a + b \quad &= -1 \\ a + b + c &= 1 \\ a + b + c &= 3 \end{aligned}$$

Thus, equating the final two linear equations, we get $1 = 3$. However, this is false. So, by contradiction, $2x^3 - x^2 + x + 3 \notin \text{span}(S)$. ✓

(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ For some (real-valued) constants a, b, c;

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & c \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} a + c & b + c \\ -a & b \end{pmatrix}$$

In order for these two matrices to be equal as suggested above, all their entries must be identical to each othe: i.e.

$$\begin{aligned} a + b + c &= 1 \\ b + c &= 2 \\ -a \qquad &= -3 \\ b \quad &= 4 \end{aligned}$$

$$\begin{aligned} 3 \quad - 2 &= 1 \\ c &= -2 \\ a \qquad &= 3 \\ b \quad &= 4 \end{aligned}$$

Wherefore, $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \text{span}(S)$ ✓

11.† Prove that $\text{span}(\{x\}) = \{ax \colon a \in F\}$ for any vector x in a vector space. Interpret this result geometrically in $\mathbb{R}^3$.

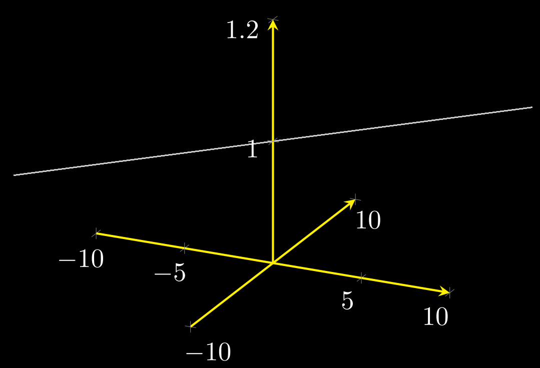
Let V be a vector space over the field F, and $\{x\} \subseteq V$:

By definition, for all $y \in \text{span}(\{x\})$, y is a linear combination of x. i.e. There exists an $a \in F$ such that $y = ax$. Thus $y \in \text{span}(\{x\})$ implies $y \in \{ax \colon a \in F\}$.

Conversely, for all $y \in \{ax \colon a \in F\}$, there exists some $a \in F$ such that $y = ax$. By definition, y is a linear combination of x. Hence, it is in $\text{span}(\{x\})$. Which means, if $y \in \{ax \colon a \in F\}$, then $y \in \text{span}(\{x\})$.

Therefore, $y \in \text{span}(\{x\})$ iff $y \in \{ax \colon a \in F\}$. In other words, $\text{span}(\{x\}) = \{ax \colon a \in F\}$.

Geometric Interpretation in $\mathbb{R}^3$: (latex is pain)



$\text{span}(\{x\}) \subseteq$ that generates $\{ax : a \in \mathbb{R}\}$ where $x \in \mathbb{R}^3$. i.e. Geometrically, it generates a straight line as shown in the diagram; a function $f \colon \mathbb{R} \to \mathbb{R}^3$ of the form $f(a) = ax$. Specifically in the ase of the above diagram, $f(a) = a(1, 1, 0)$.

13.† Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

Assume that S_1 and S_2 are subsets of a vector space V, over the field F, such that $S_1 \subseteq S_2$. For all elements $v \in \text{span}(S_1)$, there exists $u_1, u_2, \dots, u_n \in S_1$ and $a_1, a_2, \dots, a_n \in F$ such that $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. Hence, since $S_1 \subseteq S_2$, v is also a linear combination of the vectors $u_1, u_2, \dots, u_n \in S_2$. i.e. For all v, $v \in \text{span}(S_1)$ implies $v \in \text{span}(S_2)$.
So, $\text{span}(S_1) \subseteq \text{span}(S_2)$.

If $S_1 \subseteq S_2 \subseteq V$ and $\text{span}(S_1) = V$; By Theorem 1.5, $\text{span}(S_2)$ is a subspace of V. Combining this with our previous result, we get $\text{span}(S_1) = V \subseteq \text{span}(S_2) \subseteq V$. Thence, for all v, $v \in \text{span}(S_2)$ iff $v \in V$. By extensionality, $\text{span}(S_2) = V$.

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (The sume of two subsets is defined in the execises of Section 1.3.)

For all $v \in \text{span}(S_1 \cup S_2)$; there exists $u_1, u_2, \ldots, u_k \in S_1$, $u_k, u_{k+1}, \ldots, u_n \in S_2$ and $a_1, a_2, \ldots, a_n \in F$, such that $v = \sum_{i=1}^{n} a_i u_i$. Now, $\sum_{i=1}^{k} a_i u_i \in \text{span}(S_1)$ and $\sum_{i=k}^{n} a_i u_i \in \text{span}(S_2)$; because they are linear combinations of vectors in S_1 and S_2, respectively. Hence, $v = \sum_{i=1}^{n} a_i u_i = \sum_{i=1}^{k} a_i u_i + \sum_{i=k}^{n} a_i u_i \in \text{span}(S_1) + \text{span}(S_2)$. In sum, for all v, $v \in \text{span}(S_1 \cup S_2)$ implies $v \in \text{span}(S_1) + \text{span}(S_2)$.

Conversely, we apply a similar procedure: For all $v \in \text{span}(S_1) + \text{span}(S_2)$; there exists $\alpha \in \text{span}(S_1)$ and $\beta \in \text{span}(S_2)$ such that $\alpha + \beta = v$. Again, there exists $u_1, u_2, \ldots, u_k \in S_1$, $u_k, u_{k+1}, \ldots, u_n \in S_2$ and $a_1, a_2, \ldots, a_n \in F$; such that $\alpha = \sum_{i=1}^{k} a_i u_i$ and $\beta = \sum_{i=k}^{n} a_i u_i$. Summing them up, $v = \sum_{i=1}^{k} a_i u_i + \sum_{i=k}^{n} a_i u_i = \sum_{i=1}^{n} a_i u_i \in \text{span}(S_1 \cup S_2)$, since this is a linear combination of the vectors $u_1, u_2, \ldots, u_n \in S_1 \cup S_2$. We now have that for all v, $v \in \text{span}(S_1) + \text{span}(S_2)$ implies $v \in \text{span}(S_1 \cup S_2)$.

