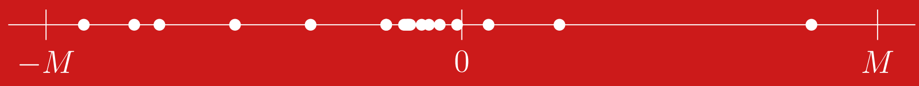

Grass

Solutions to Baby Rudin / PMA

January 2024-Today

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Chapter 1

The Real and Complex Number Systems

§1.1 Hw 1

Exercise 1.1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. If they were rational, $(r + x) - r$ and $rx \cdot 1/r$ would also be rational, a contradiction.

Exercise 1.3. Prove Proposition 1.15.

Proof. (a) The axioms (M) give

$$\begin{aligned}(1/x)(xy) &\overset{\text{M5}}{=} (1/x)(xz)\\ [(1/x)x]y &\overset{\text{M3}}{=} [(1/x)x]z\\ [x(1/x)]y &\overset{\text{M2}}{=} [x(1/x)]z\\ 1 \cdot y &\overset{\text{M5}}{=} 1 \cdot z\\ y &\overset{\text{M4}}{=} z\end{aligned}$$

(b) Fix $z = 1$ in (a).
(c) Take $z = 1/x$ in (a).
(d) Apply (c) to $(1/x)x = 1$.

Exercise 1.5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf(A) = -\sup(-A).$$

Proof. Let $i := \inf(A)$ and $s = \sup(-A)$. Notice $-i$ is a upper bound of $-A$, and $-s$ is a lower bound of A. As such, $-i \geq s$ and $i \geq -s$; the equality $i = -s$ holds.

Exercise 1.6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are irrational.

(c) If x is real, define $\mathrm{B}(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup \mathrm{B}(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup \mathrm{B}(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. (a) Notice $b^{mq} = b^{pn}$. By taking (nq)th roots, we have

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

(b) By the Corollary to Theorem 1.21,

$$b^{\frac{m}{n}+\frac{p}{q}} = b^{\frac{mq+pn}{nq}} = (b^{mq}b^{pn})^{\frac{1}{nq}} = b^{\frac{m}{n}}b^{\frac{p}{q}}.$$

(c) Let $m/n > p/q$. Then $b^{m/n} > b^{p/q}$, lest $b^{mq} \leq b^{pn}$ (a contradiction). The converse hence holds. So, $b^r = \sup \mathrm{B}(r)$.

(d) For rational t, note $b^{x+t}b^{-t}$ bounds $\mathrm{B}(x)$ from above. That is, $b^{x+t} \geq b^x b^t$. Equality holds since $b^x b^y \geq b^{x+y}$ is clear. Let $s := \sup\{b^{x+t} \mid t \leq y\}$. Then, $s(b^x)^{-1}$ bounds $\mathrm{B}(y)$ from above. As such, $s \geq b^x b^y$. So $b^x b^y \leq s \leq b^{x+y}$.

Exercise 1.8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Proof. Suppose, for contradiction, that $(\mathbb{C}, <)$ is an ordered field. Wlog, let $1 > -1$. Then $i^2 = (-i)^2 = -1 > 0$ so $-1 > 1$, a contradiction.

§1.2 Hw 2

Exercise 1.10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Proof. If $v \geq 0$, then $z^2 = u + |v|i = w$. Similarly when $v \leq 0$, we have $(\bar{z})^2 = u - |v|i = w$. We conclude that every nonzero complex number has two complex square roots.

Exercise 1.12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Proof. The result is clear by using induction on the triangle inequality (Theorem 1.33(e)).

Exercise 1.13. If x, y are complex, prove that

$$\big||x| - |y|\big| \leq |x - y|.$$

Proof. Clearly, square roots of reals preserve order. It suffices to notice

$$\begin{aligned} |x - y|^2 &= (x - y)(\bar{x} - \bar{y}) \\ &= x\bar{x} - \operatorname{Re}(x\bar{y}) + y\bar{y} \\ &\geq |x|^2 - 2|x\bar{y}| + |y|^2 \\ &= |x|^2 - 2|x||y| + |y|^2 \\ &= ||x| - |y||^2. \end{aligned}$$

Note 1.1. The *real* case is marginally simpler:

$$\big||x| - |y|\big|^2 = x^2 - 2|x||y| + y^2 \leq x^2 - 2xy + y^2 = |x - y|^2.$$

Exercise 1.15. Under what conditions does equality hold in the Schwarz inequality?

Proof. Consider the real case. Suppose that there exists $x \in \mathbb{R}$, so for all i, so $b_i = xa_i$. Then, equality is clear as

$$\left|\sum_{i=1}^n a_i b_i\right|^2 = x^2 \left|\sum_{i=1}^n a_i^2\right|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |xa_i|^2.$$

Exercise 1.17. Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

Proof. Notice that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} |\mathbf{x}-\mathbf{y}|^2 &= (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) \\ &= |\mathbf{x}|^2 - 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2. \end{aligned}$$

Therefore, $|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$. Let A be the area of the two squares having length equal to a parallelogram's respective diagonals. Also let B be the area of the two squares having side lengths equal to the parallelogram's longer and shorter sides. Then, our equality implies $A = 2B$.

Proof. Alternatively, this can be proved as a corollary of Pythagoras' Theorem. We know

$$|\mathbf{x}+\mathbf{y}|^2 = |x|^2 + |y|^2 \qquad \text{and} \qquad |\mathbf{x}-\mathbf{y}|^2 = |x|^2 + |y|^2.$$

Taking the sum,

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2.$$

Chapter 2

Basic Topology

§2.1 Theorems

"A metric space is just a space equipped with a ruler" — Me (4/5/24)

Theorem 2.30. Suppose $Y \subseteq X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

Proof. Let $p \in E$.
Assume E is open relative to Y. Then, there is $r > 0$ so $N_r(p) \cap Y \subseteq E$. Notice, $N_r(p) \cap (Y - E) = \varnothing$. So we define $G := \bigcup_{p \in E} N_{r_p}(p)$. Clearly, this superset of E is open relative to X.
Conversely, suppose $E = Y \cap G$ for some open subset G of X. Then, as $p \in G$, there is $r > 0$ with $N_r(p) \subseteq G$. As such, $N_r(p) \cap Y \subseteq E$. Hence, E is open relative to Y.

Example 2.31. Let $X = \mathbb{R}^2$. In each case, G is the combined region enclosed by the dotted lines, and Y consists of the shaded area and all points of E.

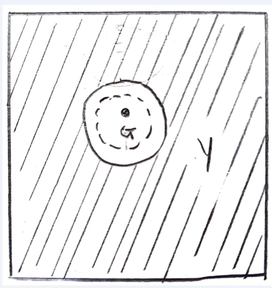


Figure 2.1: E contains only the point in black.

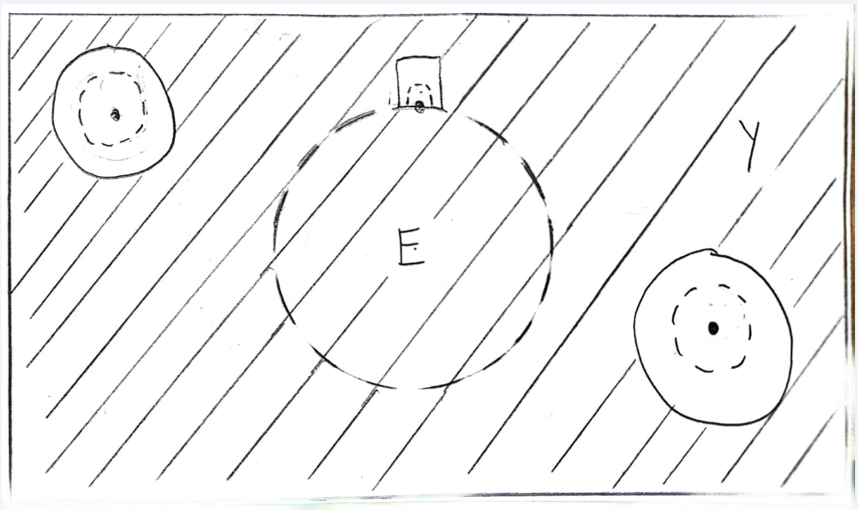


Figure 2.2: E contains the central dotted circle and the points in black.

Theorem 2.33. Suppose $K \subseteq Y \subseteq X$. Then, K is compact relative to X if and only if K is compact relative to Y.

Proof. When K is compact relative to X, let $\{F_\alpha\}$ be an open cover in Y. By theorem 2.30, there is an open cover $\{G_\alpha\}$ in X containing a finite subcover $\{G_{\alpha_n}\}$, for which $Y \cap G_\alpha$ is always F_α. So, $\{F_{\alpha_n}\}$ is a finite subcover of $\{F_\alpha\}$.
Conversely, assume K is compact relative to Y and let $\{G_\alpha\}$ be an open cover in X. Then, $\{Y \cap G_\alpha\}$ is an open cover in Y. As such, it contains a finite subcover $\{Y \cap G_{\alpha_n}\}$ in $Y \subseteq X$.

Theorem 2.34. Compact subsets of metric spaces are closed.

Proof. Let $x \in E^\complement$ be a limit point of E. Now, define the open cover

$$\mathcal{C} := \{N_{d(x,p)/2}(p) \,|\, p \in E\}.$$

Given any finite subset $\{N_{d(x,p)/2}(p_i) \,|\, 1 \leq i \leq n\}$, fix $m > 0$ as the minimum distance x is from all such p_i. Then, there is a point v in E such that $d(x, v) < m/2$. No finite subcover of $\mathcal{C}$ exists; hence E is not compact.

Theorem 2.35. Any closed subset of a compact set is compact.

Proof. Let E be a closed subset of the compact set F, and $\{G_\alpha\}$ be an open cover of E. Then, F has a finite subcover $\{G_{\alpha_n} \cup E^\complement\}$. So, $\{G_{\alpha_n}\}$ must be a finite subcover of E.

Theorem. Finite unions of compact sets are compact.

Proof. Let $\{K_i\}_{i=1}^n$ contain only compact sets, and $\{G_\alpha\}$ be an open cover of $\mathcal{K} := \bigcup_i K_i$. Each K_i has a finite subcover $\{G_{\alpha_{ij}} \mid 1 \le j \le m_i\}$. So,

$$\{G_{\alpha_{ij}} \mid 1 \le i \le n \text{ and } 1 \le j \le m_i\}$$

is a finite subcover of $\mathcal{K}$.

Note. Infinite unions of compact sets aren't necessarily compact.

Example. Consider the metric space $\mathbb{R}$. Clearly, each singleton set $\{x\}$ is compact. But since the set

$$(0,1] = \bigcup \{\{x\} \mid x \in (0,1]\}$$

does not contain the limit point 0, it is not compact.

Theorem 2.36. If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha$ is nonempty.

Proof. Assume, for contradiction, that finite intersections in $\{K_\alpha\}$ are nonempty, but $\bigcap_\alpha K_\alpha = \varnothing$. Since $\{K_\alpha\}$ can't be empty, there is some K_β. It has the open cover $\left\{K_\alpha^\complement\right\}$.

Consider any finite subset $\left\{K_i^\complement\right\}$. Then, it is disjoint from the nonempty set $K_\beta \cap \bigcap_i K_i$. As such, K_β is not compact. A contradiction.

Corollary. Let $\{K_\alpha\}$ be a set of closed sets, at least one of which is compact, such that $\{K_\alpha\}$ has the finite intersection property. Then, $\bigcap_\alpha K_\alpha$ is nonempty.

Proof. Wlog, K_0 is compact. Then, $\{K_0 \cap K_\alpha\}$ is a set of nonempty compact sets. Hence, the preceeding theorem says $\bigcap_\alpha K_\alpha$ is nonempty.

Definition. We say that a set (E,d) is *globally closed* iff for all supersets (X,d') of (E,d) (such that there is an isometric embedding of (E,d) in (X,d')), we have that E is closed relative to X.

Claim. If a set (E,d) is globally closed, it is compact.

Here is a counterexample.

Example. Consider the non-compact metric space $\mathbb{N}$ under the Euclidean norm. Suppose, for contradiction, that there exists some superset X in which $\mathbb{N}$ is not closed (such that there is an isometric embedding of $\mathbb{N}$ in X).

Let p be such a limit point. So, there exists $n_2 < n_3$, such that $d(n_2,x)$ and $d(n_3,x)$

are less than $1/2$. Consequently,

$$d(n_2, x) + d(x, n_3) < 1 \leq d(n_2, n_3).$$

A contradiction.

Thus, compactness is quite a strong condition; it is stronger than even global closure!

Theorem 2.37. If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. Suppose, for contradiction, that there exists a subset E of some compact set K, without a limit point in K. As such, E is closed. Hence, it is compact. Since E contains no limit points, for each $p \in E$ there is a minimum distance m_p, such that $m_p \leq d(p, q)$ for all $q \in E$ (that isn't p). But now the open cover $\{N_{m_p/2}(p) \,|\, p \in E\}$ has no finite subcover. A contradiction.

Theorem 2.38. If $\{I_n\}_{n=1}^{\infty}$ is a sequence of closed intervals in $\mathbb{R}$, such that $I_{n+1} \subseteq I_n$, then $\bigcap_n I_n$ is nonempty.

Proof. Let $I_n = [a_n, b_n]$. Then, $a_n \leq \sup\{a_n\}_{n=1}^{\infty} \leq b_n$ for all n.

Theorem 2.39. Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_{n+1} \subseteq I_n$, then $\bigcap_n I_n$ is nonempty.

Proof. Similarly, let I_n be the k-cell of elements $(x_1, x_2, \ldots, x_k)$, such that $a_{ni} \leq x_i \leq b_{ni}$. Also let $s_i := \sup\{a_{ni}\}_{n=1}^{\infty}$. Notice that $a_{ni} \leq s_i \leq b_{ni}$ for all n and i. That is, $(s_1, s_2, \ldots, s_k) \in \bigcap_n I_n$.

Theorem 2.40. Every k-cell is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of a k-cell I.
Notice that for each $\mathbf{x} \in I$, there exists $m(\mathbf{x}) > 0$, such that $N_{m(\mathbf{x})}(\mathbf{x})$ is contained in some G_α and

$$m(\mathbf{x}) > \frac{1}{2} \sup\{r \,|\, N_r(\mathbf{x}) \subseteq G_\alpha \text{ for some } \alpha\}.$$

Suppose, for contradiction, that $\ell := \inf\{m(\mathbf{x}) \,|\, \mathbf{x} \in E\} = 0$. By AC, we can choose a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ for which $\mathbf{x}_n := (x_{1n}, x_{2n}, \ldots, x_{kn})$ and $m(\mathbf{x}_n) < 1/n$. By iteratively applying Bolzano-Weierstrass, there exists a subsequence $\{\mathbf{x}_{n_k}\}_{k=1}^{\infty}$ such that $s_i := \lim_{k \to \infty} x_{n_k}$ always exists. Let $\mathbf{s} \in I$ be the point whose ith coordinate is s_i. So, for some $n \in \mathbb{N}$, we have $2m(\mathbf{x}_n) < m(\mathbf{s})$, satisfying

$$|s_i - x_{in}| < \frac{1}{\sqrt{k}}\big(m(\mathbf{s}) - 2m(\mathbf{x}_n)\big).$$

Hence,

$$\sum_{i=1}^{k}(s_i - x_{in})^2 < \sum_{i=1}^{k}\frac{1}{k}\big(m(\mathbf{s}) - 2m(\mathbf{x})\big)^2 = \big(m(\mathbf{s}) - 2m(\mathbf{x}_n)\big)^2.$$

That is, $|\mathbf{s} - \mathbf{x}_n| < m(\mathbf{s}) - 2m(\mathbf{x}_n)$. Now for each $|\mathbf{x}_n - \mathbf{q}| < 2m(\mathbf{x}_n)$,

$$|\mathbf{s} - \mathbf{q}| \leq |\mathbf{s} - \mathbf{x}_n| + |\mathbf{x}_n - \mathbf{q}| < m(\mathbf{s}) - 2m(\mathbf{x}_n) + 2m(\mathbf{x}_n) = m(\mathbf{s}).$$

As such, $N_{2m(\mathbf{x}_n)}(\mathbf{x}_n)$ is contained in the same G_α as $N_{m(\mathbf{s})}(\mathbf{s})$. But by definition

$$2m(\mathbf{x}_n) > \sup\{r \mid N_r(\mathbf{x}) \subseteq G_\alpha \text{ for some } \alpha\},$$

a contradiction.

Consequently, $\ell > 0$ so there exists a finite number of neighbourhoods (of radius ℓ), each contained in some G_{α_n}, that covers I.

Oh, actually I probably don't need to use Bolzano-Weierstrass. It's pretty clear that there must be coordinate-wise convergence, in order for $\{\mathbf{x}_n\}_{n=1}^{\infty}$ to converge.

Theorem 2.41. If a set in $\mathbb{R}^k$ has one of the following properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Proof.

i. Assume (a) is true. Then, let

$$a_i := \inf\{x_i \mid (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k\} \quad \text{and} \quad b_i := \sup\{x_i \mid (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k\}.$$

Now, E is a closed subset of the k-cell of points $(x_1, x_2, \ldots, x_k)$, such that $a_i \leq x_i \leq b_i$. Since k-cells are compact, so is E. That is, (b) holds.

ii. Now suppose (b) is valid. Thus, (c) follows immediately from theorem 2.37.

iii. Finally, we presume (c) to be true. If E is unbounded, then (using AC) we can construct a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of points, such that $|\mathbf{x}_{n+1}| > |\mathbf{x}_n| + 1$ for all n. Hence, it is clear that $\{\mathbf{x}_n \mid n \in \mathbb{N}\}$ contains no limit points. A contradiction. Similarly, when $\mathbf{p} \notin E$ is a limit point of E, there is a sequence $\{\mathbf{y}_n\}_{n=1}^{\infty}$, with $|\mathbf{p} - \mathbf{y}_n| < 1/n$. Therefore, $\{\mathbf{y}_n \mid n \in \mathbb{N}\}$ also has no limit points in E. Again, this is impossible; (a) must hold.

Question. When a set is unbounded, must it be non-compact?

Proof. Yes. Consider an unbounded metric space X and choose a point $x \in X$. Then, the open cover $\{N_n(x)\}$ of X has no countable subcover.

Theorem 2.42. Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. Let K be a bounded infinite subset of $\mathbb{R}^k$. Then, its closure $\overline{K}$ is compact. Hence, there exists a limit point of $\overline{K}$. This is also a limit point of K.

Theorem 2.43. Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

Proof. Suppose, for contradiction, that P is countable. Notice P is complete as it is closed in $\mathbb{R}^k$ (see note). We consider the metric space P (rather than $\mathbb{R}^k$). First fix a bijection $f\colon \mathbb{N} \to P$ and $G_n := P - \{f(n)\}$. Then, since $N_{|g-f(n)|}(g) \subseteq G_n$ for every $g \in G_n$, each G_n is an open dense subset of P. Consequently, Baire's Theorem (3.22) says $\bigcap G_n$ is nonempty, but this is impossible.

Note. Two separated subsets A and B of a metric space X do not have to satisfy $\bar{A} \cap \bar{B} = \varnothing$. Consider $A = (0, 1)$ and $B = (1, 2)$.

Theorem 2.47. A subset E of the real line $\mathbb{R}$ is connected iff it has the following property: If $x, y \in E$ and $x < z < y$, then $z \in E$.