Combining these two results, for all v, $v \in \text{span}(S_1 \cup S_2)$ iff $v \in \text{span}(S_1) + \text{span}(S_2)$. i.e. $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

15. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

First, let V be the vectors space over the field F; for all $v \in \text{span}(S_1 \cap S_2)$, there exists $u_1, u_2, \ldots, u_n \in S_1 \cap S_2$ and $a_1, a_2, \ldots, a_n \in F$, such that $v = \sum_{i=1}^{n} a_i u_i$. Which means v can be written as a linear combination of both $u_1, u_2, \ldots, u_n \in \text{span}(S_1)$ and $u_1, u_2, \ldots, u_n \in \text{span}(S_2)$. Consequently, for all $v \in \text{span}(S_1 \cap S_2)$, $v \in \text{span}(S_1) \cap \text{span}(S_2)$. i.e. $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

Examples: Let $\mathbb{R}^2$ be the vector space over the field $\mathbb{R}$.

When equal: $\text{span}(\{(0,1),(2,0)\}) \cap \text{span}(\{(0,1),(0,2)\}) = \text{span}(\{(0,1),(2,0)\} \cap \{(0,1),(0,2)\})$
When unequal: $\text{span}((1,0),(0,1) \cap \{(2,0),(0,2)\}) \neq \text{span}(\{(1,0),(0,1)\}) \cap \text{span}(\{(2,0),(0,2)\}$

$$\begin{aligned}
\text{span}((1,0),(0,1) \cap \{(2,0),(0,2)\}) &= \text{span}(\varnothing) = \{\vec{0}\} \\
&\neq \mathbb{R}^2 \\
&= \mathbb{R}^2 \cap \mathbb{R}^2 \\
&= \text{span}(\{(1,0),(0,1)\}) \cap \text{span}(\{(2,0),(0,2)\})
\end{aligned}$$

16. Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1v_1 + a_2v_2 + \cdots + a_nv_n = \vec{\mathbf{0}}$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be *uniquely* written as a linear combinaton of vectors of S.

Every vector in the span of S can be written as a linear combinaton of vectors of S: By the definition of the span, this immediately follows suit.

Uniqueness:

Let V be the vector space over F. If there exists some vector $s \in \text{span}(S)$ such that $s \neq \vec{\mathbf{0}}$ can be written as two linear combinations of S:

$$s = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{m} b_i u_i$$

where for all natural i; $v_i, u_i \in S$ and $a_i, b_i \in F$, such that $a_i \neq \mathbb{0}$ and $b_i \neq \mathbb{0}$. Then,

$$\sum_{i=1}^{n} a_i v_i + \left(-\sum_{i=1}^{m} b_i u_i\right) = \sum_{i=1}^{m} b_i u_i + \left(-\sum_{i=1}^{m} b_i u_i\right) \quad \text{(VS 3)}$$
$$\sum_{i=1}^{n} a_i v_i + \left(-\sum_{i=1}^{m} b_i u_i\right) = \vec{\mathbf{0}} \quad \text{(VS 4)}$$

Assume (without loss of generality) that there exists $u_k, u_{k+1}, \ldots, u_m \notin \{v_1, v_2, \cdots, v_n\}$ for some natural k, where $1 \leq k \leq m$. Now,

$$\sum_{i=1}^{n} a_i v_i + \left(-\sum_{i=1}^{m} b_i u_i\right) = \vec{\mathbf{0}}$$
$$\sum_{i=1}^{n} a_i v_i + \left(\sum_{i=1}^{m} (-b_i) u_i\right) = \vec{\mathbf{0}} \quad \text{by Fact 16.1}$$
$$\sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{k-1} (-b_i) u_i + \sum_{i=k}^{m} (-b_i) u_i = \vec{\mathbf{0}}$$

By the given definition of our subset S, this means that for all natural i, $a_i = -b_i = b_i = \mathbb{0}$. However, this would mean $s = \vec{\mathbf{0}}$. Which contradicts our assumption that $s \neq \vec{\mathbf{0}}$. Now, for all natural i, $u_i \in \{v_1, v_2, \cdots, v_n\}$ is guarenteed.
(Importantly, one should notice that for all such natural i and j, $v_i \neq v_j$ and $u_i \neq u_j$; as if we assume otherwise, then $a_1v_1 + a_2v_2 = \vec{\mathbf{0}}$ could imply $a_2 = (-a_1) \neq \mathbb{0}$ and $v_2 = v_1$. That would contradict the construction of S.)
Thence, $n = m$.

Restating our sums with the above infomation:

$$s = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i u_i$$

Consequently, for all natural i, there exists a (unique) natural j such that $a_i v_i = b_j u_j$ (of course, $1 \leq i, j \leq n$)
There are now three cases to consider:

1. $a_i = b_j = \mathbb{0}$. This would mean $s = \vec{\mathbf{0}}$ which contradicts our assumption that $s \neq \vec{\mathbf{0}}$. Accordingly, this case is not possible.
2. $a_i \neq b_k$ (Both $a_i \neq \mathbb{0}$ and $b_k \neq \mathbb{0}$). By (F 4), there exists the multiplicative inverse b_j^{-1} such

that $b_j \cdot b_k^{-1} = \mathbb{1}$. Then,

$$\begin{aligned}
b_j^{-1}(a_i v_i) &= b_j^{-1}(b_j u_j) && \text{by (A1)}\\
(b_j^{-1}a_i)v_i &= (b_j^{-1}b_j)u_j && \text{by (VS 6)}\\
(b_j^{-1}a_i)v_i &= (b_j^{-1}b_j)u_j && \text{by (VS 6)}\\
(b_j^{-1}a_i)v_i &= \mathbb{1}u_j && \text{by (F 3)}\\
(b_j^{-1}a_i)v_i &= u_j && \text{by (VS 5)}
\end{aligned}$$

However, now $\beta = (-\alpha(b_j^{-1}a_i)^{-1})$ is a valid solution to $\alpha v_i + \beta u_j = \vec{\mathbf{0}}$ (where $\alpha, \beta \in F$); By (F 3) again, the multiplicative inverse $(b_j^{-1}a_i)^{-1}$ and additive inverse $-\alpha$ exists such that $(b_j^{-1}a_i) \cdot (b_j^{-1}a_i)^{-1} = \mathbb{1}$ and $\alpha + (-\alpha) = \mathbb{0}$,

$$\begin{aligned}
\alpha v_i + \left(-\alpha(b_j^{-1}a_i)^{-1}\right) u_j &= \alpha v_i + \left(-\alpha \left(b_j^{-1}a_i\right)^{-1}\right) \left(\left(b_j^{-1}a_i\right) v_i\right)\\
&= \alpha v_i + \left[\left(-\alpha \left(b_j^{-1}a_i\right)^{-1}\right) \left(b_j^{-1}a_i\right)\right] v_i && \text{by (VS 6)}\\
&= \alpha v_i + \left[-\alpha \left(\left(b_j^{-1}a_i\right)^{-1} \left(b_j^{-1}a_i\right)\right)\right] v_i && \text{by (F 2)}\\
&= \alpha v_i + [-\alpha(\mathbb{1})]\, v_i && \text{by (F 4)}\\
&= \alpha v_i + [-\alpha]\, v_i && \text{by (F 3)}\\
&= \alpha v_i + (-[\alpha v_i]) && \text{by Theorem 1.2(b)}\\
&= \vec{\mathbf{0}} && \text{by (VS 4)}
\end{aligned}$$

Thus, this is in clear contradiction to our definition of S, that the only possible solution is for $\alpha = \beta = \mathbb{0}$. This case is not possible either.

3. $a_i = b_j \neq \mathbb{0}$. It follows that, from (F 4), there exists the multiplicative inverse a_i^{-1} such that $a_i \cdot a_i^{-1} = \mathbb{1}$

$$\begin{aligned}
a_i^{-1}(a_i v_i) &= a_i^{-1}(a_i u_j) && \text{by (M1)}\\
\left(a_i^{-1}a_i\right) v_i &= \left(a_i^{-1}a_i\right) u_j && \text{by (VS 6)}\\
\left(a_i a_i^{-1}\right) v_i &= \left(a_i a_i^{-1}\right) u_j && \text{by (F 1)}\\
(\mathbb{1})\, v_i &= (\mathbb{1})\, u_j && \text{by (F 4)}\\
v_i &= u_j && \text{by (VS 5)}
\end{aligned}$$

This is the only possible case.

Wherefore, we have shown that if

$$s = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{m} b_i u_i,$$

then $n = m$; for all natural i, j: $a_i = b_j$, and most critically $v_i = u_j$. $(1 \leq i, j \leq n)$
i.e. These two linear combinations are actually the same linear combination.
In other words, for all vectors $s \in \text{span}(S)$, $s \neq \vec{\mathbf{0}}$ can be uniquely written as a linear combination of vectors in S — in this case: $\{v_1, v_2, \ldots, v_n\} = \{u_1, u_2, \ldots, u_n\} \subseteq S$.

Fact 16.1: For all $m \in \mathbb{N}$; for all i: $b_i \in F$ and $u_i \in V$

(Where V is the vector space over the field F.)

$$-\sum_{i=1}^{m} b_i u_i = \sum_{i=1}^{m} (-b_i) u_i$$

Proof:

When $m = 1$, $-(b_1 u_1) = (-b_1) u_1$ by Theorem 1.2(b).
Suppose that the claim is true for some $m \in \mathbb{N}$. Then, it is true for $m + 1$ as well:

$$\begin{aligned}
-\sum_{i=1}^{m+1} b_i u_i &= -\sum_{i=1}^{m} b_i u_i + (-(b_{m+1} u_{m+1})) \\
&= \sum_{i=1}^{m} (-b_i) u_i + (-(b_{m+1} u_{m+1})) && \text{by the induction hypothesis} \\
&= \sum_{i=1}^{m} (-b_i) u_i + (-b_{m+1}) u_{m+1} && \text{by Theorem 1.2(b)} \\
&= \sum_{i=1}^{m+1} (-b_i) u_i
\end{aligned}$$

As we wanted. So, for all $m \in \mathbb{N}$, our statement is indeed true as we claimed.

17. Let W be a subspace of a vector space V. Under what conditions are there only a finite number of distinct subsets S of W such that S generates W?

W is a finite subspace:

(I) W is the zero subspace of V. Then, there are two distinct subsets of W: W itself and $\varnothing$, so that $\operatorname{span}(W) = \operatorname{span}(\varnothing)$.

(II) Or $W = \{v_1, v_2, \ldots, v_n\} \neq \varnothing$ for some $n \in \mathbb{N}$. Then, the set of all distinct subsets S of W that generate W is $A = \{\{v_i, v_j, \ldots, v_k\} \subseteq W \mid \operatorname{span}(\{v_i, v_j, \ldots, v_k\})\}$. A is now a subset of $\mathscr{P}(W)$, so $|A| \leq |\mathscr{P}(W)| = 2^n$. In other words, A is finite as desired.

If W is an infinite subspace, the vectors space V, over field F, is also infinite. Which implies $F \neq \{\mathbb{0}\}$, because otherwise: By Theorem 1.2(a), $\mathbb{1}x = \mathbb{0}x = \vec{\mathbf{0}}$, which is in contradiction with (VS 5) that states $\mathbb{1}x = x$ (for all $x \in V$). So, $\mathbb{1} \neq \mathbb{0}$.