Proof. Suppose there exists $x, y \in E$ and $z \notin E$, such that $x < z < y$. Then, $E \cap (-\infty, z)$ and $E \cap (z, \infty)$ are separated sets that partition E. Therefore, E is not connected.
Consider when E satisfies the given property, and let $A, B \subseteq E$ be two nonempty separated sets. Wlog, $\alpha < \beta$ for some $\alpha \in A$ and $\beta \in B$. Define

$$s := \sup\{a \in A \,|\, \alpha \le a < \beta\} \qquad \text{and} \qquad i := \inf\{b \in B \,|\, s < b\}.$$

Notice that $s \le \frac{s+i}{2} \le i$. Hence, $\frac{s+i}{2} \notin A \cup B$. (Otherwise $\frac{s+i}{2} \le s$ or $i \le \frac{s+i}{2}$. So $s = \frac{s+i}{2} = i$, implying A and B are not separate.) i.e. $E \neq A \cup B$.

Exercise. (JohnDS's Exercise) Let X be a metric space. Prove the following are equivalent.

(1) $E \subseteq X$ is dense.

(2) For every $\varepsilon > 0$ and all $x \in X$, there is an $p \in E$ such that $d(x, p) \le \varepsilon$.

(3) The closure of E is X.

(4) For every $x \in X$, there is a sequence $\{p_n\}_{n=1}^{\infty}$ in E, such that $x = \lim_{n\to\infty} p_n$.

Proof. Clearly, (3) implies (1).
Now assume (1) is true. Pick any $x \in X$ and $\varepsilon > 0$. If $x \in E$, simply let $p = x$. Otherwise, x is a limit point of E. Thus, such p exists by definition. As such, (2) holds.
Suppose (2) is true. By AC, there is a sequence $\{p_n\}_{n=1}^{\infty}$ with $d(x, p_n) < 1/n$. That is, $x = \lim_{n\to\infty} p_n$. So, (4) holds.
Presume (4) is satisfied. When $p_n = x$ for some n, we know $x \in E$. Otherwise, p_n is never x, which means x is a limit point of E. Consequently, (3) holds true.

§2.2 Hw 3

Exercise 2.5. Construct a bounded set of real numbers with exactly three limit points.

Proof. Let $E := \{1/n, 2/n, 3/n \,|\, n \in \mathbb{Z}^+\}$. By the Archimedean property, $\{1, 2, 3\} \subseteq E'$. In fact, equality is clear: the neighbourhood of any i/n (for $n \geq 2$) with radius

$$\min\left\{\frac{i}{n-1} - \frac{i}{n}, \frac{i}{n} - \frac{i}{n+1}\right\}$$

is disjoint from E.

Exercise 2.6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Proof. Let p be a limit point of E'. So, for any $r > 0$, there is $q \in N_r(p) \cap E'$. Hence, letting $m := \min\{d(p, q), r - d(p, q)\}$, there is also $s \in N_m(q) \cap E$. As such, p is a limit point of E, since

$$0 < d(p, s) \leq d(p, q) + d(q, s) < r.$$

Similarly, any limit point of $\overline{E}$ is a limit point of E. The converse is clear from $E \subseteq \overline{E}$.
However, limit points of E are not necessarily limit points of E'. For instance, let $E := \{1/n \,|\, n \in \mathbb{Z}^+\} \subseteq \mathbb{R}$. Then, $E' = \{1\}$ whilst $E'' = \varnothing$.

Exercise 2.8. Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. Yes, every point p of an open set $E \subseteq \mathbb{R}^2$ is a limit point of E. There is $m > 0$ with $N_m(p) \subseteq E$. Thus, for any $r > 0$, the set $N_r(p) \cap E$ must be nonempty, for it contains the nonempty set $N_r(p) \cap N_m(p)$.
But this does not extend to closed sets in $\mathbb{R}^2$. Consider the closed set $\{1\}$; it has

no limit points.

Exercise 2.10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{(if } p \neq q) \\ 0 & \text{(if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. Notice that (a) and (b) holds immediately, and (c) is clear as $d(p,r)+d(r,q)$ is either 1 or 2. Therefore, d is a metric on X.
Let $E \subseteq X$ and $p \in E$. Observe that $N_{1/2}(p) = \{p\} \subseteq E$. All subsets of X must be open, and, with exception of $\varnothing$, be not closed.
Lastly, E is compact iff E is finite. If E is compact but infinite, then the open cover of singleton subsets of E (i.e. $\{\{p\} \mid p \in E\}$) has no finite subcover. The converse is clear as this open cover is already finite.

Exercise 2.11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2 - y^2|, \\ d_4(x,y) &= |x - 2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof.

(a) Since $(1-0)^2 = 1$ is larger than $(1-0.5)^2 + (0.5-0)^2 = 0.5$, we have that d_1 is not a metric.

(b) Parts (a) and (b) are immediate. Moreover, (c) also holds, because for any real numbers x, y and z:

$$\begin{aligned} |x-y| &\leq |x-z| + 2\sqrt{|x-z||z-y|} + |z-y|, \\ \sqrt{|x-y|} &\leq \sqrt{|x-z|} + \sqrt{|z-y|}. \end{aligned}$$

That is, d_2 is a metric.

(c) Part (a) doesn't hold as $|1^2 - (-1)^2| = 0$. Hence, d_3 is not a metric.

(d) Part (a) doesn't hold as $|1 - 2(0.5)| = 0$. Therefore, d_4 is not a metric.

(e) Again, parts (a) and (b) are clear. Notice

$$|x-y| \leq |x-z| + |z-y| + 2|x-z||z-y| + |x-y||x-z||z-y|.$$

Hence, adding $|x-y||x-z| + |x-y||z-y| + |x-y||x-z||z-y|$ to both sides, then factoring,

$$|x-y|(1+|x-z|)(1+|z-y|) \leq (|x-z| + 2|x-z||z-y| + |z-y|) \\ (1+|x-y|).$$

Therefore,

$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}.$$

Accordingly, d_5 is a metric.

§2.3 Hw 4

Exercise 2.12. Let $K \subseteq \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel Theorem).

Proof. Let $\{G_\alpha\}$ be an open cover of K. Then, $N_{1/n}(0) \subseteq G_{\alpha_{n+1}}$, for some $n \in \mathbb{N}$ and α_{n+1}. Furthermore, given any $1 \leq i \leq n$, there exists α_i for which $1/i \in G_{\alpha_i}$. Hence, $\{G_{\alpha_i} \,|\, 1 \leq i \leq n+1\}$ is a finite subcover.

Exercise 2.16. Regard $\mathbb{Q} :=$ as the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Proof. Notice $|p-q|$ is always less than $\sqrt{3}-\sqrt{2}$. Now consider any limit point l of E. Then $2 \leq l^2 \leq 3$, lest $N_{|l-\sqrt{2}|}(l)$ or $N_{|l-\sqrt{3}|}(l)$ is disjoint from E. Since there is no rational square root of 2 or 3, we have $l \in E$.
We have that E is not closed relative to $\mathbb{R}$, since the limit point $\sqrt{2} \notin E$. Thus, E is not compact.
Finally, E is indeed open in $\mathbb{Q}$, because for all $p \in E$, either $N_{|p-\sqrt{2}|}(p)$ or $N_{|p-\sqrt{3}|}(p)$ is contained in E.

Exercise 2.17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Proof. Let $a = 0.4777\ldots$ and $b = 0.7444\ldots$
An $a \leq x \leq b$ cannot be obtained by changing only the first digit, since

$$0.777\cdots > b > a \quad \text{and} \quad 0.444\cdots < a < b.$$

It also can't be obtained by changing the first digit and some others, as

$$0.777\ldots 74 \geq b > a \quad \text{and} \quad 0.444\ldots 47 \leq a < b.$$

(The digits, after what has been written out explicitly, are omitted.)
Nor does keeping the first digit the same:

$$0.4777\ldots 74 < a < b \quad \text{and} \quad 0.7444\ldots 47 > b > a$$

Hence no such x exists; E is not dense.
Let the decimal expansion of the limit point L contain some other digit not 4 or 7, first in the jth digit. Then, there exists $c \in E$ such that $|c - L| < 10^{-j}$. But now

$$1 > 10^j |c - L| \geq 1.$$

A contradiction. As such, E must be closed, and hence compact.
Furthermore, E is perfect, thus uncountable[a]. For any $y \in E$, define $y^{(i)}$ to be the ith digit of y, and

$$y^{[i]} := \begin{cases} 4 & \text{if } y^{(i)} = 7, \\ 7 & \text{if } y^{(i)} = 4. \end{cases}$$

So, let z_n be the number whose ith digit is $y^{(i)}$ if $1 \leq i \leq n$, and $y^{[i]}$ otherwise. Then, for each n we have

$$|y - z_n| < \frac{4}{10^{n+1}} < \frac{1}{10^n}.$$

[a]Alternatively, we can use diagonalization, or the fact that ${}^{\mathbb{N}}\{4,7\} \approx \mathscr{P}(\mathbb{N}) \succ \mathbb{N}$.

Exercise 2.22. A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$. Recall $\mathbb{Q}$ is dense in $\mathbb{R}$. So, for each i, there exists $|x_i - p_i| < \frac{\varepsilon}{\sqrt{k}}$. Therefore,

$$|\mathbf{x} - \mathbf{p}| = \sqrt{\sum_{i=1}^{k} (x_i - p_i)^2} < \sqrt{\sum_{i=1}^{k} \frac{\varepsilon^2}{k}} = \varepsilon.$$

Since $\mathbb{R}^k$ contains the countable dense subset $\mathbb{Q}^k$, it is separable.

Example. Consider $[0,1]$ and the countable base

$$\{N_r(p) \subseteq [0,1] \,|\, p, r \in \mathbb{Q}\}.$$

Let x be a point in some open subset S of $[0,1]$. Accordingly, $N_\varepsilon(x) \subseteq S$ for some $\varepsilon > 0$. Furthermore, $\varepsilon/2 > n^{-1}$ for some $n \in \mathbb{N}$. By the density of the rationals in the reals, there exists some rational $|p - x| < n^{-1}$. Consequently,

$$x \in N_{n^{-1}}(p) \subseteq N_\varepsilon(x) \subseteq S.$$

Exercise 2.25. Prove that every compact metric space K has a countable base, and that K is therefore separable.

Proof. For each n, let $\{N_{n^{-1}}(x_{i,n}) \,|\, 1 \leq i \leq m_n\}$ be a finite subcover of $\{N_{n^{-1}}(x) \,|\, x \in K\}$. Fix $\varepsilon > 0$ and $x \in K$. So, there exists n and i, for which $\varepsilon/2 > n^{-1}$ and $x \in N_{n^{-1}}(x_{i,n})$. Hence $N_{n^{-1}}(x_{i,n}) \subseteq N_\varepsilon(x)$. As such,

$$\bigcup_n \{N_{n^{-1}}(x_{i,n}) \,|\, 1 \leq i \leq m_n\}$$

is a countable base. Furthermore, x is a limit point of the countable dense subset

$$\bigcup_n \{x_{i,n} \,|\, 1 \leq i \leq m_n\},$$

if it is not some $x_{i,n}$.

§2.4 Hw 5

Exercise 2.27. Define a point p in a metric space X to be a *condensation point* of a set $E \subseteq X$ if every neighbourhood of P contains uncountably many points of E. Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all the condensation points of E. Prove that P is perfect and that at most countably many points are not in P. In other words, show that $P^\complement \cap E$ is at most countable.

Proof. We consider E as our metric space for this proof. Let x be a limit point of P and $\varepsilon > 0$. Then, there exists $p \in P$ with $|x - p| < \varepsilon/2$. For each of the uncountably many $|p - q| < \varepsilon/2$, we have $|x - q| \leq |x - p| + |p - q| < \varepsilon$. So, P is closed. Since it is clear that P contains only limit points of itself, P is perfect.

Now suppose, for contradiction, that $P^\complement \cap E$ is uncountable. Let $n \in \mathbb{N}$ and $r_n := 0.5^n/n$. For some point $e_n \in N_{r_{n-1}}(e_{n-1})$, the neighbourhood $N_{r_n}(e_n)$ is uncountable. Otherwise, $N_{r_{n-1}}(e_{n-1})$ would be countable, since

$$\{N_{r_{n-1}}(e_{n-1}) \cap N_{r_n}(m_1 r_n, m_2 r_n, \ldots, m_k r_n) \,|\, m_1, m_2, \ldots, m_k \in \mathbb{N}\}$$

covers $N_{r_{n-1}}(e_{n-1})$. Therefore, $\{e_n\}$ is Cauchy, since

$$d(e_n, e_m) \leq \sum_{i=n}^{m-1} d(e_n, e_{n+1}) \leq \sum_{i=1}^{\infty} \frac{0.5^i}{N} = \frac{1}{N},$$

for every $n \geq m \geq N$. By the completeness of $\mathbb{R}^k$, it converges to some limit $\mathfrak{e}$. Furthermore, it is clear that $\mathfrak{e}$ is a condensation point.

Chapter 3

Numerical Sequences and Series

§3.1 Hw 5

Exercise 3.1. Prove that the convergence of $\{s_n\}$ implies the convergence of $\{|s_n|\}$. Is the converse true?

Proof. Let $\{s_n\}$ be a sequence in a metric space X converging to x, and y any point of X. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d(s_n, x) < \varepsilon$. So by the triangular inequality,

$$|d(s_n, y) - d(x, y)| \leq d(s_n, x) < \varepsilon.$$

Hence, $\{d(s_n, y)\}$ converges to $d(x, y)$.

The converse cannot hold. A counterexample is the alternating series.

Exercise 3.2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

Proof. Let $\varepsilon > 0$ and $c := 1/2 - \varepsilon$. Notice that $n^2 + n + 1/4 > n^2 + n$ implies $1/2 - \left(\sqrt{n^2+n} - n\right) > 0$. Moreover, there is some $N > \frac{c^2}{1-2c}$. So, for $n \geq N$,

$$\begin{aligned} \frac{c^2}{1-2c} &< n, \\ n^2 + 2cn + c^2 &< n^2 + n, \\ n + c &< \sqrt{n^2+n}, \\ \left|\frac{1}{2} - \left(\sqrt{n^2+n} - n\right)\right| &< \varepsilon. \end{aligned}$$

In other words, $\lim_{n\to\infty}(\sqrt{n^2+n} - n) = 1/2$.

Exercise 3.3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$

Proof. Notice that $\sqrt{2+\sqrt{2}} < \sqrt{4} = 2$. So, assume $s_n \leq 2$ for $n = k$, and consider $n = k + 1$. Thus,

$$s_{n+1} \leq \sqrt{2 + \sqrt{2}} < 2.$$

That is, s_n is always less than 2.
Furthermore, since $\{s_n\}$ is nonnegative, it is monotonically increasing. By the Monotone Sequence Theorem (3.14), $\{s_n\}$ converges.

§3.2 Theorems

The following exercises were suggested by DarQ (or, xxdarqxx). The first is exercise 2.4.7 of Abbott.

Exercise (Limit Superior). Let $\{a_n\}$ be a bounded sequence.

(a) Prove that the sequence defined by $y_n := \sup\{a_k \mid k \geq n\}$ converges.

(b) The *limit superior* of $\{a_n\}$, or $\limsup a_n$, is defined by

$$\limsup a_n := \lim y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf a_n$ and briefly explain why it exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof.

(a) Let $\varepsilon > 0$. For $L := \inf\{y_n \mid n \in \mathbb{N}\}$, recall there exists $N \in \mathbb{N}$ such that $y_N - L < \varepsilon$. So for $n \geq N$, since $\{y_n\}$ is clearly nonincreasing,

$$y_n - L \leq y_N - L < \varepsilon.$$

Hence, $\lim y_n = L$.

(b) We can define $z_n := \inf\{a_k \mid k \geq n\}$, and hence,

$$\limsup a_n := \lim z_n.$$

This exists for any bounded sequence $\{a_n\}$, because it is just $\sup\{y_n \mid n \in \mathbb{N}\}$.

The proof is similar to that of (a).

(c) Clearly, $y_n \geq z_n$ for each $n \in \mathbb{N}$. Accordingly, $\lim y_n \geq \lim z_n$. i.e.

$$\limsup a_n \geq \liminf a_n.$$

An example for which the strict inequality $\limsup a_n > \liminf a_n$ holds. For the sequence $\{(-1)^n\}$, we see that

$$\limsup(-1)^n = 1 > -1 = \liminf(-1)^n.$$

(d) Suppose $L := \limsup a_n = \liminf a_n$ and let $\varepsilon > 0$. Recall the above results:

$$\limsup a_n = \inf\{y_n \,|\, n \in \mathbb{N}\} \qquad \text{and} \qquad \liminf a_n = \sup\{z_m \,|\, m \in \mathbb{N}\}.$$

There hence exists $N, M \in \mathbb{N}$ such that

$$a_k - L < y_N - L < \varepsilon \qquad \text{and} \qquad L - a_k < L - z_M < \varepsilon$$

for any $k \geq \max\{N, M\}$. i.e. $|a_k - L| < \varepsilon$. So, $\lim a_k = L$.
Conversely, assume $\mathscr{L} = \lim a_k$ exists, and again, let $\varepsilon > 0$. Pick $K \in \mathbb{N}$, such that $|a_k - L| < \varepsilon/2$ for all $k \geq K$. Notice that for each $k \geq K$, there is $j \geq k$ with $|y_k - a_j| < \varepsilon/2$. As such,

$$|y_k - \mathscr{L}| \leq |y_k - a_j| + |a_j - \mathscr{L}| < \varepsilon.$$

In other words, $\mathscr{L} = \limsup a_k$. It can be similarly shown that $\mathscr{L} = \liminf a_k$.

Exercise. Prove that the definitions of lim sup and lim inf in Rudin and Schröder/Abbott are equivalent.
That is, let $\{s_n\}$ be a sequence of real numbers, and E the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. Now define $s^* := \sup E$ and $s_* = \inf E$. Also define $y_n := \sup\{a_k \,|\, k \geq n\}$ and $z_n := \inf\{a_k \,|\, k \geq n\}$; $\mathsf{s}^\circ := \lim y_n$ and $\mathsf{s}_\circ := \lim z_n$. Then, it holds that

$$s^* = \mathsf{s}^\circ \qquad \text{and} \qquad s_* = \mathsf{s}_\circ.$$

Proof. If $\{s_n\}$ is unbounded from above, then $s^* = \mathsf{s}^\circ = \infty$ is trivial. So, consider when $\{a_n\}$ is bounded above. For each $k \in \mathbb{N}$, pick $N_k \in \mathbb{N}$ such that $|y_{N_k} - \mathsf{s}^\circ| < \frac{1}{2k}$. Furthermore, since y_n is supremal, $|y_{N_k} - a_{n_k}| < \frac{1}{2k}$ for some $n_k \geq N_k$. Hence, $|a_{n_k} - \mathsf{s}^\circ| < \frac{1}{k}$. As such, the subsequence[a] $\{a_{n_k}\}$ converges to s°. Therefore, $s^* \geq \mathsf{s}^\circ \in E$.
To prove $s^* \leq \mathsf{s}^\circ$, first choose any subsequence $\{a_{n_k}\}$ that converges to some limit

x. Then, it is clear that $y_m \geq x$ for all $m \in \mathbb{N}$, because y_m bounds the subsequence $\{a_{n_{m+k}}\}$ from above. Accordingly, $\mathsf{s}^\circ \geq s^*$.
Consequently, equality holds. The proof of $s_* = \mathsf{s}_\circ$ is similar.