Now, for all subsets $S = \{u_1, u_2, \ldots, u_m\}$ of W that generates W (where $m \in \mathbb{N}$), and for all $n_1, n_2, \ldots, n_m \in \mathbb{N}$; the set

$$O = \left\{ \left(\sum_{i=1}^{n_1} \mathbb{1} \right) u_1, \left(\sum_{i=1}^{n_2} \mathbb{1} \right) u_2, \ldots, \left(\sum_{i=1}^{n_m} \mathbb{1} \right) u_m \right\} \neq \{\vec{\mathbf{0}}\}$$

is also a distinct subset of W that generates W. Since this is true for all $n_1, n_2, \ldots, n_m \in \mathbb{N}$, there are an infinite number of distinct subsets of W which generates W.

1.5 Linear Dependence and Linear Independence

1. Label the following statements as true or false. ✓ Nice

   (a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

   (b) Any set containing the zero vector is linearly dependent.

   (c) The empty set is linearly dependent.

   (d) Subsets of linearly dependent sets are linearly dependent.

   (e) Subsets of linearly independent sets are linearly independent.

   (f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = \vec{\mathbf{0}}$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

   (a) False. E.g.: Given the linearly dependent subset $S = \{(1,0),(0,1),(2,0)\}$ of $\mathbb{R}^3$, $(0,1)$ is not a linear combination of other vectors in S. ✓

   (b) True. $\mathbb{1} \cdot \vec{\mathbf{0}} = \vec{\mathbf{0}}$. ✓

   (c) False. The empty set is linearly independent. It is vacuously true that *for all* vectors $v_1, v_2, \ldots, v_n \in \varnothing$ and all constants $a_1, a_2, \ldots, a_n$ in the field F, $a_1v_1 + a_2v_2 + \cdots + a_nv_n = \vec{\mathbf{0}}$ implies $a_1 = a_2 = \cdots = a_n = \mathbb{0}$. ✓

   (d) False. Given the linearly dependent subset $S = \{(1,0),(0,1),(2,0)\}$ of $\mathbb{R}^3$, the subset $S' = \{(1,0)\}$ of S is linearly independent. ✓

   (e) True. ✓

   (f) True. By definition. ✓

5. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

   Where $a_i \in F$ for all i;

$$\sum_{i=0}^{n} a_i x^i = \mathbb{0}$$
$$\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} \mathbb{0} \cdot x^i$$

   So, now we can create a system of linear equations by comparing the coefficents of x^i that immediately gives us the solutions to our a_i:

$$\begin{aligned} a_0 \qquad\qquad\qquad &= \mathbb{0} \\ a_1 \qquad\qquad &= \mathbb{0} \\ a_2 \qquad &= \mathbb{0} \\ &\vdots \\ a_n &= \mathbb{0} \end{aligned}$$

   Therefore, we see that there are only trivial representations of $\mathbb{0}$ as a linear combination of vectors in the set $\{1, x, x^2, \ldots, x^n\}$.

   `"Wait, what even is` x `here?"`. See `https://en.wikipedia.org/wiki/Polynomial_ring#:~:text=Over%20a%20field%2C%20every%20nonzero,r%20such%20that%20q%20%3D%20pr.`

6. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is $\mathbb{1}$ in the ith row and jjth column. Prove that $\{E^{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is linearly independent.

The representations of $\mathbf{O}$ as a linear combination of $\{E^{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ can be written in the form:

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}E^{ij}\right) = \mathbf{O}.$$

All entries $\left(a_{ij}E^{ij}\right)_{i,j} = a_{ij}\cdot\mathbb{1} = a_{ij}$. While $\left(a_{ij}E^{ij}\right)_{u,v} = a_{ij}\cdot\mathbb{0} = \mathbb{0}$ for all naturnal numbers $u \ne i$ or $v \ne j$ still. Now, it is clear that the resultant matrix from the double sum above is the M such that $M_{i,j} = a_{i,j}$ for all natural numbers ij (where $1 \le i \le m$ and $1 \le j$). As a result, for all such i, j; if $M = \mathbf{O}$, then $M_{i,j} = a_{ij} = \mathbb{0}$. Hence, there only exists trivial representations of $\mathbf{O}$ as linear combinations of vectors in $\{E^{ij} \mid 1 \le i \le m, 1 \le j \le n\}$.

9.† Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Let the vector space V be over the field F,

($\Longrightarrow$) Assume $\{u, v\}$ is linearly dependent.
There are 3 cases to consider for the possible (nontrivial) linear representations of $\vec{\mathbf{0}}$ in this context:

i. There exists some nonzero $a \in F$ such that $au = \vec{\mathbf{0}}$. By (F 4) there exists the additive inverse a^{-1} such that $a \cdot a^{-1} = \mathbb{1}$. Thus, by (F 3) and Theorem 1.2(c), $\left(a \cdot a^{-1}\right)u = \mathbb{1}u = u = a^{-1}\cdot\vec{\mathbf{0}} = \vec{\mathbf{0}}$. Indeed, this means that they are constant multiples of each other: $\mathbb{0}\cdot v = \vec{\mathbf{0}} = u$ (Theorem 1.2(a)).

ii. There exists some nonzero $a \in F$ such that $av = \vec{\mathbf{0}}$. This is identical to the case above, except with the positions of u and v switched.

iii. There exists some nonzero $a_u, a_v \in F$ such that $a_u u + a_v v = \vec{\mathbf{0}}$. If u and or v are the zero vectors, then it is equivalent to the cases above. Now consider the vectors u, v being nonzero. Then, $u \ne v$, because otherwise this singleton set would be linearly independent (see fact). Given that the above holds true,

$$a_u^{-1}(a_u u + a_v v) = a_u^{-1}\cdot\vec{\mathbf{0}}$$
$$u + \left(a_u^{-1}\cdot a_v\right)v = \vec{\mathbf{0}}$$
$$u = -\left[\left(au^{-1}\cdot a_v\right)v\right]$$
$$u = \left[-au^{-1} + (-a_v)\right]v$$

Therefore, u is indeed a constant multiple of v.

So, indeed if $\{u, v\}$ is linearly dependent, then u or v is a multiple of the other.

($\Longleftarrow$) On the other hand, now suppose u or v is a multiple of the other (such that $u \ne v$). Without loss of generality, we can assert that u is a multiple of v — i.e. $u = kv$ for some (nonzero) $k \in F$ — since our pick of u, v are arbitrary. Anyways, then there exists a nontrivial linear representation of $\vec{\mathbf{0}}$, when $a_v = -(a_u \cdot k)$ (which is nonzero for all $a_u \ne \mathbb{0}$): $a_u u + [-(a_u\cdot k)]v = (a_u\cdot k)v + [-(a_u\cdot k)]v = \left(a_u\cdot k + [-(a_u\cdot k)]\right)v = \mathbb{0}v = \vec{\mathbf{0}}$. Wherefore, if $\{u, v\}$ is u or v is a multiple of the other, then $\{u, v\}$ is linearly dependent.

So, $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other (such that $u \ne v$).

11. Let $S = \{u_1, u_2, \dots, u_n\}$ be a linearly independent subset of a vector space V over the field Z_2. How many vectors are there in span(S)? Justify your answer.

The set S has cardinality $n \in \mathbb{N}$ by definition. Hence, each and every element of span(S) can be written as $a_1u_1 + a_2u_2 + \cdots + a_nu_n$, where $a_1, a_2, \dots, a_n \in Z_2$. i.e.: They are all either $\mathbb{1}$ or $\mathbb{0}$. Recall that $\mathbb{1} \cdot u_i = u_i$ and $\mathbb{0} \cdot u_i = u_i$ for all natural $i \leq n$. With this, the cardinality of span(S) is 2^n, because we have two choices for each a_i. For all i, we can choose to set $a_i = \mathbb{1}$, or $a_i = \mathbb{0}$. Wherefore, there are $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ choices to be made, and hence 2^n elements of span(S). Since S is linearly independent, these representations of nonzerovectors in span(S) as a linear combination of vectors in S are unique by question 16 from the previous section. In the case of the zero vector in span(S), it is not repeated in our selection as $a_1u_1 + a_2u_2 + \cdots + a_nu_n = \vec{\mathbf{0}}$ implies $a_i = \mathbb{0}$ for all i. Which occurs in only one of our above 2^n choices.

12. Prove Theorem 1.6 and its corollary.

Theorem 1.6:

Let V be the vector space over the field F:
If S_1 is linearly dependent, then there exists some nontrivial representation of $\vec{\mathbf{0}}$ as a linear combination of vectors in S_1. Since $S_1 \subseteq S_2$, that is also a nontrivial representation of $\vec{\mathbf{0}}$ as a linear combination of vectors in S_2. Hence, S_2 is linearly dependent.

Corollary:

The corollary is simply the contrapositive of the conditional statement of Theorem 1.6, and thus, it must be true as well!

13. Let V be a vector space over a field of characteristic not equal to two.

 (a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
 (b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

Let V be a vector space over the field F of characteristic not equal to two, as well as $a_1, a_2, a_3, b, c, d \in F$:

(a)

($\implies$) Assume $\{u, v\}$ is linearly independent. If

$$a_1(u + v) + a_2(u - v) = \vec{\mathbf{0}}$$
$$(a_1 + a_2)u + (a_1 - a_2)v = \vec{\mathbf{0}}$$

Then, by our assumption

$$a_1 - a_2 = \mathbb{0} \qquad a_1 + a_2 = \mathbb{0}$$
$$a_1 = a_2 \qquad (\mathbb{1} + \mathbb{1})a_1 = \mathbb{0}$$
$$a_1 = \mathbb{0} \quad \text{since } \mathrm{char}(F) \neq 2,\ \mathbb{1} + \mathbb{1} \neq \mathbb{0}$$
$$a_2 = \mathbb{0}$$

So, the only representations of $\vec{\mathbf{0}}$ as linear combinations of $u+v$ and $u-v$ are trivial. By definition, $\{u+v, u-v\}$ is linearly independent.

$(\Longleftarrow)$ Conversely, suppose that $\{u+v, u-v\}$ is linearly independent. We can form the below equation to find the possible linear combinations of u, v that give $\vec{\mathbf{0}}$:

$$cu + dv = \vec{\mathbf{0}}$$

Now, as $\text{char}(F) \neq 2$, we can let $a_2 = \frac{c-d}{\mathbb{1}+\mathbb{1}}$ and $a_1 = c - a_2 = d + a_2$. So, $c = a_1 + a_2$ while $d = a_1 - a_2$:

$$(a_1 + a_2)u + (a_1 - a_2)v = \vec{\mathbf{0}}$$
$$a_1(u+v) + a_2(u-v) = \vec{\mathbf{0}}$$

By our assumption, $a_1 = a_2 = \mathbb{0}$ is guarenteed. Hence, implying that $c = d = \mathbb{0}$ as well.

Again, the only representations of $\vec{\mathbf{0}}$ as linear combinations of u and v are trivial. By definition, $\{u, v\}$ is linearly independent.