[a]Since we can just choose the least such N_k and n_k, AC is not necessary here.

Theorem 3.33 (Root Test). Given $\sum a_n$, put $\alpha := \limsup \sqrt[n]{|a_n|}$. Then,

(a) if $\alpha < 1$, then $\sum a_n$ converges;
(b) if $\alpha > 1$, then $\sum a_n$ diverges;
(c) if $\alpha = 1$, the test gives no information.

Proof.

(a) First assume $\alpha < 1$. We pick $K \in \mathbb{N}$, such that

$$\frac{\alpha+1}{2} > \sup\left\{ \sqrt[k]{|a_k|} \,\middle|\, k \geq K \right\}$$

Then, $\left(\frac{\alpha+1}{2}\right)^k > |a_k|$ for all $k \geq K$. So, by the comparison test (thm 3.25) $\sum a_n$ converges absolutely.

(b) Now suppose $\alpha > 1$. Then, there is a subsequence $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Choose $K \in \mathbb{N}$, such that $\sqrt[n_k]{|a_{n_k}|} > 1$ whenever $k \geq K$. Thus, $|a_{n_k}| > 1$. This implies $a_n \not\to 0$. So $\sum a_n$ must diverge.

Theorem 3.34 (Ratio Test). The series $\sum a_n$

(a) converges if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$,
(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Proof.

(a) First assume $\alpha := \limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$. Then, pick $K \in \mathbb{N}$ such that

$$\beta := \frac{\alpha+1}{2} > \sup\left\{ \left|\frac{a_{k+1}}{a_k}\right| \,\middle|\, k \geq K \right\}.$$

Then, $|a_{k+1}| < \beta |a_k|$ for every $k \geq K$. So, $|a_{K+n}| < \beta^n |a_K|$. By the comparison test (thm 3.25), $\sum a_n$ converges absolutely.

(b) Now suppose $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$. i.e. $|a_n| \geq |a_{n_0}|$. Then, $a_n \not\to 0$ so $\sum a_n$ is divergent.

Question. Is it possible for the root test to fail, but for the ratio test to be conclusive? Is there such a series, which (a) converges? (b) diverges?

Proof. Yes it is possible. Clearly, the alternating series $\{\sum(-1)^n\}$ fails the root test, but the ratio test tells us it diverges.
But, no such convergent series can be found. Suppose a series $\sum a_n$ converges by the ratio test. We use the notation of the above proof (for thm 3.34). Notice that

$$\limsup \sqrt[n]{|a_{K+n}|} \leq \limsup \beta \sqrt[n]{|a_K|} = \beta < 1.$$

Hence, the root test also says ($\sum a_{K+n}$ and therefore) $\sum a_n$ converges.

Question. Is it possible for the ratio test to fail, but for the root test to be conclusive? Is there such a series, which (a) converges? (b) diverges?

Proof. Yes it is possible.

(a) Consider the sequence defined by $a_{2n+1} := a_{2n} := 2^{-n}$. Since $\frac{a_{2n+1}}{a_{2n}} = 1$ and $\frac{a_{2n}}{a_{2n-1}} = \frac{1}{2}$ for every $n \in \mathbb{N}$, we notice the ratio test fails. However, as $\limsup \sqrt[n]{2^{-2n}} = 1/4$, the root test tells us the series $\sum a_n$ is convergent.

(b) Now let $b_{2n} := 2^{-n}$ and $b_{2n+1} := 2^n$. Then, $\left|\frac{b_{2n+1}}{b_{2n}}\right| = 2^{2n}$ and $\left|\frac{b_{2n}}{b_{2n-1}}\right| = 2^{1-2n}$. Hence, the ratio test is inconclusive. But since $\limsup \sqrt[n]{|b_n|} = \lim \sqrt{2}^n = \infty$, the root test implies the series $\sum b_n$ diverges.

Question. Is it possible for the root and ratio tests to fail simultaneously? Is there such a series, which (a) converges? (b) diverges?

Proof. Yes, it is possible. They both fail for the alternating harmonic series $\left\{\sum \frac{(-1)^n}{n}\right\}$ (which converges) and harmonic series $\{\sum \frac{1}{n}\}$ (which diverges).
Let $\varepsilon > 0$. Since $(\varepsilon + 1)^k > 1 + k\varepsilon \to \infty$, we see that

$$\frac{(k+1) - k}{(\varepsilon+1)^k - (\varepsilon+1)^{k-1}} = \frac{1}{\varepsilon(\varepsilon+1)^{k-1}} \to 0.$$

By the Stolz-Cesaro theorem, we pick $K \in \mathbb{N}$ such that $k + 1 < (\varepsilon + 1)^k$, for every $k \geq K$. Simplifying this gives $\sqrt[k]{k+1} - 1 < \varepsilon$. Hence, $\sqrt[k]{k+1} \to 1$, i.e. $\sqrt[k]{\frac{1}{k+1}} \to 1$. As claimed, the root test fails.
The ratio test also fails as $\limsup \frac{n}{n+1} = 1$, but $\frac{n}{n+1} = 1 - \frac{1}{n+1} < 1$ for all $n \in \mathbb{N}$.

Theorem 3.37. For any sequence $\{c_n\}$ of positive numbers,

$$\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n},$$
$$\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}.$$

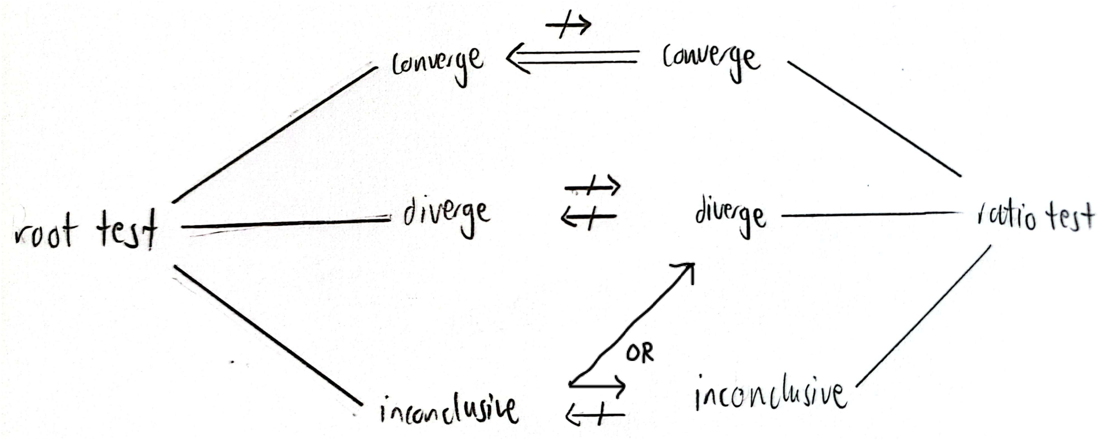


Figure 3.1: The interplay between the root and ratio tests.

Proof. Let $\alpha := \limsup \frac{c_{n+1}}{c_n}$ and $\varepsilon > 0$. Then, pick $K \in \mathbb{N}$, such that

$$\alpha + \varepsilon > \sup \left\{ \frac{c_{k+1}}{c_k} \,\middle|\, k \geq K \right\}.$$

As such, $c_{k+1} < (\alpha + \varepsilon) c_k$ for each $k \geq K$. That is, $c_{K+n} < (\alpha+\varepsilon)^n c_K$. So,

$$0 \leq \sqrt[K+n]{c_{K+n}} < (\alpha+\varepsilon)^{\frac{n}{K+n}} \sqrt[K+n]{c_K} \to \alpha + \varepsilon.$$

This implies

$$\limsup \sqrt[n]{c_n} = \limsup \sqrt[K+n]{c_{K+n}} \leq a + \varepsilon.$$

Consequently, $\limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$. The proof for $\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n}$ is similar.

Question. Are there series $\{a_n\}$ and $\{b_n\}$ for which

$$\limsup(a_n + b_n) < \limsup a_n + \limsup b_n?$$

Proof. Yes, consider $a_n := (-1)^n$ and $b_n := -(-1)^n$. Then, we note that

$$\limsup(a_n + b_n) = 0 \qquad \text{and} \qquad \limsup a_n + \limsup b_n = 1 + 1 = 2.$$

Question. Is there a series $\{a_n\}$ such that $\limsup \left| \frac{a_{n+1}}{a_n} \right| = 0$?

Proof. Yes, let $a_n := 0.5^{x!}$. Then, $\limsup \left| \frac{a_{n+1}}{a_n} \right| = \frac{0.5^{(x+1)!}}{0.5^{x!}} = 0.5^{x+1} = 0.$

Exercise. Assume that absolute convergence implies convergence in an ordered field $\mathbb{F}$. Then, is $\mathbb{F}$ complete?

Theorem 3.42 (Dirichlet's Test). Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(c) $\lim b_n = 0$.

Then, $\sum a_n b_n$ converges.

Proof. Let $\varepsilon > 0$ and fix an upper bound $\mathcal{B} > 0$ of $|A_n|$. We pick $N \in \mathbb{N}$, such that

$$b_m - b_n < \frac{\varepsilon}{2\mathcal{B}} \qquad \text{and} \qquad |b_m| < \frac{\varepsilon}{4\mathcal{B}},$$

for $n \geq m \geq N$. Then, using summation by parts (thm 3.41),

$$\begin{aligned}\left|\sum_{j=m}^{n} a_j b_j\right| &\leq \sum_{j=m}^{n-1} |A_j||b_j - b_{j+1}| + |A_n||b_n| + |A_{m-1}||b_m| \\ &\leq \mathcal{B} \sum_{j=m}^{n-1} (b_j - b_{j+1}) + 2\mathcal{B}|b_m| \\ &= \mathcal{B}(b_m - b_n) + 2\mathcal{B}|b_m| \\ &< \mathcal{B} \cdot \frac{\varepsilon}{2\mathcal{B}} + 2\mathcal{B} \cdot \frac{\varepsilon}{4\mathcal{B}} = \varepsilon.\end{aligned}$$

Consequently, $\sum a_n b_n$ converges absolutely.

Theorem 8.29 of Tom Apostol's Mathematical Analysis:

Exercise (Abel's Test). The series $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\{b_n\}$ is a monotonic convergent sequence.

Proof. Suppose that $\sum a_n$ converges and $\{b_n\}$ is a monotonic convergent sequence. Hence, limit laws imply $\{A_n b_n\}$ and $\{A_{n-1} b_n\}$ converge to a common limit L. Thus, $\lim(A_n b_n - A_{n-1} b_n) = 0$. Now, let $\varepsilon > 0$ and $\mathcal{B}$ be an upper bound of $A_n := \sum_{k=0}^{n} a_k$. So, pick $N \in \mathbb{N}$, such that

$$|A_n b_n - A_m b_m| < \frac{\varepsilon}{3}, \qquad |A_m b_m - A_{m-1} b_m| < \frac{\varepsilon}{3}, \qquad \text{and} \qquad |b_m - b_n| < \frac{\varepsilon}{3\mathcal{B}}$$

for $n \geq m \geq N$. Therefore,

$$\begin{aligned}\left|\sum_{j=m}^{n} a_j b_j\right| &\leq \sum_{j=m}^{n} |A_j||b_j - b_{j+1}| + |A_n b_n - A_{m-1} b_m| \\ &\leq \mathcal{B}|b_m - b_n| + |A_n b_n - A_m b_m| + |A_m b_m - A_{m-1} b_m| \\ &< \mathcal{B} \cdot \frac{\varepsilon}{3\mathcal{B}} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

As such, $\sum a_n b_n$ converges.

§3.3 Hw 6

Exercise 3.6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$;

(b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$;

(c) $a_n = (\sqrt[n]{n} - 1)^n$;

(d) $a_n = \dfrac{1}{1+z^n}$, for complex values of z.

Proof.

(a) Notice the partial sums simplify as follows:

$$\sum_{i=1}^{n} a_i = \sum_{i=2}^{n+1} \sqrt{i} - \sum_{i=1}^{n} \sqrt{n} = \sqrt{n+1} - 1.$$

Hence, the series $\sum a_n$ diverges to ∞.

(b) This series converges by (a) and the comparison test.

(c) Since $\limsup \sqrt[n]{n} - 1 = 1 - 1 = 0$, the root test ensures $\sum(\sqrt[n]{n} - 1)^n$ converges.

(d) If $|z| > 1$, then

$$\left|\frac{1+z^n}{1+z^{n+1}}\right| = \left|\frac{\frac{1}{z^{n+1}} + \frac{1}{z}}{\frac{1}{z^{n+1}} + 1}\right| \to \frac{1}{|z|} < 1.$$

Hence, the ratio test implies $\sum a_n$ converges. But when $|z| \le 1$, then $\frac{1}{1+z^n} \not\to 0$ by limit laws, i.e. $\sum a_n$ is divergent.

Exercise 3.7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \ge 0$.

Proof. Notice that $a_n \ge 1/n^2$ and $a_n \le 1/n^2$ respectively imply $a_n \ge \sqrt{a_n}/n$ and $1/n^2 \ge \sqrt{a_n}/n$. So,

$$\sum \frac{\sqrt{a_n}}{n} \le \sum a_n + \sum \frac{1}{n^2}$$

converges.

Exercise 3.11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \qquad \text{and} \qquad \sum \frac{a_n}{1+n^2 a_n} \quad ?$$

Proof.

(a) Suppose, for contradiction, that $\sum \frac{a_n}{1+a_n}$ converges. Then, since $a_n > 0$ and $\frac{a_n}{1+a_n} = 1 - \frac{1}{1+a_n}$, the sequence $\{1 + a_n\}$ is (eventually) nonincreasing, converging to 1. As such, Abel's Test implies $\sum a_n = \sum \frac{a_n}{1+a_n} \cdot (1 + a_n)$ converges. A contradiction.

(b) Let $N \in \mathbb{N}$. Because $s_n \to \infty$, there is a $k \in \mathbb{N}$ with $s_{N+k} > 2s_N$. Furthermore, $\{s_n\}$ is increasing. Hence,

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}} > \frac{1}{2}.$$

i.e. $\sum \frac{a_n}{s_n}$ diverges (to infinity).

(c) Again, $\{s_n\}$ is increasing. Thus,

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

holds for every $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and pick $N \in \mathbb{N}$, such that $s_m > 1/\varepsilon$ if $m \geq N$. Then,

$$\sum_{j=m+1}^{n} \frac{a_j}{s_j^2} \leq \sum_{j=m+1}^{n} \frac{1}{s_{j-1}} - \frac{1}{s_j} = \frac{1}{s_m} - \frac{1}{s_n} \leq \frac{1}{s_m} < \varepsilon.$$

The Cauchy Criterion is, therefore, met; $\sum \frac{a_n}{s_n^2}$ converges.

(d) If there is a lower bound $l > 0$ of $\{a_n\}$, then

$$\frac{a_n}{1+na_n} = \frac{1}{n}\left(1 - \frac{1}{1+na_n}\right) \geq \frac{1}{n}\left(1 - \frac{1}{1+l}\right).$$

By the p-series test (thm 3.28), $\sum \frac{a_n}{1+na_n} \to \infty$. When $a_n \to 0$, it is still possible for divergence to occur: consider $a_n := 1/n$. Then,

$$\sum \frac{a_n}{1+na_n} = \sum \frac{1}{2n} \to \infty.$$

But, convergence can, too, occur. We let

$$a_n := \begin{cases} 1 & \text{if } n = 2^k \text{ for some integer } k, \\ \frac{1}{n^2} & \text{otherwise.} \end{cases}$$

Then,

$$\sum a_n \geq \sum 1 = \infty.$$

Simultaneously,

$$\sum \frac{a_n}{1 + na_n} \leq \sum \frac{1}{n(n+1)} + \sum \frac{1}{1 + 2^n} \leq \sum \frac{1}{n^2} + \sum \frac{1}{2^n}.$$

So, $\sum \frac{a_n}{1+na_n}$ must converge. For an illustration, see Figure 3.2.

The latter series is simpler. Since

$$\sum \frac{a_n}{1 + n^2 a_n} = \sum \frac{1}{n^2} \left(1 - \frac{1}{1 + n^2 a_n}\right) \leq \sum \frac{1}{n^2},$$

convergence is evident.

Note. Since $b_n := \frac{a_n}{1+na_n} = \frac{1}{n}\left(1 - \frac{1}{1+na_n}\right)$, we might be tempted to think that $na_n \to 0$ for $\sum b_n$ to converge. But, our example above shows that this is unnecessary; we even had $\limsup a_n = \limsup 2^n = \infty$. It is, however, necessary for $\liminf a_n = 0$, lest the series diverges by the comparison test.

Naturally, this leads us to the following question:

Question. Let $\{a_n\}$ be a positive sequence, such that $\sum a_n \to \infty$.

(a) Is it possible for $na_n \to 0$?

(b) If so, is it plausible that we simultaneously have

$$\sum \frac{a_n}{1 + na_n}$$

converging?

Proof.

(a) Yes, simply let $a_n := (n \log n)^{-1}$.

(b) No. Suppose (a) holds and pick $N \in \mathbb{N}$, such that $na_n < 1$, for all $n \geq N$. Then,

$$\frac{a_n}{1 + na_n} > \frac{1}{2} a_n.$$

Hence, (b) cannot hold:

$$\sum \frac{a_n}{1 + na_n} \geq \frac{1}{2} \sum_{n \geq N} a_n = \infty.$$

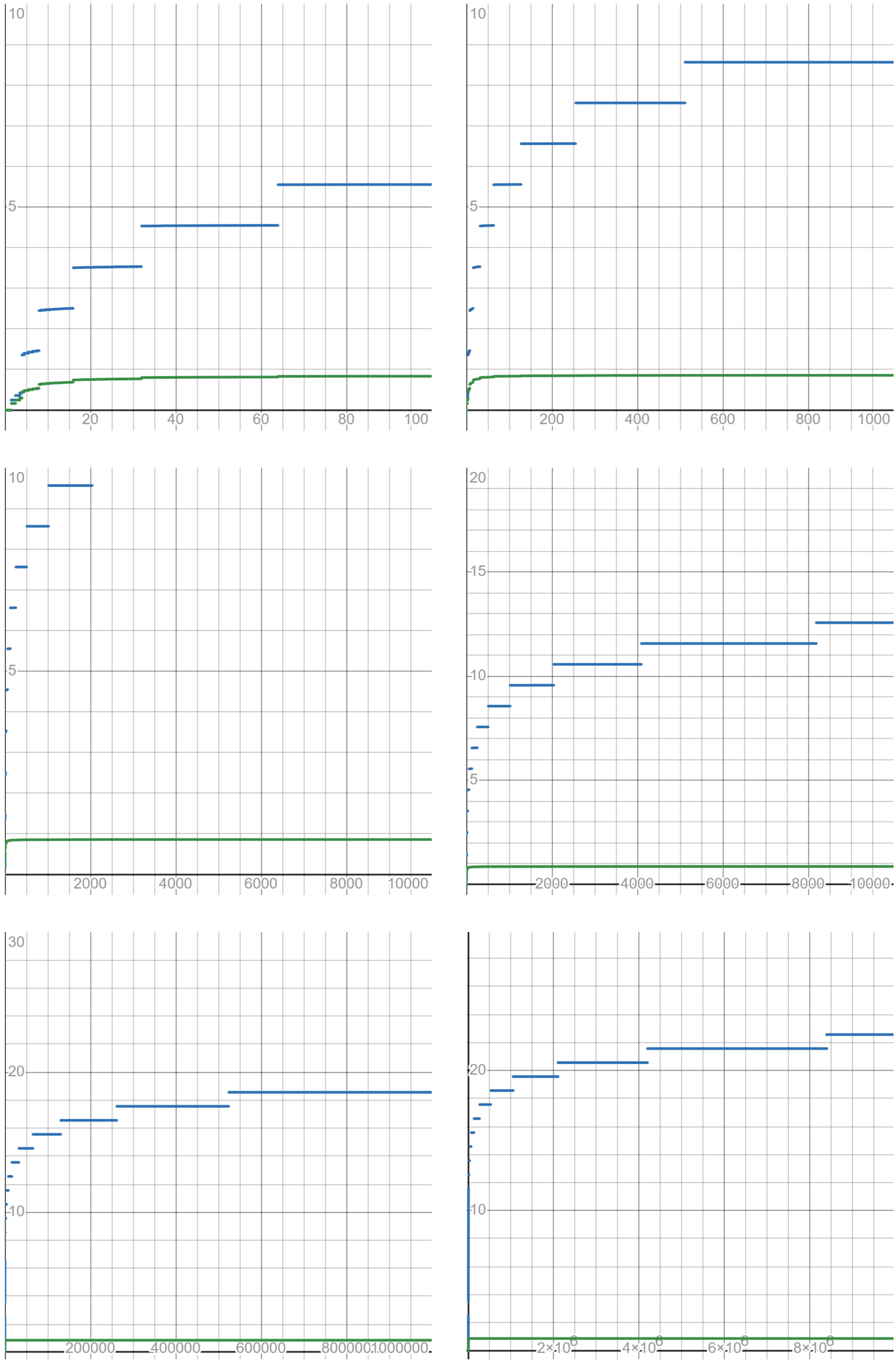


Figure 3.2: An illustration for when $\sum a_n$ (blue) diverges and $\sum \frac{a_n}{1+na_n}$ (green) converges (Desmos).