Combining our conditional statements which we have proven true, it must hence also be true that; $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

(b)

$(\Longrightarrow)$ Presume $\{u, v, w\}$ is linearly independent. Now, if

$$a_1(u+v) + a_2(u+w) + a_3(v+w) = \vec{\mathbf{0}},$$
$$\text{then} \quad (a_1 + a_2)u + (a_1 + a_3)v + (a_2 + a_3)w = \vec{\mathbf{0}}$$

As a result, from our presumtion, we can form and solve the following system of linear equations:

$$\begin{aligned} a_1 + a_2 \quad\quad &= \vec{\mathbf{0}} \\ a_1 \quad\quad + a_3 &= \vec{\mathbf{0}} \\ a_2 + a_3 &= \vec{\mathbf{0}} \end{aligned}$$

$$\begin{aligned} a_2 - a_3 &= \vec{\mathbf{0}} \\ a_1 \quad\quad + a_3 &= \vec{\mathbf{0}} \\ a_2 + a_3 &= \vec{\mathbf{0}} \end{aligned}$$

$$\begin{aligned} a_1 \quad + \quad\quad a_3 &= \vec{\mathbf{0}} \\ a_2 - \quad\quad a_3 &= \vec{\mathbf{0}} \\ (\mathbb{1} + \mathbb{1})a_3 &= \vec{\mathbf{0}} \end{aligned}$$

Since $\text{char}(F) \neq 2$, we can conclude that $a_1 = a_2 = a_3$, as illustrated below

$$\begin{aligned} a_1 \quad\quad &= -a_3 \\ a_2 \quad &= \ \ a_3 \\ a_3 &= \ \ \vec{\mathbf{0}} \end{aligned}$$

Hence, as all representations of $\vec{\mathbf{0}}$ as linear combinations of $u+v$, $u+w$, and $v+w$ are trivial; $\{u+v, u+w, v+w\}$ is linearly independent.

$(\Longleftarrow)$ In contrast, now assume $\{u+v, u+w, v+w\}$ is linearly independent. We again form the equation that gives the possible linear combinations of $u+v$, $u+w$, and $v+w$ that equal $\vec{\mathbf{0}}$,

$$bu + cv + dw = \vec{\mathbf{0}}$$

Now, let $a_2 = \frac{d+b-c}{\mathbb{1}+\mathbb{1}}$, $a_3 = c - b + a_2$, and $a_1 = b - a_2$. This is allowed since $\text{char}(F) \neq 2$. It follows that $b = a_1 + a_2$, $c = a_1 + a_3$, $d = a_2 + a_3$:

$$\begin{aligned} a_1 &= b - a_2 & a_2 &= \frac{d+b-c}{\mathbb{1}+\mathbb{1}} & a_3 &= c - b + a_2 \\ b &= a_1 + a_2 & d &= (\mathbb{1}+\mathbb{1})a_2 + c - b & c &= b - a_2 + a_3 \\ & & d &= a_2 + (c - b + a_2) & c &= (a_1 + a_2) - a_2 + a_3 \\ & & d &= a_2 + a_3 & c &= a_1 + a_3 \end{aligned}$$

Thence,

$$\begin{aligned} (a_1 + a_2)u + (a_1 + a_3)v + (a_2 + a_3)w &= \vec{\mathbf{0}} \\ a_1(u+v) + a_2(u+w) + a_3(v+w) &= \vec{\mathbf{0}} \end{aligned}$$

By our assumption, $a_1 = a_2 = a_3 = \vec{\mathbf{0}}$. Consequently, $b = c = d = \vec{\mathbf{0}}$ as well. Once more, there exists only trivial representations of $\vec{\mathbf{0}}$ as a linear combination of $u+v$, $u+w$, and $v+w$. Which means $\{u+v, u+w, v+w\}$ is linearly independent.

Accordingly, combining our results like before, $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

18. Let S be a set of nonzero polynomials in $P(F)$ such that no two have the same degree. Prove that S is linearly independent.

Let $a_{i,n}, b_n \in F$ such that $a_{n,n} \neq \mathbb{0}$, for all natural numbers n and i where $1 \leq i \leq n$. Every vector in S can be written as some

$$\sum_{i=1}^{n} a_{i,n} x^i$$

Now, for all $m \in \mathbb{N}$; if

$$\sum_{n=1}^{m} \left(b_n \cdot \sum_{i=1}^{n} a_{i,n} x^i \right) = \mathbb{0},$$

$$\text{then} \quad \sum_{n=1}^{m} \left[\sum_{i=1}^{m} (b_n \cdot a_{i,n})\, x^i \right] = \mathbb{0} \qquad \text{where } a_{i,n} = 0 \text{ if } i > n$$

$$\sum_{i=1}^{m} \left[\sum_{n=1}^{m} (b_n \cdot a_{i,n})\, x^i \right] = \mathbb{0}$$

$$\sum_{i=1}^{m} \left[\left(\sum_{n=1}^{m} b_n \cdot a_{i,n} \right) x^i \right] = \mathbb{0}$$

In order for the double sum to equal $\mathbb{0}$: For all i, the coefficent of x^i must be $\mathbb{0}$. In other words, for all i, n: $b_n \cdot a_{i,n} = \mathbb{0}$. Which means for all i, n; $b_n = \mathbb{0}$ and/or $a_{i,n} = \mathbb{0}$. However, it is impossible that $a_{i,n} = \mathbb{0}$ for all i, n — because $a_{n,n} \neq \mathbb{0}$ by definition. Hence, it must be that $b_n = \mathbb{0}$ for all n.

Wherefore, the only representations of $\mathbb{0}$ as a linear combination of vectors in S are trivial. So, S is indeed linearly independent.

19. Prove that if $\{A_1, A_2, \cdots, A_k\}$ is a linearly independent subset of $M_{n\times n}(F)$, then $\{A_1^t, A_2^t, \cdots, A_k^t\}$ is also linearly independent.

Self-Extension: If $\{A_1, A_2, \cdots, A_k\}$ is a linearly independent subset of $M_{n\times m}(F)$, then $\{A_1^t, A_2^t, \cdots, A_k^t\}$ is a linearly independent subset of $M_{m\times n}(F)$

We know that since $\{A_1, A_2, \cdots, A_k\}$ is linearly independent,

$$\sum_{i=1}^{k} a_i A_i = \mathbf{O} \implies a_i = \mathbb{0} \quad \text{for all } 0 \leq i \leq k$$

Now, using the tranpose:

$$\sum_{i=1}^{k} a_i A_i = \mathbb{0}$$

$$\iff \left(\sum_{i=1}^{k} a_i A_i \right)^t = \mathbb{0}^t$$

$$\iff \sum_{i=1}^{k} a_i A_i^t = \mathbb{0} \implies a_i = \mathbb{0} \quad \text{for all } 0 \leq i \leq k$$

Wherefore, once again, we have that the only representations of $\mathbb{0}$ as a linear combination of $A_1^t, A_2^t, \cdots, A_k^t$ are trivial. So, $\{A_1^t, A_2^t, \cdots, A_k^t\}$ is indeed linearly independent.