Exercise 3.12. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges.
[This gives a converse of (a), but under the additional assumption that $n a_n \to 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|n a_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:
If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then, $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

Proof.

(a) Let $\varepsilon > 0$ and pick $M \in \mathbb{N}$, such that $|s_m - s| < \varepsilon/2$ for all $m \geq M$. Then choose $N \geq M$ with

$$\frac{|s_i - s|}{N} < \frac{\varepsilon}{2M},$$

for every $0 \leq i \leq M-1$. So, for $n \geq N$,

$$\begin{aligned}\left|\frac{s_0+s_1+\cdots+s_n}{n+1}-s\right| &< \sum_{i=1}^{M-1}\frac{|s_i-s|}{N}+\sum_{i=M}^{n}\frac{|s_i-s|}{n+1}\\ &< \left(1-\frac{1}{M}\right)\cdot\frac{\varepsilon}{2}+\left(1-\frac{M}{n+1}\right)\cdot\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

Hence, $\lim \sigma_n = s$.

(b) Consider the alternating sequence $\{(-1)^n\}$, which is clearly divergent. Then, $|\sigma_n| \leq \frac{1}{n+1}$ so $\lim \sigma_n = 0$.

(c) Yes, let

$$s_n := \begin{cases} k & \text{if } n = k^3 \text{ for some integer } k,\\ (-1)^n & \text{otherwise.}\end{cases}$$

Pick any integer n and suppose $k^3 \leq n < (k+1)^3$. Then,

$$|\sigma_n| \leq \frac{1+(1+2+\cdots+k)}{2^k} < \frac{k^2+k+1}{k^3} = \frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}.$$

Thus, $\lim \sigma_n = 0$. Simultaneously, as $\lim s_{k^3} = \infty$, we know $\limsup s_n = \infty$.

(d) As expected, we notice

$$\begin{aligned}s_n-\sigma_n &= \frac{1}{n+1}\left[(n+1)s_n-s_0-s_1-\cdots-s_n\right]\\ &= \frac{1}{n+1}\sum_{k=1}^{n} s_n - s_{k-1}\\ &= \frac{1}{n+1}\sum_{k=1}^{n}\sum_{j=k}^{n} a_j\\ &= \frac{1}{n+1}\sum_{k=1}^{n} k a_k.\end{aligned}$$

From (a), we deduce $\lim \frac{1}{n+1}\sum_{k=1}^{n} k a_k = \lim(n a_n) = 0$. So, $\lim s_n = \lim \sigma_n$; the sequence s_n converges.

(e) First notice that, if $n > m$, then

$$\sigma_n-\sigma_m = \frac{m-n}{m+1}\cdot\frac{(s_0+s_1+\cdots+s_n)}{n+1}+\frac{s_{m+1}+s_{m+2}+\cdots+s_n}{n+1}$$

and

$$\frac{1}{n-m}\sum_{i=m+1}^{n}(s_n-s_i) = s_n-\frac{s_{m+1}+s_{m+2}+\cdots+s_n}{n-m}.$$

As such,

$$\begin{aligned}
&\frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i) \\
&= -\frac{s_0 + s_1 + \cdots + s_n}{n+1} + \left(\frac{m+1}{n+1} - 1\right) \cdot \frac{s_{m+1} + s_{m+2} + \cdots + s_n}{n-m} + s_n \\
&= -\frac{s_0 + s_1 + \cdots + s_n}{n+1} - \frac{s_{m+1} + s_{m+2} + \cdots + s_n}{n+1} + s_n \\
&= s_n - \sigma_n.
\end{aligned}$$

For these i,

$$|s_n - s_i| \le \frac{1}{i+1}\sum_{k=i+1}^{n}(i+1)|a_k| \le \frac{1}{i+1}\sum_{k=i+1}^{n}|ka_k| \le \frac{(n-i)M}{i+1}.$$

The latter inequality hence follows. Now, fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \le \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then,

$$\begin{aligned}
m + m\varepsilon \le n - \varepsilon \qquad &\text{and} \qquad n - \varepsilon < (M+1) + M + 1, \\
\frac{m+1}{n-m} \le \frac{1}{\varepsilon} \qquad &\text{and} \qquad \varepsilon > \frac{n-m-1}{m+2}.
\end{aligned}$$

From the latter, we deduce $|s_n - s_i| < M\varepsilon$. Now pick $N \in \mathbb{N}$, such that

$$|\sigma_n - \sigma| < \varepsilon \qquad \text{and} \qquad |\sigma_n - \sigma_m| < \varepsilon^2,$$

for all $n \ge m \ge N$. Consequently,

$$\begin{aligned}
|s_n - \sigma| &\le |s_n - \sigma_n| + |\sigma_n - \sigma| \\
&\le |\sigma_n - \sigma| + \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{i=m+1}^{n}|s_n - s_i| \\
&< \varepsilon + \frac{\varepsilon^2}{\varepsilon} + \frac{(n-m)M\varepsilon}{n-m} = (M+2)\varepsilon.
\end{aligned}$$

So, $\limsup|s_n - \sigma| \le (M+2)\varepsilon$ (and $\liminf|s_n - \sigma| \ge 0$ is self-explanatory). Since ε was arbitrary, $\lim s_n = \sigma$.

Exercise 3.16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$, by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \dots).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \qquad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Proof.

(a) As the discriminant $(-2\sqrt{\alpha})^2 - 4(1)(\alpha) = 0$, we know $x_n^2 + \alpha \geq 2\sqrt{\alpha}x_n$. i.e. $x_{n+1} \geq \sqrt{\alpha}$. Hence, $x_n \geq \sqrt{\alpha}$ for all $n \in \mathbb{N}$, making $x_{n+1} \leq x_n$ clear. Now letting $L := \lim x_n$, we see that $L = \frac{1}{2}\left(L + \frac{\alpha}{L}\right)$. Thus $L = \sqrt{\alpha}$ follows.

(b) We see that

$$\varepsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{x_n^2 - 2\sqrt{\alpha}x_n + \alpha}{2x_n} = \frac{\varepsilon_n^2}{2x_n}.$$

The given inequality hence holds, since $\{x_n\}$ is bounded below by $\sqrt{\alpha}$. Accordingly,

$$\varepsilon_2 < \frac{\varepsilon_1^2}{\beta} = \beta\left(\frac{\varepsilon_1}{\beta}\right)^2.$$

Furthermore, presuming $\varepsilon_{k+1} < \beta(\varepsilon_1/\beta)^{2^k}$, we notice

$$\varepsilon_{k+2} < \frac{\varepsilon_{k+1}^2}{\beta} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^{k+1}}.$$

(c) Fix $\alpha = 3$ and $x_1 = 2$. Then, since $81/25 > 3$,

$$\frac{\varepsilon_1}{\beta} = \frac{\sqrt{3}}{3} - \frac{1}{2} < \frac{\sqrt{81/25}}{3} - \frac{1}{2} = \frac{1}{10}.$$

Moreover, because $4 > 3$,

$$\varepsilon_5 < 2\sqrt{3}\left(\frac{1}{10}\right)^{2^4} < 4 \cdot 10^{-16}, \qquad \varepsilon_6 < 2\sqrt{3}\left(\frac{1}{10}\right)^{2^5} < 4 \cdot 10^{-32}.$$

Exercise 3.20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Proof. Let $\varepsilon > 0$. Pick $I \in \mathbb{N}$, such that

$$d(p_n - p_m) < \frac{\varepsilon}{2} \qquad \text{and} \qquad d(p_{n_I} - p) < \frac{\varepsilon}{2},$$

for every $n \geq m \geq n_I$. Then, the full sequence $\{p_n\}$ converges to p, since

$$d(p_n - p) \leq d(p_n - p_{n_I}) + d(p_{n_I} - p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Exercise 3.22 (Baire's theorem). Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap G_n$ is not empty. (In fact, it is dense in X.)

Proof. Pick a sequence of points $\{g_n\}$, such that $d(g_n, g_{n-1}) < 0.5^n/n$ and $g_n \in G_n \cap \bigcap_{m=1}^{n-1} N_{r_m}(g_m)$, where $N_{r_m} \subseteq G_m$. Notice that, for $n \geq m \geq N$,

$$d(g_n, g_m) \leq \sum_{i=n}^{m-1} d(g_n, g_{n+1}) \leq \sum_{i=1}^{\infty} \frac{0.5^i}{N} = \frac{1}{N}.$$

i.e. $\{g_n\}$ is Cauchy. Hence, by Completeness it converges to some limit $\mathfrak{g}$. Furthermore, since $g_n, g_{n+1}, \ldots \in N_{r_n}(g_n)$, it follows that $\mathfrak{g} \in N_{r_n}(g_n)$ for each n. Hence, $\mathfrak{g} \in \bigcap G_n$. See Figure 3.3 for an illustration. Now, let $x \in X$ and $\varepsilon > 0$; pick $g_1 \in G_1 \cap N_{\varepsilon/2}(x)$. By choosing $r_1 \leq \varepsilon/2$, we have $\mathfrak{g} \in N_{\varepsilon/2}(g_1)$. So, $\mathfrak{g} \in N_{\varepsilon}(x)$; the intersection $\bigcap G_n$ is dense in X.

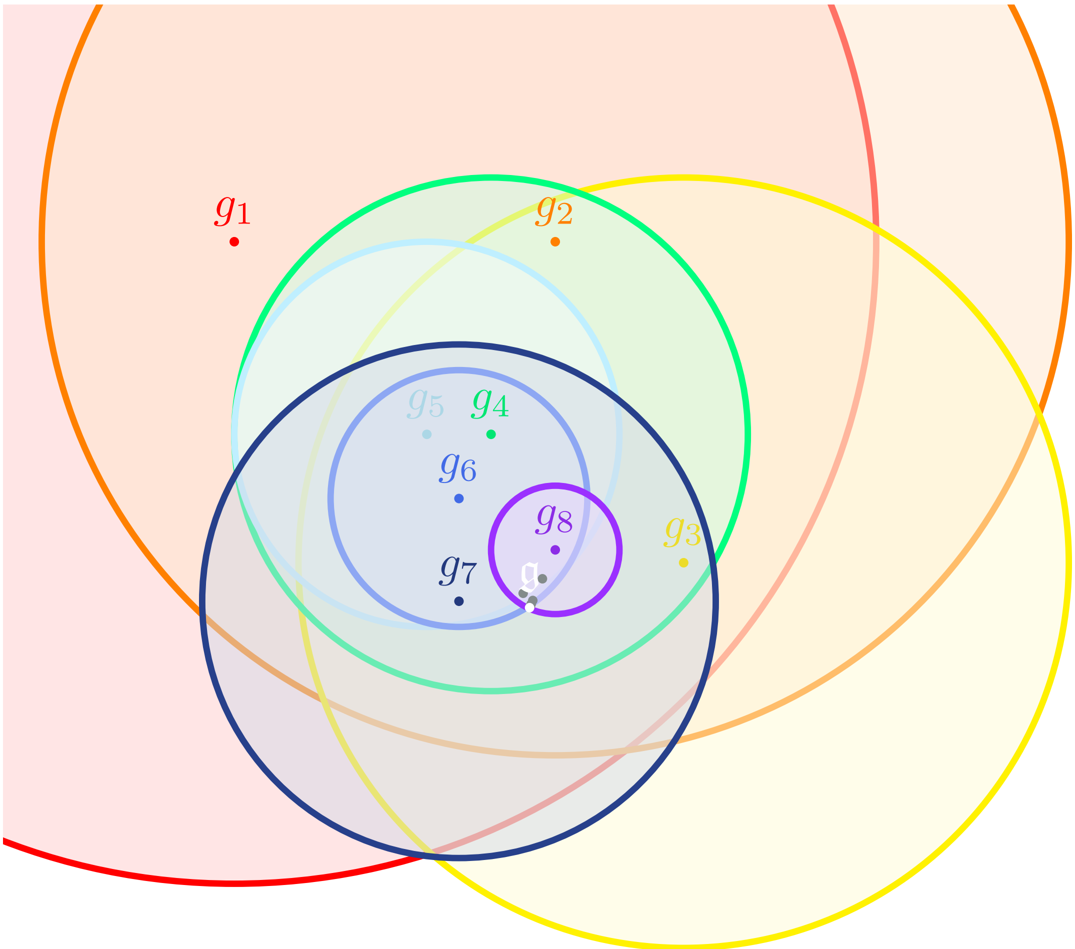


Figure 3.3: An illustration of the above procedure to obtain $\mathfrak{g} \in \bigcap G_n$.

Exercise 3.23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

Proof. Let $\varepsilon > 0$. We pick $N \in \mathbb{N}$, such that

$$d(p_n, p_m) < \frac{\varepsilon}{2} \qquad \text{and} \qquad d(q_n, q_m) < \frac{\varepsilon}{2},$$

for any $n \geq m \geq N$. Then, the reverse triangular inequality implies

$$\begin{aligned} |d(p_n, q_n) - d(p_m, q_m)| &\leq |d(p_n, q_n) - d(q_n, p_m)| + |d(q_n, p_m) - d(p_m, q_m)| \\ &\leq d(p_n, p_m) + d(q_n, q_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, by the completeness of $\mathbb{R}$, the Cauchy sequence $\{d(p_n, q_n)\}$ converges.

Exercise 3.24. Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e. a distance preserving mapping) of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

Proof.

(a) As illustrated below, all three conditions of being an equivalence relation are satisfied.

Reflexivity: The zero sequence $\{d(p_n, p_n)\}$ always converges to zero.

Symmetry: This is clear, since d is a metric.

Transitivity: Let $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$ be Cauchy sequences, such that

$$\lim d(p_n, q_n) = \lim d(q_n, r_n) = 0.$$

By the triangle inequality and the Squeeze Theorem, it is clear that $\lim d(p_n, r_n) = 0$.

(b) Let $\varepsilon > 0$, $\{p_n\}, \{b_n\} \in P$, and $\{q_n\}, \{d_n\} \in Q$. We pick $N \in \mathbb{N}$, such that

$$d(p_n, b_n) < \frac{\varepsilon}{2} \qquad \text{and} \qquad d(q_n, d_n) < \frac{\varepsilon}{2},$$

for each $n \geq N$. Then, $\lim d(p_n, q_n) = \lim d(b_n, d_n)$, because

$$\begin{aligned} |d(p_n, q_n) - d(b_n, d_n)| &\leq |d(p_n, q_n) - d(q_n, b_n)| + |d(q_n, b_n) - d(b_n, d_n)| \\ &\leq d(p_n, b_n) + d(q_n, d_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

i.e. $\Delta\colon X^* \times X^* \to \mathbb{R}_0^+$ is well-defined. In fact, since d is a metric, so is Δ.

(c) Let $\{P_r\}$ be a Cauchy sequence in X^*. We shall construct a sequence $\{q_k\}$ whose limit is $\lim P_r$.

For each $r \in \mathbb{N}$, choose a $\left\{p_i^{(r)}\right\} \in P_r$. Fix $k \in \mathbb{N}$ and pick the least $N_k \in \mathbb{N}$, such that $\Delta(P_n, P_m) < 1/k$, for any $n \geq m \geq N_k$. Now pick $I_k \in \mathbb{N}$, such that $d\left(p_i^{(N_k)}, p_j^{(N_k)}\right) < 1/k$, for all $i \geq j \geq I_k$. Hence, define $q_k := p_{I_k}^{(N_k)}$. We proceed to verify that $\{q_k\}$ is Cauchy.

Fix $\alpha \geq \beta \geq k$. We notice $\Delta(P_{N_\alpha}, P_{N_\beta}) < 1/k$. So, pick $\Gamma \geq I_\alpha, I_\beta$ such that, for $\gamma \geq \Gamma$, we have $d\left(p_\gamma^{(N_\alpha)}, p_\gamma^{(N_\beta)}\right) < 2/k$. Then,

$$d(q_\alpha, q_\beta) \leq d\left(p_{I_\alpha}^{(N_\alpha)}, p_\gamma^{(N_\alpha)}\right) + d\left(p_\gamma^{(N_\alpha)}, p_\gamma^{(N_\beta)}\right) + d\left(p_\gamma^{(N_\beta)}, p_{I_\beta}^{(N_\beta)}\right) < \frac{4}{k}.$$

As such, $\{q_k\}$ is a Cauchy sequence in X and we let $Q \in X^*$ denote its equivalence class. Finally, we show that $Q = \lim P_r$.

Fix $k \in \mathbb{N}$ and $r \geq N_k$. Pick $\Lambda \geq N_k$, such that $d\left(p_{I_\mu}^{(N_\mu)}, p_{I_\lambda}^{(N_\lambda)}\right) < 1/k$, for $\lambda \geq \mu \geq \Lambda$. By leastness, $N_\lambda \geq N_\mu \geq N_k$. Therefore, $d\left(p_\lambda^{(r)}, p_\delta^{(N_\mu)}\right) < 2/k$ for some $\delta \geq I_\mu$. As $\delta \geq I_\mu$, we see that $d\left(p_\delta^{(N_\mu)}, p_{I_\mu}^{(N_\mu)}\right) < 1/k$. Thus,

$$d\left(p_\lambda^{(r)}, q_\lambda\right) \leq d\left(p_\lambda^{(n)}, p_\delta^{(N_\mu)}\right) + d\left(p_\delta^{(N_\mu)}, p_{I_\mu}^{(N_\mu)}\right) + d\left(p_{I_\mu}^{(N_\mu)}, p_{I_\lambda}^{(N_\lambda)}\right) < \frac{4}{k}.$$

In other words, $\Delta(P_r, Q) \leq 4/k$ for $r \geq N_k$. So, $\lim P_r = Q$. See Figures 3.4 and 3.5 for an illustration.

(d) Since every P_p is a constant sequence whose terms are all p,

$$\Delta(P_p, P_q) := \lim_{n\to\infty} d(p, q) = d(p, q).$$

(e) Let $\{p_n\} \in P \in X^* - \varphi(X)$ and $\varepsilon > 0$. Then, pick $N \in \mathbb{N}$, such that $d(p_n, p_m) < \varepsilon/2$ for all $n \geq m \geq N$. Accordingly, $\Delta(P, P_{p_m}) = \lim_{n\to\infty} d(p_n, p_m) < \varepsilon$; the isomorphic embedding $\varphi(X)$ is dense in X^*. For the following proof, we shall reuse the notation defined in (c). Assume X is complete and notice that $\{q_k\}$ converges to a limit $\mathfrak{q} \in X$. So, $Q = P_{\mathfrak{q}} \in X^*$.

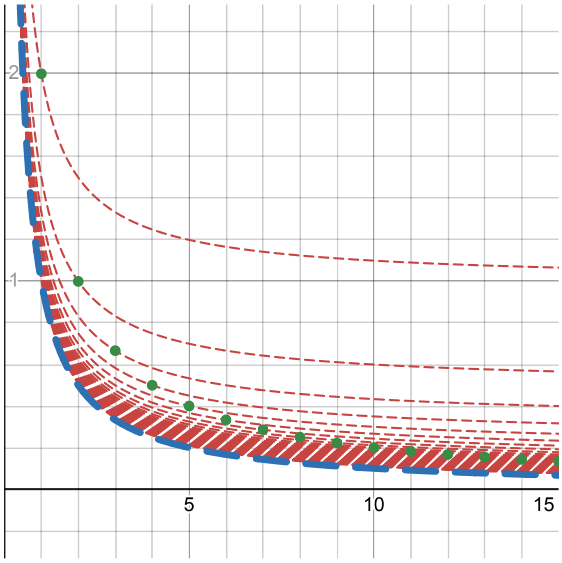


Figure 3.4: A Cauchy sequence, whose equivalence class is the limit of the Cauchy sequence of equivalence classes of the red Cauchy sequences, in $\mathbb{R} - \{0\}$ (Desmos).

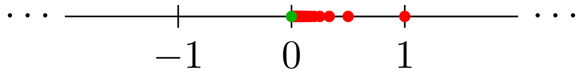


Figure 3.5: The picture of Figure 3.4 in the completion $(\mathbb{R} - \{0\})^* \cong \mathbb{R}$.

Claim. Let X be a complete metric space and $E \subseteq X$. Then, E is complete iff E is closed (relative to X).

Proof. Let x be a limit point of the complete metric space E. Now, pick a sequence $\{p_n\}$ in E that converges to x. Since $\{p_n\}$ is a Cauchy sequence in E, we are certain that its limit $x \in E$. Hence, E is closed in X.
Conversely, let $\{q_n\}$ be a Cauchy sequence in the closed set E. Then, by the completeness of X, it converges to some $x \in X$. Since this is a limit point of E, it is included in E. As such, E is complete.

Note. A metric space X is complete iff if it (more accurately, $\varphi(X)$) is closed relative to X^*.

Claim. A metric space X is globally closed iff it is complete.

Proof. If X is globally closed, it is closed relative to its completion. Hence X would be complete, by the preceding claim. Conversely, consider when X is complete and $X \subseteq Y$. Let y be a limit point of X. So, we pick a sequence $\{x_n\}$ in X that converges to y. As $\{x_n\}$ is a Cauchy sequence in X, its limit $y \in X$. Therefore, X is globally closed.

Claim. (Revised) If a metric space X is both globally closed/complete and bounded, it is compact.

Proof. This is false. Consider the discrete metric on $\mathbb{N}$, hence giving us a complete metric space. Then, the open cover consisting of all neighbourhoods $N_{1/2}(n)$ has no finite subcover.

Claim. A compact metric space X is globally closed/complete.

Proof.

Claim. A compact metric space X is bounded.

Claim. Let the metric space X be perfect, complete, and bounded. Then, X is compact.

Chapter 4

Continuity

§4.1 Theorems

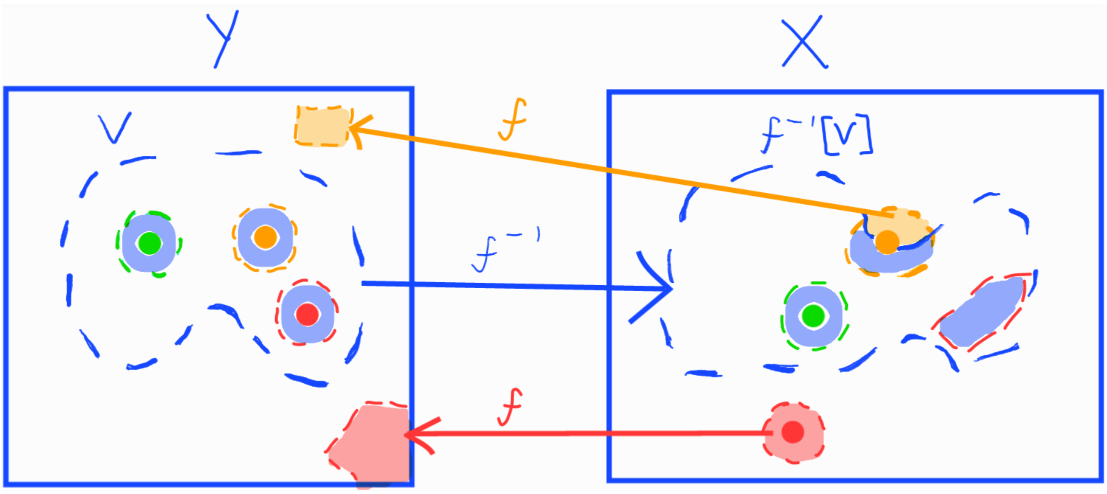


Figure 4.1: An illustration of topological continuity: f is continuous at ●, but is discontinuous at ● and ●.

Theorem 4.7. Let X, Y, Z be metric spaces, $E \subseteq X$, $f\colon E \to Y$ and $g\colon f[E] \to Z$. If f is continuous at a point $p \in E$ and g is continuous at the point $f(p)$, then $h := g \circ f$ is continuous at p.

Proof. Let $\varepsilon > 0$. By continuity, we can pick $\delta, \eta > 0$, such that $g[N_\delta(f(p))] \subseteq N_\varepsilon(h(p))$ and $f[N_\eta(p)] \subseteq N_\delta(f(p))$. Hence, $h[N_\eta(p)] \subseteq N_\varepsilon(h(p))$.

Claim. Let $f\colon X \to Y$ and $g\colon Y \to Z$. Then, if [find necessary conditions], then $\lim_{x\to p}(g \circ f)(x) = \lim_{y\to q} g(y)$. (What about the converse?)

Theorem 4.8. Let X and Y be metric spaces. Then, any function $f\colon X \to Y$ is continuous iff $f^{-1}[V]$ is open for every open set V.

Proof. Consider when $f^{-1}[V]$ is open for every open set V. So, let $\varepsilon > 0$ and $x \in X$. Since $f^{-1}[N_\varepsilon(f(x))]$ is open, it contains $N_\delta(x)$ for some $\delta > 0$. Thus, $f[N_\delta(x)] \subseteq N_\varepsilon(f(x))$, i.e. f is continuous at x.
Now consider when $N_r(x) \nsubseteq f^{-1}[V]$ for some open set V, point $x \in f^{-1}[V]$, and *all* $r > 0$. So, for each $n \in \mathbb{N}$, pick $z_n \in N_{1/n}(x) - f^{-1}[V]$. Furthermore, as V is open, $N_\varepsilon(f(x)) \subseteq V$ for some $\varepsilon > 0$. Now, $d_Y(f(z_n), f(x)) \geq \varepsilon$. Hence, f is not continuous at x.

Proof. A direct proof for the ($\implies$) direction. Let $f\colon X \to Y$ be continuous, V an open subset of Y, and $x \in f^{-1}[V]$. Thus, $f[N_\delta(x)] \subseteq N_\varepsilon(f(x)) \subseteq V$ for some $\delta > 0$ and $\varepsilon > 0$. i.e. $N_\delta(x) \subseteq f^{-1}[V]$. So, $f^{-1}[V]$ is open.

Corollary (Baby Rudin page 87)**.** A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}[C]$ is closed in X for every closed set C in Y.

Theorem 4.14. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f[X]$ is compact.

Proof. Let $\{G_\alpha\}$ be an open cover of $f[X]$. By theorem 4.8, there is a finite cover $\{f^{-1}[G_{\alpha_n}]\}$ of X. Now $\{G_{\alpha_n}\}$ is a finite subcover of $f[X]$, which must hence be compact.

Observation. Any continuous function, from a compact metric space X into a metric space Y, must be bounded.

Claim. Suppose f is a continuous mapping of a complete metric space X into a metric space Y. Then $f[X]$ is complete.

Claim. If f is a continuous mapping of a closed metric space X into a metric space Y, then $f[X]$ does not have to be closed.

Theorem 4.16 (The extreme value theorem)**.** Suppose f is a continuous real function on a compact metric space X, and

$$M := \sup_{p\in X} f(p), \qquad m := \inf_{p\in X} f(p).$$

Then there exists $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Proof. This is a corollary of the preceding theorem.

Question. Does the extreme value theorem hold if X is just a complete metric space?

Theorem 4.17. Suppose f is a continuous bijection of a compact metric space X into a metric space Y. Then the inverse mapping defined by

$$f^{-1}(f(x)) := x$$

is a continuous bijection of Y into X.

Proof. Let $C \subseteq X$ be closed and $\{c_n\}$ be a sequence in C, such that $f(c_n) \to y \in \overline{E}$. Since Y is compact, there is a convergent c_{n_k}. Hence, $f[C]$ is closed. The corollary to theorem 4.8 implies the continuity of f^{-1}.

Question. If X and Y are metric spaces, such that X is complete. Then, must bounded and continuous functions $f\colon X \to Y$ be uniformly continuous?

Proof. No. A counterexample[a]: consider the bounded and continuous function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \sin(x^2)$. Let $\delta > 0$ and recall that $\lim_{x\to\infty} \sqrt{x} = 1$. So, pick $n \in \mathbb{N}$ such that $\sqrt{\pi/2 + 2n\pi} - \sqrt{2n\pi} < \delta$. We notice $\left|f\left(\sqrt{\pi/2 + 2n\pi}\right) - f\left(\sqrt{2n\pi}\right)\right| = 1$, meaning f cannot be uniformly continuous.

[a] For a *failed* counterexample, see Figure 6.1.

An alternative.

Proof. Consider the bounded continuous function $L\colon \mathbb{R}_0^+ \to \mathbb{R}$ defined by

$$L(x) := (-1)^n n \left(x - \sum_{i=1}^{n} 1/i\right) + \frac{(-1)^n + 1}{2},$$

for $x \in [\sum_{i=1}^{n-1} 1/i, \sum_{i=1}^{n} 1/i]$. See Figure 4.2 for an illustration.

Definition. Let $c_{n,m} := (\underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}, c, 0, 0, \ldots, 0) \in \mathbb{R}^m$. We write c_n for $c_{n,n}$.

Claim. Let f be a continuous mapping of a complete bounded metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof. This is false. Let the metric $\mathfrak{d}$ on $\mathfrak{X} := \bigcup_n \{c_n \mid c = 1, 1 + 1/n\}$ be defined by $\mathfrak{d}(c_n, \mathsf{c}_m) := |c_{n,m} - \mathsf{c}_m|$, for $m \geq n$. Clearly, $(\mathfrak{X}, \mathfrak{d})$ is bounded, as $\mathfrak{X} \subset \bigcup_n N_2(0_n)$. Moreover, $\mathfrak{X}$ is a set of isolated points. Hence, it is complete and $f\colon (\mathfrak{X}, \mathfrak{d}) \to (\mathbb{R}, |\cdot|)$ defined by $f(c_n) = \lceil c \rceil$ is continuous. But it is not uniformly continuous:

$$\mathfrak{d}(1_n, (1 + 1/n)_n) = 1/n \qquad \text{and} \qquad |f((1_n) - f(1 + 1/n)_n)| = 1$$

for all n. See Figure 4.3 for an illustration.

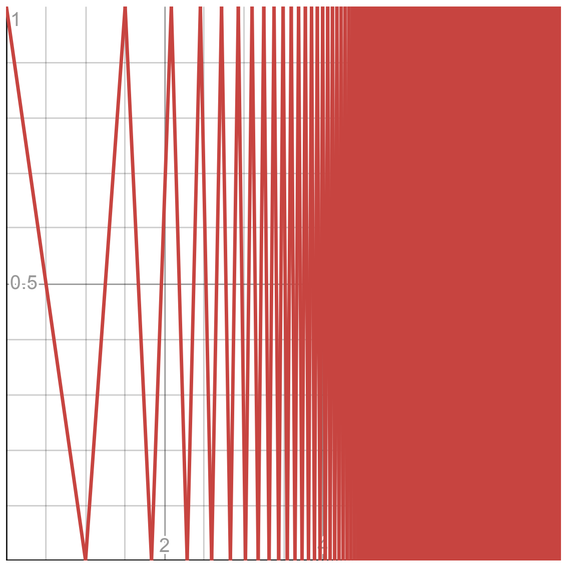
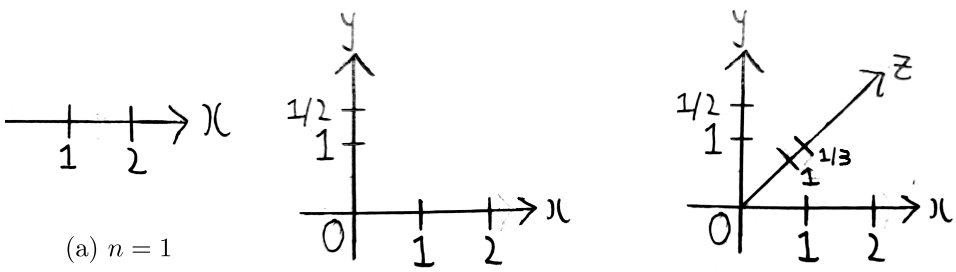


Figure 4.2: An illustration of L, which is obtained by adjoining line segments nx of horizontal width $1/n$ together (Desmos).



(a) $n = 1$

(b) $n = 1, 2$

(c) $n = 1, 2, 3$

Figure 4.3: An illustration of the set $\mathfrak{X}$, up to $n = 3$.

I thought of using infinite coordinates, but I didn't think of any nice metrics. The ℓ^p sequence spaces are useful tools to have for such situations!

Remark (Courtesy of James and Outsider from Mathcord)**.** An easier counterexample can be found in ℓ^p spaces — the set of all sequences $\{x_n\}$ for which $\sum |x_n|^p < \infty$. For $p \geq 1$, it is endowed with the metric $p(\{x_n\}, \{y_n\}) \coloneqq (\sum |x_n - y_n|^p)^{1/p}$. For $0 < p < 1$, it is endowed with the metric $p(\{x_n\}, \{y_n\}) \coloneqq \sum |x_n - y_n|^p$.

Remark. As an aside, a uniformly continuous function f on metric spaces can be unbounded. Consider the inclusion map $\iota\colon (\mathbb{N}, \mathsf{d}) \to (\mathbb{N}, |\cdot|)$, where d represents the discrete metric.

Claim. Let X and Y be metric spaces, such that every sequence in X has a convergent subsequence. Then, every continuous function $f\colon X \to Y$ is uniformly continuous.

Proof. Let $\varepsilon > 0$, $s_x \coloneqq \sup\{\delta > 0 \,|\, d(f(x), f(y)) < \varepsilon \text{ if } d(x,y) < \delta\}$, and $i \coloneqq \inf\{\delta_x \,|\, x \in X\}$. Suppose, for contradiction, that $f\colon X \to Y$ is not uniformly continuous. i.e. $i = 0$. Hence, pick a sequence $\{x_n\}$, such that it converges to some $p \in X$ and $s_{x_n} \to 0$. By continuity, there is $\delta > 0$ such that $d(f(x), f(p)) < \varepsilon/2$, for $x \in N_\delta(p)$. Choose $N \in \mathbb{N}$, such that $d(x_n, p) < \delta/2$ for $n \geq N$. Now $s_{x_n} \geq \delta/2$, a contradiction. The claim is therefore true.

Question. If every sequence in X has a convergent subsequence, then must X be compact?

Proof. Consider a cover $\{N_{r_x}(x)\}$ of X, where $r_x > 0$. Pick $x_n \notin \bigcup_{i=1}^{n-1} N_{r_{x_i}}(x_i)$, such that $r_{x_n} \geq \sup\{r_x/2 \mid x \notin \bigcup_{i=1}^{n-1} N_{r_{x_i}}(x_i)\}$. Suppose, for contradiction, that $\{x_n\}$ is infinite. Wlog, $\{x_n\}$ converges to some $p \in X$. Thus, since $d(x_n, x_{n+1}) > r_{x_n}$, we have that $r_{x_n} \to 0$. But then $r_{x_m} < r_p/2$ for some m, where $p \notin \bigcup_i N_{r_{x_i}}(x_i)$. A contradiction.

Corollary. The following statements are equivalent, for a *metric space* X.

(a) X is compact.

(b) Every open cover of X contains a finite subcover.

(c) Every sequence in X has a convergent subsequence.

(d) Every infinite subset of X contains a limit point in X.

Note. (d) does not imply that X contains only a finite number of isolated points. For instance, consider $X = \{0\} \cup \{1/n \,|\, n \in \mathbb{N}\}$.

Theorem 4.19. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof. This follows from the preceding claim. Alternatively, let $\varepsilon > 0$ and, for each $x \in X$, pick $\delta_x > 0$ such that $f[N_{\delta_x}(x)] \subseteq N_{\varepsilon/2}(f(x))$. By compactness, there is a finite subcover $\{N_{\delta_{x_n}/2}(x_n)\}$ of X. So, $f\left[N_{\min_n\{\delta_{x_n}/2\}}(x)\right] \subseteq N_\varepsilon(f(x))$ for every $x \in X$. i.e. f is uniformly continuous on X.

Question. If all continuous mappings f, from a metric space X into a metric space Y, are uniformly continuous, must X then be compact? What if the space Y is compact and infinite?

Proof. No to both. Let d denote the discrete metric. Consider the non-compact space $(\mathbb{R}, \mathsf{d})$, and the compact space $(\mathbb{R}, |\cdot|)$. Since $f[N_1(x)] = \{f(x)\}$ for all $x \in \mathbb{R}$, every function $f\colon (\mathbb{R}, \mathsf{d}) \to (\mathbb{R}, |\cdot|)$ is uniformly continuous.

Theorem 4.22. If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f[E]$ is connected.

Proof. Let $f\colon X \to Y$ be continuous; A and B be separated subsets of $f[E]$. For $x \in \overline{f^{-1}[A]}$, we have $f(x) \in \bar{A}$ by continuity. So, $x \notin f^{-1}[B]$. By symmetry, we conclude that $f^{-1}[A]$ and $f^{-1}[B]$ are separated.