20. Let $f, g \in F(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $F(\mathbb{R}, \mathbb{R})$.

We know that $e^t > 0$ for all $t \in \mathbb{R}$. Hence, for all $r, s, t \in \mathbb{R}$, $f(t) = e^{rt} > 0$ and $g(t) = e^{st} > 0$. Consequently,

$$af(t) + bg(t) = 0$$
$$ae^{rt} + be^{st} = 0e^{rt} + 0e^{st}$$

Comparing coefficients of e^{rt} and e^{st}, $a = b = 0$. So, there exists only trivial representations of 0 as linear combinations of $f(t)$ and $g(t)$. Which means f and g are linearly independent.

Remarks:
Gist of better ans: If $\{f, g\}$ is linearly dependent, then we have $f = kg$. But this means $1 = f(0) = kg(0) = k \cdot 1$ and hence $k = 1$. And $e^r = f(1) = kg(1) = e^s$ means $r = s$.

21. Let S_1 and S_2 be disjoint linearly independent subsets of V. Prove that $S_1 \cup S_2$ is linearly dependent if and only if $\operatorname{span}(S_1) \cap \operatorname{span}(S_2) \neq \{\vec{\mathbf{0}}\}$.

Proof. Assume S_1 and S_2 are disjoint linearly independent subsets of the vector space V over the field F so that $S_1 \cup S_2$ is linearly dependent. Then, there exists some natural k with the vectors $u_1, u_2, \ldots, u_n \in S_1 \cup S_2$ and scalars $a_1, a_2, \ldots, a_n \in F$ not all zero such that

$$\sum_{i=1}^{n} a_i u_i = \vec{\mathbf{0}}.$$

In other words, since each $u_i \in S_1 \cup S_2$ is either in S_1 or S_2, this means that we can assume $u_1, u_2, \ldots, u_m \in S_1$, $u_{m+1}, u_{m+2}, \ldots, u_n \in S_2$ and $a_1, a_2, \ldots, a_m, \ldots, a_n \in F$ without loss of generality, so

$$\sum_{i=1}^{m} a_i u_i + \sum_{i=m+1}^{n} a_i u_i = \vec{\mathbf{0}}$$

for some natural $m \leq n$. Equivalently, we know that

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=m+1}^{n} (-a_i) v_i.$$

Thus, as $\sum_{i=1}^{m} a_i v_i \in \operatorname{span}(S_1)$ and $\sum_{i=m+1}^{n} (-a_i) v_i \in \operatorname{span}(S_2)$, their intersection $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$ must contain this vector as well. Hence, $\operatorname{span}(S_1) \cap \operatorname{span}(S_2) \neq \{\vec{\mathbf{0}}\}$ because $\sum_{i=1}^{m} a_i u_i = \sum_{i=m+1}^{n} (-a_i) u_i \neq \vec{\mathbf{0}}$ by virtue of the scalars a_i being not all zero.

Conversely, suppose that $\operatorname{span}(S_1) \cap \operatorname{span}(S_2) \neq \{\vec{\mathbf{0}}\}$ where S_1 and S_2 are still disjoint linearly independent subsets of the vector space V over the field F. Consequently, we see that there exists the naturals m and n such that the following sum is a nonzero vector in $\operatorname{span}(S_1) \cap \operatorname{span}(S_2)$:

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{n} b_i v_i$$

where the scalars $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in F$ are not all zero, $u_1, u_2, \ldots, u_m \in S_1$ and

$v_1, v_2, \ldots, v_n \in S_2$. Thus, we simply reverse what we previously did:

$$\sum_{i=1}^{m} a_i u_i - \sum_{i=1}^{n} b_i v_i = \vec{\mathbf{0}},$$

$$\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{n} (-b_i) v_i = \vec{\mathbf{0}}.$$

By the simple re-indexing of letting $-b_i = a_{m+i}$ and $v_i = u_{m+i}$, we get that

$$\sum_{i=1}^{m+n} a_i u_i = \vec{\mathbf{0}}.$$

Therefore, there indeed exists a nontrivial representation of $\vec{\mathbf{0}}$ as a linear combination of vectors in $S_1 \cup S_2$ (recall that at least one of the scalars is nonzero). Thence, $S_1 \cup S_2$ is linearly dependent.

Wherefore, $S_1 \cup S_2$ is linearly dependent if and only if $\text{span}(S_1) \cap \text{span}(S_2) \neq \{\vec{\mathbf{0}}\}$. □

1.6 Bases and Dimension

Self-Proof of Theorem 1.8. Let V be a vector space over the field F and $u_1, u_2, \ldots, u_n$ be distinct vectors in V.