Theorem 4.23 (The intermediate value theorem)**.** Let f be a continuous real function on the interval $[a, b]$. If c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Proof. Suppose, for contradiction, that $c \notin f(a, b)$. By the preceding theorem, since $[f(a), c)$ and $(c, f(b)]$ are separated sets, $[a, b]$ is not connected. A contradiction.

Definition. If X is a metric space and $E \subseteq X$, the *interior* $\operatorname{Int}_X(E)$ (or simply $\operatorname{Int}(E)$) of E is the set of all interior points of E, relative to X.

Exercise (From Eric)**.** A metric space E is disconnected (i.e. not connected) iff it is the union of two nonempty disjoint open subsets of itself.

Proof. Let E be disconnected; $E = A \cup B$ for some nonempty separated sets A and B. Notice that $\operatorname{Int}(A)^\complement \cap \operatorname{Int}(B)^\complement \subseteq \bar{B} \cap \bar{A} = \varnothing$. So, E is the union of its disjoint open subsets $\operatorname{Int}(A)$ and $\operatorname{Int}(B)$. Conversely, let $E = C \cup D$ for some open nonempty disjoint subsets C and D. Suppose, for contradiction, that $\bar{C} \cup D \neq \varnothing$. But now $C \cap N_\varepsilon(p) \neq \varnothing$, for some $N_\varepsilon(p) \subseteq D$. A contradiction.

Theorem 4.30. Let f be monotonic on (a, b). Then, the set of points of (a, b) at which f is discontinuous at is at most countable.

§4.2 (Self) Limits at infinity for metric spaces?

Definition 4.31. Let X be a metric space with $x, y \in X$. Then, the set $\mathbb{L}_{x,y}$ of all points z such that

$$d(x,z) = d(x,y) + d(y,z) \qquad \text{or} \qquad d(x,z) = d(y,z) - d(x,y)$$

is the *line induced by* x, y.

Definition 4.32. Let f be a function from a metric space X into the metric space Y, and $a, b \in X$. A point $y \in Y$ is the *limit of f at $\infty_{a,b}$* iff for each $\varepsilon > 0$ there is $M \geq 0$, such that $d(f(x), p) < \varepsilon$ whenever $d(a,x) = d(a,b) + d(b,x) \geq M$. A point $y \in Y$ is the *limit of f at $-\infty_{a,b}$* iff for each $\varepsilon > 0$ there is $M \geq 0$, such that $d(f(x), p) < \varepsilon$ whenever $d(a,x) = d(b,x) - d(a,b) \geq M$.

Is this definition consistent with the typical definition of infinite limits in $\mathbb{R}$? Does this definition obey limit laws?

§4.3 Hw 7

Exercise 4.1. Suppose f is a real function defined on $\mathbb{R}$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

Proof. No. Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Even though

$$\lim_{h \to 0} [f(h) - f(-h)] = \lim_{h \to 0} [1 - 1] = 0,$$

we notice that $\lim_{x \to 0} f(x) = 1 \neq 0 = f(0)$.

Exercise 4.2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f[\overline{E}] \subseteq \overline{f[E]}$$

for every set $E \subseteq X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f[\overline{E}]$ can be a proper subset of $\overline{f[E]}$.

Proof. Let f be a continuous mapping of a metric space X into a metric space X. Pick any limit point x of E and sequence $\{p_n\}$ in E that converges to x. By continuity, $f(x) = \lim_{n\to\infty} f(p_n) \in \overline{f[E]}$. Hence, $f[\overline{E}] \subseteq \overline{f[E]}$ is clear.
An example for when $f[\overline{E}] \neq \overline{f[E]}$. Let d denote the discrete metric. Consider the continuous function $f\colon ([0,1], \mathsf{d}) \to (\mathbb{R}, |\cdot|)$, defined by $f(x) := x$ for $x \in (0,1)$ and $f(0) := f(1) := 1$. So, $f\overline{(0,1)} = f[0,1] = (0,1]$, but $\overline{f(0,1)} = \overline{(0,1)} = [0,1]$.

Question. Is it possible for $\left|\overline{f[E]} - f[\overline{E}]\right| = |\mathbb{R}|$?

Proof. Yes! For the inclusion map $\iota\colon \mathbb{Q} \to \mathbb{R}$, we see that $\iota[\overline{\mathbb{Q}}] = \mathbb{Q}$ but $\overline{\iota[\mathbb{Q}]} = \mathbb{R}$.

Exercise 4.4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f[E]$ is dense in $f[X]$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Proof. Let p be a limit point of E and select a sequence $\{q_n\}$ in E, that converges to p. Then, since $g(q_n) = f(q_n)$ for all n,

$$g(p) = \lim_{n\to\infty} g(q_n) = \lim_{n\to\infty} f(q_n) = f(p)$$

by continuity.

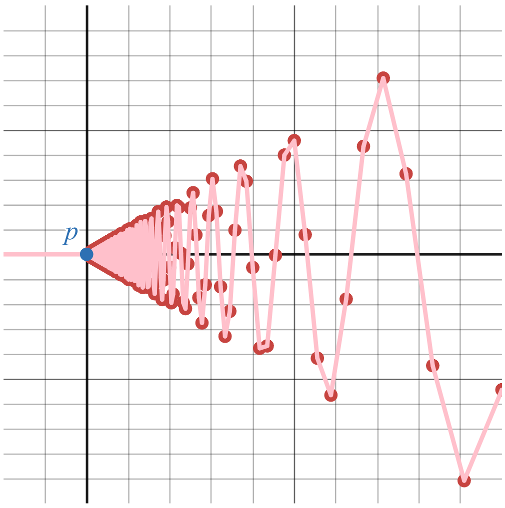


Figure 4.4: An illustration of g in the case that $E = 0 \cup \{1/n \,|\, n \in \mathbb{N}\}$ and $f(x) := \lim_{z\to x^-} \frac{\sin(x)}{x}$ (Desmos).

Exercise 4.5. If f is a real continuous function on a closed set $E \subseteq \mathbb{R}$, prove that there exist continuous real functions g on $\mathbb{R}$, such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to $\mathbb{R}$.) Show that the result become false if the word "closed" is omitted. Extend the result to vector valued functions.

Proof. Let $f\colon E \to \mathbb{R}$ be a continuous function, where E is a closed subset of $\mathbb{R}$, and pick $x \in \mathbb{R}$. Hence, define $l_x := \max\{p \in E \mid p \leq x\}$ and $u_x := \min\{p \in E \mid p \geq x\}$. Now we have the function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \left[\dfrac{f(u_x) - f(l_x)}{u_x - l_x}\right](x - l_x) + f(l_x) & \text{if } x \in (\min E, \max E) - E, \\ f(\min E) & \text{if } x \leq \min E, \\ f(\max E) & \text{if } x \geq \max E. \end{cases}$$

See Figure 4.4 for an illustration. Since line segments are continuous, so is g. (For limit points $p \in E$, fix $\varepsilon > 0$ and let $[a \pm b]_S := [a - b, a + b] \cap S$. By continuity, $f[p \pm \delta]_E \subseteq [f(p) \pm \varepsilon]_{\mathbb{R}}$ for some $\delta > 0$. Wlog, $p \pm \delta \in E$. Furthermore, $\min\{f(l_x), f(u_x)\} \leq g(x) \leq \max\{f(l_x), f(u_x)\}$. That is, $g[p \pm \delta]_{\mathbb{R}} = f[p \pm \delta]_E$.)

The word "closed" is indeed essential. Let $f\colon (0,1) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then, as $\lim_{x \to 0^+} f(x)$ does not exist, it has no continuous extension.

Now for vector-valued functions $f\colon E \to \mathbb{R}^n$, we have ________________.

Question. Let X and Y be metric spaces, and E a closed subset of X. Does every continuous function $f\colon E \to Y$ have a continuous extension to X?

Proof. No. Consider the function $f\colon [0,1] - \mathbb{Q} \to \mathbb{R}$ given by $f(x) := \frac{1}{x}$. Let $x \in [0,1] - \mathbb{Q}$ and $\varepsilon > 0$. Then, for $\delta := x - \frac{x}{1+\varepsilon x}$, we have $N_\delta(x) \subseteq \left[\frac{x}{1+\varepsilon x}, \frac{x}{1-\varepsilon x}\right]$. Hence, $f[N_\delta(x)] \subseteq N_\varepsilon(f(x))$. As such, f is continuous. But since f is unbounded, it has no continuous extensions to $\mathbb{R} - \mathbb{Q}$.

Question. Let X and Y be metric spaces, and E be a *complete* subset of X. Do all continuous functions $f\colon E \to Y$ have a continuous extension to X?

Proof. No. A counterexample: the inclusion map $\iota\colon (\{0,1\}, |\cdot|) \to (\mathbb{R}, \mathsf{d})$, where d represents the discrete metric, has no continuous extension to $(\mathbb{R}, |\cdot|)$.

Question. Let X and Y be metric spaces, and E be an complete subset of X that contains at least one limit point p. Do all continuous functions $f\colon E \to Y$ have a continuous extension to X?

Proof. No. Let $S = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$. We see that any extension of the inclusion map $\iota\colon (S, |\cdot|) \to (S, |\cdot|)$ to $(\mathbb{R}, |\cdot|)$ always has a jump discontinuities. For instance, at $\sup\{1/2 \leq x < 1 \mid f(x) = 1/2\}$.

Question. Let X and Y be metric spaces, and E a compact subset of X. Do all continuous functions $f\colon E \to Y$ have a continuous extension to X?

Proof. No. The same counterexample applies, as in the preceding question.

Observation. Even if E is a perfect compact subset of X, not all continuous functions $f\colon E \to Y$ must have a continuous extension to X.

Proof. Consider the Euclidean metric and the map $f\colon [0,1] \cup [2,3] \to \{1,3\}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 3 & \text{if } x \in [2,3]. \end{cases}$$

Any extension of f to $\mathbb{R}$ must clearly be discontinuous at $\sup\{1 \le x < 3 \mid f(x) = 1\}$.

Question. Let X and Y be metric spaces, and let E be a perfect compact subset of X. Then, if $f\colon E \to Y$ is continuous and maps limit points of E to limit points of Y, must it have a continuous extension to X?

Proof. No. See the observation below.

Observation. It is not necessary for every function from a subset E of a metric space X to a metric space Y to have a continuous extension to X, even when E and Y are both perfect and compact.

Proof. Consider the Euclidean metric and the identity map id on $[-2,-1] \cup [1,2]$. Similarly, every extension of id to $\mathbb{R}$ is discontinuous at $\sup\{x \in [-1,1] \mid \mathrm{id}(x) = -1\}$.

We notice that the source of the above counterexamples is, very informally, a hole in our codomain that completeness does not rectify. See Figure 4.5 for an illustration.

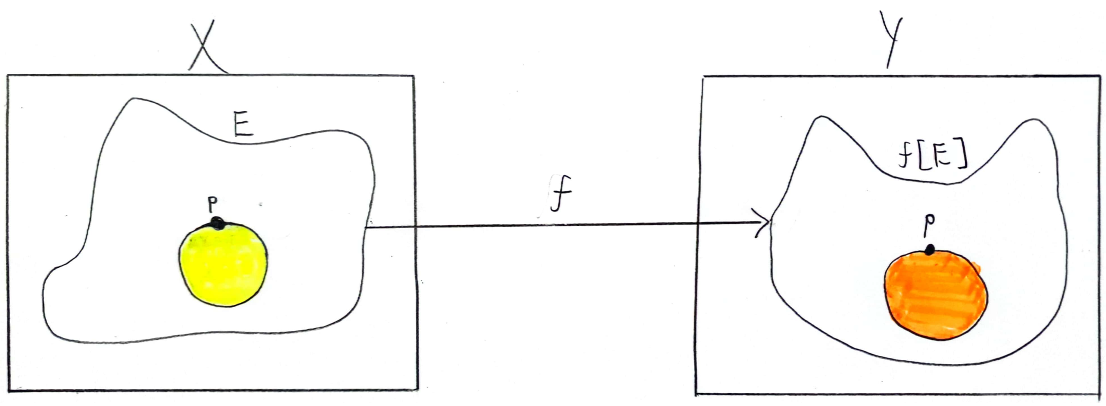


Figure 4.5: When ● $\subseteq X - E$ and ● $\not\subseteq Y$, there may be no continuous extension of f to X.

Exercise 4.7. If $E \subseteq X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by: $f(0,0) := g(0,0) := 0$, $f(x,y) := xy^2/(x^2+y^4)$, $g(x,y) := xy^2/(x^2+y^6)$ if $(x,y) \neq (0,0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighbourhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

Proof. Suppose, for contradiction, that f is unbounded on $\mathbb{R}^2$. Then, for some sequence $\{(x_n, y_n)\}$,

$$\left|\frac{1}{f(x_n,y_n)}\right| = \frac{|x_n|}{y_n^2} + \frac{y_n^2}{|x_n|} \to 0.$$

Hence $\frac{|x_n|}{y_n^2}, \frac{y_n^2}{|x_n|} \to 0$. But taking their product, we have $1 \to 0$, a contradiction. Moreover, since $f(1/n^2, 1/n) = 1/2 \not\to 0$, the function f is not continuous at $(0,0)$. For g, it is unbounded in every neighbourhood of $(0,0)$ because $g(1/n^3, 1/n) = n/2 \to \infty$. See Figure 4.6 for an illustration.
By limit laws, it is clear that f and g are be continuous on all $(x,y) \neq (0,0)$. So, fix nonzero $a, b \in \mathbb{R}$ and consider the line ℓ defined by $\mathbf{r}_\lambda = \lambda(a,b)$, for $\lambda \in \mathbb{R}$. Since

$$f(\mathbf{r}_\lambda) = \frac{\lambda ab^2}{a^2 + \lambda^4 b^4} \qquad \text{and} \qquad g(\mathbf{r}_\lambda) = \frac{\lambda ab^2}{a^2 + \lambda^4 b^6}$$

are continuous with respect to λ, the continuity of $f|_\ell$ and $g|_\ell$ at $(0,0)$ is certain.

§4.4 Hw 8

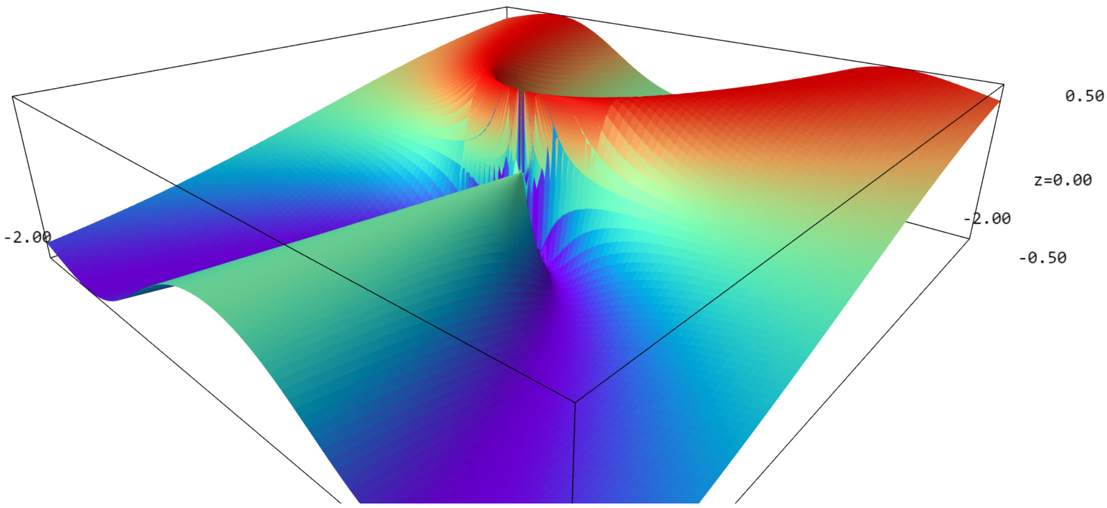
Question. Let E and Y be metric spaces, such that for all $\delta > 0$ there is a finite cover $\{N_\delta(p_n)\}$ of E. Then, if $f\colon E \to Y$ is uniformly continuous, must $\{d(f(x), y)\}_{x \in X}$ be bounded for each $y \in Y$?

Proof. Yes. Pick $\delta > 0$, such that $f[N_\delta(x)] = N_1(f(x))$ for all $x \in E$. Now choose a finite cover $\{N_\delta(p_n)\}_{1 \le n \le N}$ of E. We see that $\{d(f(x), y)\}_{x \in X}$ is bounded by

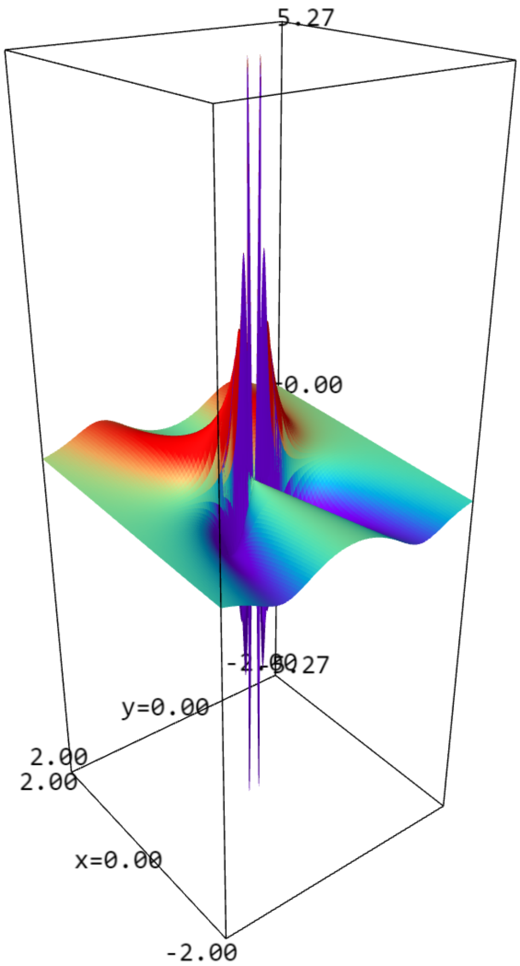
$$1 + \sum_{n=1}^{N-1} d(f(x_n), f(x_{n+1})) + d(f(x_N), y).$$

Question. Let f be a uniformly continuous mapping of a subset E of a compact metric space X into a metric space Y. Then, must f be bounded?

Proof. Yes. Fix $\delta > 0$. By compactness, there is a finite cover $\{N_{\delta/2}(x_n)\}$ of E. So, pick $p_n \in N_{\delta/2}(x_n)$. Then, $\{N_\delta(p_n)\}$ covers E. The preceding question implies



(a) $f(x, y)$



(b) $g(x, y)$

Figure 4.6: An illustration of f and g in Sage.

f is bounded.

Observation. The compactness of a metric space X is strictly stronger than the criteria that it must have a finite cover $\{N_\delta(x_n)\}$, for all $\delta > 0$. For instance, consider $(0, 1)$.

Exercise 4.8. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}$. Prove that f is bounded on E.
Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Proof. The result follows from the preceding question. The conclusion need not be true when E is unbounded: The identity function on $\mathbb{R}$ provides an example.

Exercise. Let X and Y be metric spaces, such that Y is complete. Prove that $f\colon E \to Y$ has a uniformly continuous extension from X to X^*.