First assume that $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis of V. By definition, every $v \in V$ can be expressed as a linear combination of vectors of β because β generates V. The trickier part is to prove the uniqueness of such a linear combination. When there are two linear combinations that are identical to some $v \in V$, this means that there are some subsets A and B of $\mathbb{N}$ containing *some* natural numbers less than n so that

$$v = \sum_{i \in A} a_i u_i = \sum_{i \in B} b_i u_i,$$

$$\sum_{i \in A \cap B} (a_i - b_i) u_i + \sum_{i \in A - B} a_i u_i + \sum_{i \in B - A} (-b_i) u_i = \vec{\mathbf{0}}. \tag{1}$$

Either $A = B$ or $A \neq B$ must hold. Consider $A \neq B$; then one and only one of $A - B$ or $B - A$ is nonempty, i.e. there is some natural k in precisely one of the aforementioned sets, with exactly one of $a_k \neq \mathbb{0}$ or $b_k \neq \mathbb{0}$, in each respective case. Notice that the coefficient of u_k must thus be nonzero. In other words, there would be a nontrivial representation of $\vec{\mathbf{0}}$ as a linear combination of vectors in β. This would contradict our assumption that β is a basis of V — that is, β is linearly independent. Hence, it must be that $A = B$. Consequently, we can state equation (1) now as

$$\sum_{i \in A \cap B} (a_i - b_i) u_i = \vec{\mathbf{0}}.$$

Again, by virtue of the fact that the basis β is linearly independent, this must be a trivial representation of $\vec{\mathbf{0}}$ (as vectors in β). So, $a_i = b_i$. Which means that $\sum_{i \in A} a_i u_i$ is the exact same representation of the vector $v \in V$ as $\sum_{i \in B} b_i u_i$. i.e. uniqueness holds true.

Conversely, now suppose that each $v \in V$ can be uniquely expressed as a linear combination of vectors in β. Therefore, the zero vector can also be uniquely written as $\sum_{i=1}^{k} a_k$ for some natural k. Since $a_1 = a_2 = \cdots = a_k = \mathbb{0}$ is clearly one such possible combination of scalars and uniqueness is presumed, this must be the only possible combination (of coefficients), which is trivial. Thereupon, β is a basis of V.

Wherefore, β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors in β. □

Others

Already halted latexing everything out as it was way too time consuming. So these are my random collections of linear algebra stuff that I wanted/needed to latex out.

Theorem 0.9. *test*

Others

Already halted latexing everything out as it was way too time consuming. So these are my random collections of linear algebra stuff that I wanted/needed to latex out.

Theorem 2.32. *For any differential operator $p(D)$ of order n, the null space of $p(D)$ is an n-dimensional subspace of C^∞.*

Self-Proof of Theorem 2.32. Let m_j be the number of times $D - c_j I$ is repeated in $p(D)$. We first claim that the set[a] $S_n := \{t^{i_j} e^{e_j t} \in C^\infty \,|\, 1 \le j \le n \;\;\&\;\; 1 \le i_j \le m_j - 1\}$ is a basis for the null space of any nth order differential operator $p(D) := (D - c_1 I)(D - c_2 I) \cdots (D - c_n I)$. When $n = 1$, this is just Theorem 2.30[b]. So, assume that this is true for any differential operator $p(D)$ of a particular nth order. Then, for any differential operator $p(D)$ of order $n+1$, suppose that it has some $0 \le k \le n+1$ repeated roots. That is, $p(D) = (D - c_1 I)^{m_1}(D - c_2 I)^{m_2} \cdots (D - c_k I)^{m_k}$ for some naturals m_i. For $p(D)(y) = 0$, it simplifies to $z' - c_1 z = 0$ by having $z := (D - c_1 I)^{m_1 - 1}(D - c_2 I)^{m_2} \cdots (D - c_k)^{m_k}$. Therefore, by Theorem 2.30, $z := (D - c_1 I)^{m_1 - 1}(D - c_2 I)^{m_2} \cdots (D - c_k)^{m_k} = Ae^{c_1 t}$ — $(\star)$ for some $A \in \mathbb{C}$.

Notice $(D - c_1 I)(t^{m_1 - 1} e^{c_1 t}) = (m_1 - 1)t^{m_1 - 2} e^{c_1 t}$. By repetition, $(D - c_1 I)^{m_1 - 1}(t^{m_1 - 1} e^{c_1 t}) = (m_1 - 1)! e^{c_1 t}$. Continuing, $(D - c_j I)((m_1 - 1)! e^{c_1 t}) = (m_1 - 1)!(c_1 - c_j) e^{c_1 t}$. Again repeating this, $(D - c_1 I)^{m_1 - 1}(D - c_2 I)^{m_2} \cdots (D - c_k I)^{m_k}(t^{m_1 - 1} e^{c_1 t}) = Ce^{c_1 t}$ eventually, where we define $C := (m_1 - 1)! \prod_{j=1}^{k} (c_1 - c_j)^{m_j}$ for convenience. Hence, $t^{m_1 - 1} e^{c_1 t}$ is a solution for $(\star)$. Furthermore, since $Ce^{c_1 \cdot 0} = C \ne 0$ in the case that $t = 0$, it is certainly not the zero function. As such, $t^{m_1 - 1} e^{c_1 t}$ cannot be expressed as a linear combination of functions in S_n; implying the linear independence of S_{n+1}.

Presume f is a solution to $p(D)(y) = 0$. Thence, f satisfies $(\star)$ for some value of $A \in \mathbb{C}$, and so does $\frac{A}{C} t^{m_1 - 1} e^{c_1 t}$ for the same value of A by the above result. Consequently, $(D - c_1 I)^{m_1 - 1}(D - c_2 I)^{m_2} \cdots (D - c_k I)^{m_k} \left(f - \frac{A}{C} t^{m_1 - 1} e^{c_1 t}\right) = 0$. By our initial assumption / induction hypothesis, $f - \frac{A}{C} t^{m_1 - 1} e^{c_1 t}$ is a linear combination of functions in S_n. Accordingly, f is a linear combination of functions in S_{n+1}. That is to say, $\text{span}(S_{n+1}) = \mathrm{N}(p(D))$. Now, S_{n+1} is a basis for $N(p(D))$. In other words, the initial claim is true of $n+1$ too. Wherefore, it is true for each $n \in \mathbb{N}$ by induction. □

[a] C^∞ is the set of all functions $\mathbb{R} \to \mathbb{C}$ that has derivatives (wrt a real variable t) of all orders.
[b] It states that the solution space for $y' + a_0 y = 0$ is of dimension 1 and has $\{e^{-a_0}\}$ as a basis.