Proof. Let p be a limit point of X^* and pick $\varepsilon > 0$. So, there is a sequence $x_n \to p$, and $\delta > 0$ for which $f[N_\delta(x)] \subseteq N_\varepsilon(f(x))$ is true of all $x \in X$. Pick $N \in \mathbb{N}$, such that $d(x_n, p) < \delta/2$ for $n \geq N$. Then, $d(x_m, x_n) < \delta$ for $m, n \geq N$. As such, $d(f(x_m), f(x_n)) < \varepsilon$. Moreover, the limit $\lim_{x\to p} f(x)$ exists, since $f[X \cap N_{\delta/2}(p)] \subseteq N_{2\varepsilon}(\lim f(x_n))$. Hence, we have the uniformly continuous extension $g\colon X^* \to Y$ defined by $g(p) := \lim_{x\to p} f(x)$.

Exercise 4.13. Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X (see exercise 5 for terminology). (Uniqueness follows from exercise 4.) Could the range space $\mathbb{R}$ be replaced by $\mathbb{R}^k$? By any compact metric space?

Proof. Yes to all three. As the preceding self-exercise shows, this can even be extended to any complete metric space Y.

Corollary (Lecture 8). If $f\colon D \to \mathbb{R}$ is uniformly continuous, where $D \subseteq \mathbb{R}$, then there is a unique continuous function $\tilde{f}\colon \overline{D} \to \mathbb{R}$, where $\tilde{f}(x) = f(x)$ for all $x \in D$.

Exercise 4.14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof. Wlog, $f(0) > 0$. Let $i := \inf\{x \mid \text{if } y \geq x, \text{ then } f(y) \leq y\}$. Clearly, $f(i) - i = \lim_{x\to i^+} f(x) - x \leq 0$. Now, pick $x_n \in [i - 1/n, i)$ such that $f(x_n) > x_n$. Then, $f(i) - i = \lim_{n\to\infty} f(x_n) - x_n \geq 0$. Hence, $f(i) = i$.

Exercise 4.15. Call a mapping of X into Y *open* iff $f[V]$ is open in Y whenever V is open in X. Prove that every continuous open mapping $f\colon \mathbb{R} \to \mathbb{R}$ is monotonic.

Proof. Suppose, for contradiction, that there exists $a < b < c$, such that $f(b) > f(a), f(c)$. Then, let $f(M) := \max f|_{[a,c]}$ and $\delta := \min\{M-a, c-M\}$. We see that $f[N_\delta(M)]$ is not open, as $f(M)$ is non-interior. A contradiction.

Exercise 4.18. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/n & \text{if } x = m/n. \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple (jump or removable) discontinuity at every rational point.

Proof. Let $\{m_k/n_k\}$ be a sequence converging to $x \in \mathbb{R}$, that excludes x. Fix $N \in \mathbb{N}$ and $\delta := \min\{|m/n - x| : 1 \le n \le N,\ m \in \mathbb{Z},\ m/n \neq x\}$. Then, for some $K \in \mathbb{N}$, if $k \ge K$, then $|m_k/n_k - x| < \delta$. i.e. $n_k > N$. So, $\lim_{n\to\infty} f(m_k/n_k) = 0$. Hence, at every irrational point, f is continuous. But, for all $0 < |p/q - m/n| < 1/n$, we see that $|f(p/q) - f(m/n)| \ge 1/(n^2+n)$. Therefore, at each rational point is a removable discontinuity.

§4.5 Other exercises

Exercise 4.19. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some $x \in (a,b) \cup (b,a)$. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Proof. Assume, for contradiction, that $L := \lim f(x_n) < f(x)$ for some $x \in \mathbb{R}$ and sequence $x_n \to x^-$. Pick $M > [f(x) - L]^{-1}$ and x_{k_n}, such that $|f(x_{k_n}) - L| < \frac{1}{Mn}$. Now choose a rational $r \in (L + 1/M, f(x))$ and a sequence $s_n \in (x_{k_n}, x)$ with $f(s_n) \to r$. But $f(x) \neq r$ even though $s_n \to x$, a contradiction.

Exercise 4.20. If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{x \in E} d(x,z).$$

(a) Prove that $\rho_E(x) = 0$ iff $x \in \overline{E}$.

(b) Prove that ρ_E is uniformly continuous, by showing that

$$\rho_E(x) - \rho_E(y) \le d(x,y)$$

for all $x, y \in X$.

Exercise 4.21. Suppose K and F are disjoint sets in a metric space X, and K is compact, and F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$, if $p \in K$ and $q \in F$.

Exercise 4.22. Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}$$

for $p \in X$. Show that f is a continuous function whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of exercise 3: Every closed set $A \subseteq X$ is $Z(f)$ for some continuous $f\colon X \to \mathbb{R}$. Setting

$$V = f^{-1}[0, 1/2] \qquad \text{and} \qquad W = f^{-1}(1/2, 1],$$

show that V and W are open and disjoint, and that $A \subseteq V$ with $B \subseteq W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

Exercise 4.23. A real-valued function $f\colon (a, b) \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

whenever $a < x < b$ and $a < y < b$ and $0 < \lambda < 1$.

(a) Prove that every convex function is continuous.

(b) Prove that every increasing convex function of a convex function is convex, (For example, if f is convex, so is e^f.)

(c) If f is convex and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof.

(a) Let $\varepsilon > 0$ and fix $\alpha < x < \beta$. Choose $\delta > 0$ such that

$$\frac{x - t}{\beta - t} f(\beta) < \varepsilon \qquad \text{and} \qquad \frac{x - t}{x - \alpha}[f(\alpha) - f(x)] < \varepsilon,$$

for $t \in (x - \delta, x)$. Since $x = \frac{x-\alpha}{\beta-\alpha} \cdot \beta + \frac{\beta-x}{\beta-\alpha} \cdot \alpha$, two inequalities follow:

$$f(x) - f(t) < f(x) - \frac{\beta - x}{\beta - t} f(t) \leq \frac{x - t}{\beta - t} f(\beta)$$
$$f(t) - f(x) \leq \frac{x - t}{x - \alpha}[f(\alpha) - f(x)]$$

So, $|f(x) - f(t)| < \varepsilon$.

(c) From the same decomposition of x, notice that

$$\frac{f(x)-f(\alpha)}{x-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq \frac{f(x)-f(\beta)}{x-\beta}.$$

by subtracting $-f(\alpha)$ and $-f(\beta)$, respectively, from

$$f(x) \leq \frac{x-\alpha}{\beta-\alpha} f(\beta) + \frac{\beta-x}{\beta-\alpha} f(\alpha).$$

(b) Let $f\colon X \to Y$ and $g\colon Y \to \mathbb{R}$ be convex, where $X, Y \subseteq \mathbb{R}$. Then, if g is increasing,

$$gf(\lambda x + (1-\lambda)y) \leq g(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda gf(x) + (1-\lambda)gf(y)$$

for each $x, y \in X$ and $0 < \lambda < 1$.

Question. Are there two convex functions which, when composed, is no longer convex?

Exercise 4.24. Assume that f is a continuous real function defined in (a, b) such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

Proof. × Fix $x, y \in (a, b)$ and $\lambda \in (0, 1)$. Let $t_1 := \frac{x+y}{2}$ and

$$t_{n+1} := \begin{cases} \frac{x+t_n}{2} & \text{if } \lambda x + (1-\lambda)y < t_n, \\ \frac{t_n+y}{2} & \text{if } \lambda x + (1-\lambda)y > t_n. \end{cases}$$

Clearly, $t_n = \lambda_n x + (1-\lambda_n)y$ for some $\lambda_n \in (0, 1)$. Since $t_n \to \lambda x + (1-\lambda)y$,[a] we must have $\lambda_n \to \lambda$. Hence,

$$f(t_n) \leq \lambda_n f(x) + (1-\lambda_n) f(y)$$

implies that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, by continuity.
× Oops this sequence might not converge to t, actually.

[a] to be proven

Remark. Let $x < t < y$ and define $t_1 := \frac{x+y}{2}$ with

$$t_{n+1} = \begin{cases} \frac{x+t_n}{2} & \text{if } t < t_n, \\ \frac{t_n+y}{2} & \text{if } t > t_n. \end{cases}$$

We have that $t_n = 2^{-n}(a_n x + b_n y)$, for some positive integers a_n and b_n that sum to 2^n.

Proof. Assume that this is true for n. Then,

$$t_{n+1} := \begin{cases} \frac{(a_n+2^n)x+b_n y}{2^{n+1}} & \text{if } t < t_n, \\ \frac{a_n x+(b_n+2^n)y}{2^{n+1}} & \text{if } t > t_n. \end{cases}$$

Since $a_n + 2^n + b_n = 2^{n+1}$, the result holds for $n + 1$.

Proof (4.24). Fix $x, y \in (a, b)$ and $\lambda \in (0, 1)$ and $t := \lambda x + (1 - \lambda)y$. Let

$$t_{n+1} := \frac{t_{i_n} + t_{j_n}}{2},$$

where $t_{i_n} := \max\{x\} \cup \{t_m \leq t \,|\, m \leq n\}$ and $t_{j_n} = \min\{t_m \geq t \,|\, m \leq n\} \cup \{y\}$. Since $t_n \in \{t_{i_n}, t_{j_n}\}$, strong induction implies that, for each n, there exists a positive integer a_n such that

$$t_n = 2^{-n}[a_n x + (2^n - a_n)y] \qquad \text{and} \qquad f(t_n) \leq 2^{-n}[a_n f(x) + (2^n - a_n)f(y)].$$

Moreover, $t_{j_n} - t_{i_n} = 2^{-n}(y - x)$ entails that $t_n \to t$. So, $2^{-n}a_n \to \lambda$. By continuity,

$$f(\lambda x + (1 - \lambda y)) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Exercise 4.25. If $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$ and $\mathbf{y} \in B$.

(a) If K is compact and C is closed in $\mathbb{R}^k$, prove that $K + C$ is closed.

(b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbb{R}$ whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbb{R}$.

Proof.

(a) Let X be a normed vector space; $K \subseteq X$ be compact and $C \subseteq$ be closed. Let $k_n + c_n \to x \in X$ be a sequence in $K + C$. As K is compact, there is a subsequence $k_{m_n} \to k \in K$. Since C is closed, $c_{m_n} \to m - k \in C$ so $x \in K + C$.

(b) Let $C_1 = \mathbb{Z}$ and $C_2 = \alpha\mathbb{Z}$.

Exercise 4.26. Suppose X, Y, Z are metric spaces, and Y is compact. Let $f\colon X \to Y$, let $g\colon Y \to Z$ be continuous and injective, and put $h := g \circ f$.

(a) Prove that f is uniformly continuous, if h is uniformly continuous.

(b) Prove also that f is continuous, if h is continuous.

(c) Show (by modifying example 4.21 or finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Proof.

(a) Since Y is compact, so is $g[Y]$. Hence, g^{-1} is uniformly continuous. Clearly, the composition of two uniformly continuous maps must be uniformly continuous. i.e. $f = g^{-1} \circ h$ is uniformly continuous.

(b) Since the composition of two continuous maps is continuous, f must be continuous.

(c) Consider the bijective continuous function $g\colon [0, 2\pi) \to [-1, 1]$ from example 4.21, defined by $g(t) := (\cos(t), \sin(t))$. And let $f\colon [0, 2\pi] \to [0, 2\pi)$ satisfy

$$f(t) := \begin{cases} t & \text{if } t \in [0, 2\pi), \\ 0 & \text{if } t = 2\pi \end{cases}.$$

Even though f is discontinuous, $h(t) = (\cos(t), \sin(t))$ is uniformly continuous.

Observation 4.33. Let X, Y, Z be metric spaces; $f\colon X \to Y$ and $g\colon Y \to Z$.

(a) When f is uniformly continuous and g is continuous, $g \circ f$ does not have to be uniformly continuous. (E.g. put $f(x) := x^2$ for $X = (0, 1)$ and $g(x) := x^{-1/2}$.)

(b) When g is uniformly continuous and f is continuous, $g \circ f$ does not have to be uniformly continuous. (The same example applies.)

(c) If f and g are both uniformly continuous, then $g \circ f$ is clearly continuous.

Theorem 4.34 (Outsider). Let f be a continuous, injective, and bounded map of a subset X of $\mathbb{R}$ into another metric space Y. Then, f is uniformly continuous.

Chapter 5

Differentiation

§5.1 (Self) The gradient of functions

Let's try to extend the definition of the derivative slightly, for fun!

Definition. Let $f: X \to \mathbb{R}$, where X is a metric space. Then, we say that f is *differentiable* at a limit point p of X, with derivative $f'(p)$, iff the limit

$$f'(p) := \lim_{x \to p} \frac{f(x) - f(p)}{d(x, p)}$$

exists.

Typically, the idea of a derivative invokes the notion of a best linear approximation. But, in metric spaces, we lack a notion of linearity to speak of. So, this is closer to being the gradient of a function than a derivative. Anyways, we now see if the usual theorems hold.

Theorem 5.2 (the continuity of derivatives). If $f: X \to \mathbb{R}$ is differentiable at a point $p \in X$, then f is continuous at p.

Proof. Let $\varepsilon > 0$. Pick $0 < \delta < \varepsilon(f'(p) + \varepsilon)^{-1}$, such that

$$|f(x) - f(p) - d(x, p)f'(p)| < \varepsilon d(x, p)$$

for all $d(x, p) < \delta$. Now,

$$\begin{aligned} |f(x) - f(p)| &\leq |f(x) - f(p) - d(x, p)f'(p)| + d(x, p)f'(p) \\ &< (f'(p) + \varepsilon)d(x, p) < \varepsilon. \end{aligned}$$

So, f is continuous at p.

Theorem 5.3 (sums, products and quotients). Suppose $f, g\colon X \to \mathbb{R}$ are differentiable at a point $p \in X$. Then, $f+g$, fg, and f/g are all differentiable at p (assuming $g(p) \neq 0$ in the final case). In fact,

(a) $(f+g)'(p) = f'(p) + g'(p)$,

(b) $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$,

(c) $(f/g)'(p) = \dfrac{g(p)f'(p) - g'(p)f(p)}{g(p)^2}$.

Theorem 5.5 (chain rule). Suppose $f\colon X \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are both differentiable at $p \in X$. Then, $(f \circ g)'(p) = f'(g(p))g'(p)$.

Claim. There exists two functions $f\colon X \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$, with a point $p \in X$, such that $(f \circ g)'(p)$ exists, but neither f nor g is continuous at p.

Theorem. If $f\colon X \to \mathbb{R}$ is differentiable at $p \in X$ and has a local extremum at p, then $f'(p) = 0$.

§5.2 (Self) Investigating derivatives in normed spaces

Definition 5.6 (Schröder 17.24). Let X and Y be normed spaces, $\Omega \subseteq X$ be open, $f\colon \Omega \to Y$, and $x \in \Omega$. Then, f is called *differentiable at* x iff there is a continuous linear function $L\colon X \to Y$ so that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $t \in X$ with $\|x - t\| < \delta$ we have

$$\|f(x) - f(t) - L(x-t)\| \leq \varepsilon \|x - t\|.$$

We set $Df(x) := L$ and call it the *Fréchet derivative* of f at x.

I have seen the above definition instead have that L is bounded.

Definition 5.7. Let X and Y be normed spaces. A linear transformation $T\colon X \to Y$ is *bounded* iff there exists $c > 0$ such that $\|T(x)\| \leq c\|x\|$, for all $x \in X$.

Naturally, we ask the following:

Question 5.8. Let X and Y be normed spaces. Is a linear transformation $T\colon X \to Y$ continuous iff it is bounded?

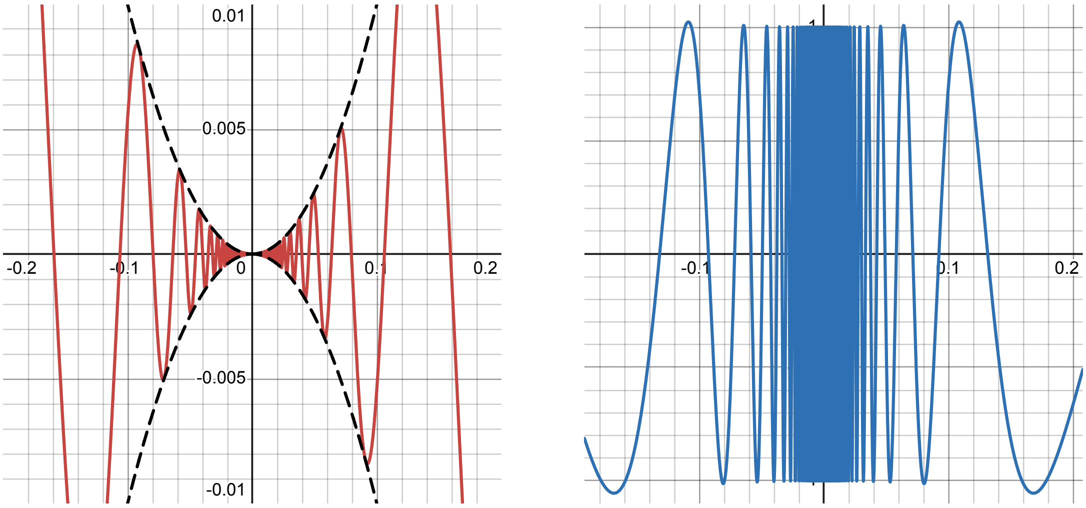
Proof. Conversely, $\lim_{t\to x}\|T(x-t)\| \leq c \lim_{t \to x}\|x - t\| = 0$.

On the topic of bounded functions, I have come across the follow fact. So, let's try to prove it!

Observation 5.9. For finite dimensional normed spaces X and Y, every linear operator $T\colon X \to Y$ is continuous.

§5.3 (Self) A squeeze theorem for derivatives?

The following is a function I found long ago, which turned out to be a classic example!



(a) $f(0) := 0,\ f(x) := x^2 \sin(1/x)$ (b) $f'(0) = 0,\ f'(x) = 2x\sin(1/x) - \cos(1/x)$

Figure 5.1: A function that has discontinuous derivative (Desmos).

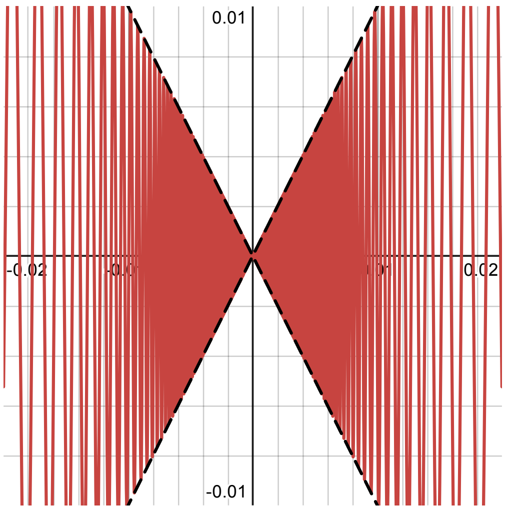


Figure 5.2: The same function but x^2 is replaced with x, i.e. $g(0) := 0$ and $g(x) := x\sin(1/x)$ (Desmos).

We see that $f(x)$ is bounded by $\pm x^2$, while $g(x)$ is bounded by $\pm x$. The former is differentiable at zero, while the latter is not. Perhaps $\pm x^2$ having the same derivative at zero is the key to the differentiability of f at zero. Hence, we naturally question whether such boundedness can, in general, give us more information on the derivative and differentiability of a function, at a point.

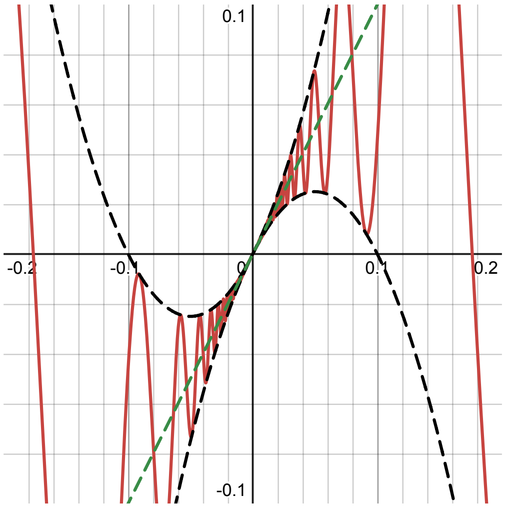


Figure 5.3: In fact, we can make the derivative at zero take on any value, such as with $h(0) := 0$ and $h(x) := x^2 \sin(1/x) + cx$ (Desmos).

Definition 5.10. Let $f, g, h \colon \mathbb{R} \to \mathbb{R}$. We say that f is bounded by a pair of functions g and h iff $\min\{g, h\} \leq f \leq \max\{g, h\}$.

Question 5.11. Let $f, \downarrow, \uparrow \colon \mathbb{R} \to \mathbb{R}$ be differentiable at zero — where $f(0) = \downarrow(0) = \uparrow(0) = 0$ — such that $\downarrow$ is monotonically decreasing and $\uparrow$ is monotonically increasing. If f is bounded by $\downarrow$ and $\uparrow$ in some $N_\delta(0)$, then must $\downarrow'(0) \leq f'(0) \leq \uparrow'(0)$?

Proof. Yes, since

$$\frac{\downarrow(t)}{t} \leq \frac{f(t)}{t} \leq \frac{\uparrow(t)}{t}$$

for any $t \in N_\delta(0)$.

Question 5.12. Let $\downarrow, \uparrow \colon \mathbb{R} \to \mathbb{R}$ be differentiable at zero, such that $\downarrow(0) = \uparrow(0)$ and $c := \downarrow'(0) = \uparrow'(0)$; $\downarrow$ is monotonically decreasing and $\uparrow$ is monotonically increasing. If $f \colon \mathbb{R} \to \mathbb{R}$ is bounded by $\downarrow$ and $\uparrow$ in some $N_\delta(0)$, and $f(0) = 0$, then must f be differentiable at zero? (More specifically, must $f'(0) = c$?)

Proof. Yes, by the Squeeze theorem.

Claim 5.13. Let $X \subseteq \mathbb{R}$ and $f \colon X \to \mathbb{R}$. If f is differentiable at some $x \in X$, then f is differentiable in some neighbourhood of x

Proof. This is false. Consider the function f below. The preceding result (5.12) implies that f (bounded by $\pm x^2$) is differentiable at zero. Yet, it is not differentiable at any $1/n$.

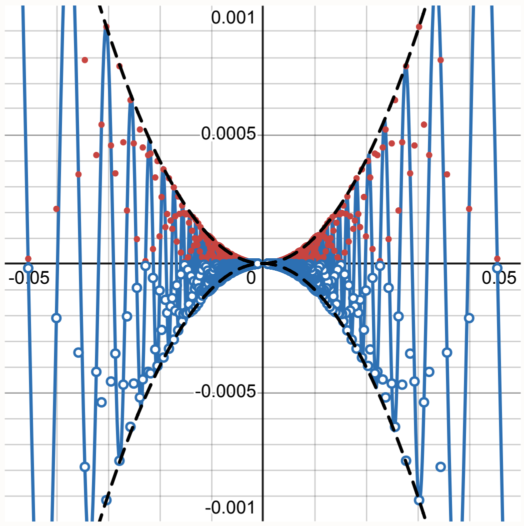


Figure 5.4: The red points ● and the blue circles ○ filled by white illustrate the jump discontinuities of this function. Everywhere else on the (blue) curve is differentiable. (Desmos)

Claim 5.14. Let $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$ be continuous. If f is differentiable at some $x \in X$, then f is differentiable in some neighbourhood of x.

Proof. This is false. Consider the continuous function f below. The preceding result (5.12) implies that f (bounded by $\pm x^2$) is differentiable at zero. But, again, f is not differentiable at any $1/n$.

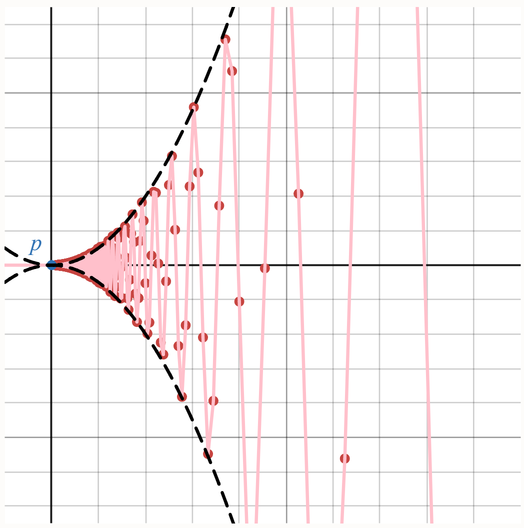


Figure 5.5: This function is differentiable at all points except the ones in red ●. (Desmos)

Claim 5.15. Let $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$ be uniformly continuous. If f is differentiable at some $x \in X$, then f is differentiable in some neighbourhood of x.

Proof. This is false. Consider the uniformly continuous function f below. The preceding result (5.12) implies that f (bounded by $\pm e^{-1/x}$) is differentiable at zero. But, again, f is not differentiable at any $1/n$.

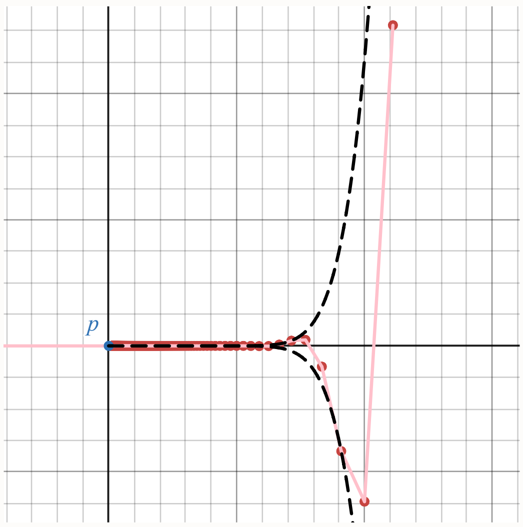


Figure 5.6: This function is differentiable at all points except the ones in red ●. (Desmos)

Remark 5.16. Let $F \subseteq X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$. If f is continuous and $f|_{X-F}$ is uniformly continuous, where F is finite, then f is uniformly continuous — simply choose the smallest δ.
Let $\{E_i\}$ be a partition of X. Similarly, if f is continuous and each $f|_{E_i}$ is uniformly continuous, then f is uniformly continuous.

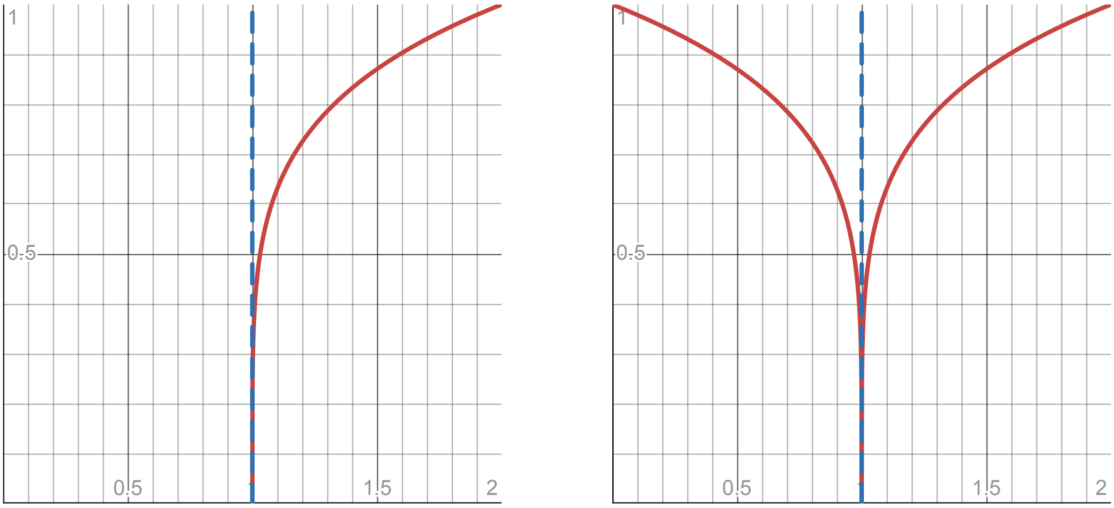
§5.4 (Self) Continuity of the derivative

Observation 5.17. (Figure 5.3) For a differentiable function $f\colon X \to \mathbb{R}$ (where $x \in X \subseteq \mathbb{R}$), the condition that $\lim_{t\to x} f'(t) = \infty$, provides no information on the value of $f'(x)$.

Claim 5.18. Let $x \in X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$ be continuous. If f is differentiable at all $t \neq x$ such that $\lim_{t\to x} f'(t) \in \mathbb{R}$, then $f'(x) = \lim_{t\to x} f'(t)$. That is, f is differentiable and f' is continuous at x.

§5.5 (Self) When are derivatives bounded?

Note 5.19. Even if a function $f\colon X \to \mathbb{R}$ has a complete bounded domain $X \subseteq \mathbb{R}$ and is itself bounded, its derivative does not have to be bounded — courtesy of oscillations and cusps. Illustrations are provided by Figures 4.4, 5.2, and 5.7.



(a) A one-sided cusp. (b) A two-sided cusp.

Figure 5.7: Some examples of cusps (Desmos).

Definition 5.20. Let $f\colon X \to \mathbb{R}$ be a continuous function where $X \subseteq \mathbb{R}$. We say that f' *is unbounded by oscillation*, at $x \in X$, iff there exist sequences $u_n \to x$ and $v_n \to x$, such that

(a) f is differentiable at all v_n, and

(b) $\lim_{n\to\infty}|f'(v_n)| = \infty$, and

(c) $f(u_{n+1}) - f(u_n) = (-1)^n$ for all n or $f(u_{n+1}) - f(u_n) = (-1)^{n+1}$ for all n.

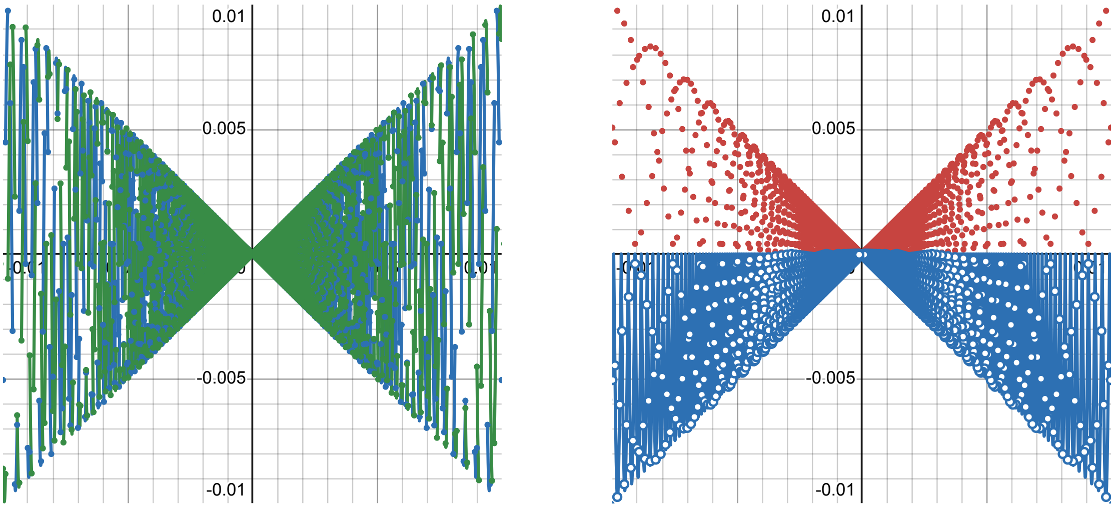
Does this definition make sense? For what it's worth, the functions in Figures 4.4, 5.2, and 5.8 satisfy this definition — their derivatives are unbounded by oscillation at $x = 0$. So, maybe this definition is fine. But only time (and more experimentation) will tell.

Definition 5.21. Let $f\colon X \to \mathbb{R}$ be a continuous function where $X \subseteq \mathbb{R}$. We say that f has a *(vertical) cusp* at x iff there is a sequence $v_n \to x$, such that $f'(v_n)$ is always positive or always negative and $\lim_{n\to\infty} f'(v_n) = \pm\infty$.

Indeed, the functions in Figures 5.7, 5.9, and 6.1 have a cusp under this definition. However, let $f\colon \mathbb{R}_0^+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is rational,} \\ -\sqrt{x} & \text{if } x \text{ is irrational.} \end{cases}.$$

Our intuition tells us that there is a cusp at $x = 0$; see 5.10. Yet, our definition implies that there are no cusps! Are we on the cusp of defeat? (Yes I had to make that pun.)



(a) (Desmos) (b) (Desmos)

Figure 5.8: Discontinuous functions whose derivatives are unbounded by oscillation at $x = 0$.

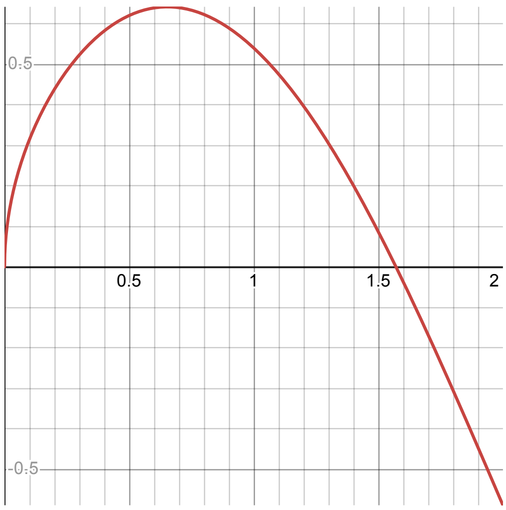


Figure 5.9: Another cusp, given by the function $f(x) \coloneqq \sqrt{x}\cos(x)$ (Desmos).

Not really; since f is differentiable nowhere, for our purposes — of considering when a derivative is bounded — we do not care for such functions. But, under our definition,

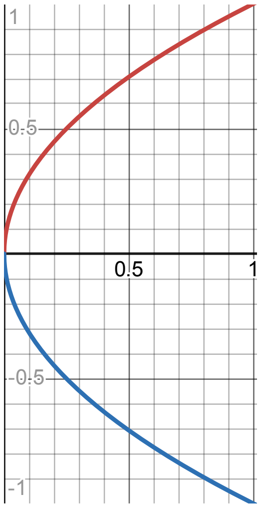


Figure 5.10: On the cusp of defeat? (Desmos)

Figure 5.11 does show a strange cusp. [continue investigating]

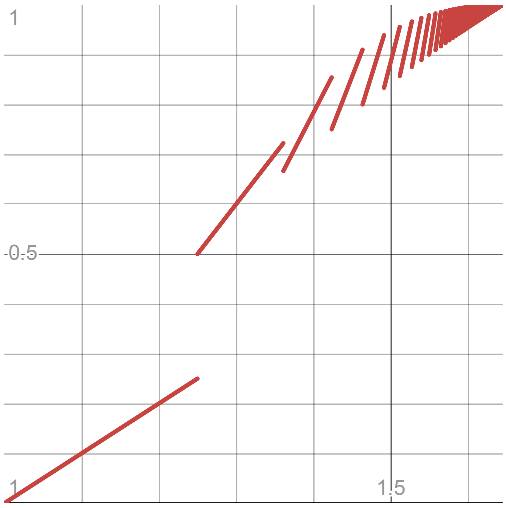


Figure 5.11: A cusp at $x = \sum 1/n^2$, created by line segments of gradient n and length $1/n^2$ (Desmos).

Question 5.22. Let $f\colon X \to \mathbb{R}$ be a continuous function where $X \subseteq \mathbb{R}$. If f has no cusps and at no point is its derivative f' is unbounded by oscillations, then must f' be bounded?

§5.6 (Self) Derivatives and constant functions

Question 5.23. Let the function $f\colon S \to \mathbb{R}$ be infinitely differentiable on $S \subseteq \mathbb{R}$. If there is a $x \in S$, such that $f^{(n)}(x) = 0$ for all n, then must f evaluate to zero in a small neighbourhood around x?

Question 5.24. Let the function $f\colon S \to \mathbb{R}$ be infinitely differentiable on $S \subseteq \mathbb{R}$. If there is a $x \in S$, such that $f^{(n)}(x) = 0$ for all n, then must f be the zero function on S?

§5.7 (Self) A collection of examples

Observation 5.25. There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable only at zero.

Proof. The function $f(x) \coloneqq 1_{\mathbb{Q}} \cdot x^2$ is differentiable at zero, by 5.12.

Observation 5.26. There is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable at zero and nowhere else.

Proof. Consider the well-known Weierstrass function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) \coloneqq \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ — where $a \in (0,1)$ and b is a positive odd integer, such that $ab > 1 + 3\pi/2$. Recall that it is continuous but differentiable nowhere. Therefore, the continuous function $g\colon \mathbb{R} \to \mathbb{R}$ with $g(x) \coloneqq x^2 f(x)(1-a)$ is differentiable at zero and nowhere else, by 5.12.

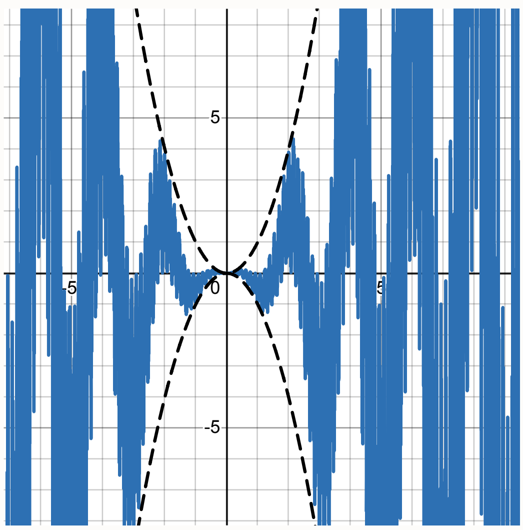


Figure 5.12: The graph of $g(x)$ against x. (Desmos)

Note 5.27. The set of differentiable points can even be made to contain a limit point.

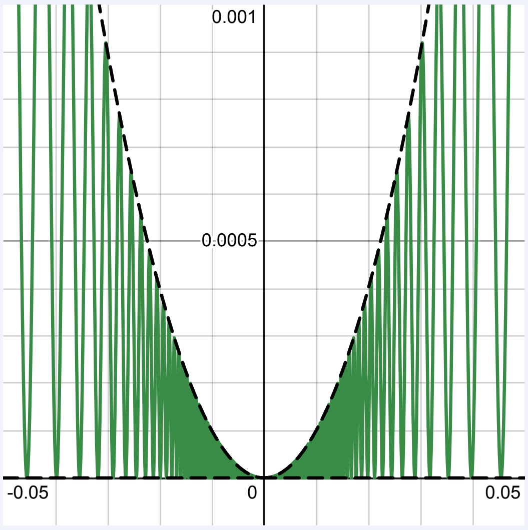


Figure 5.13: (Desmos)

It is also possible to make it an infinite set with no limit points.

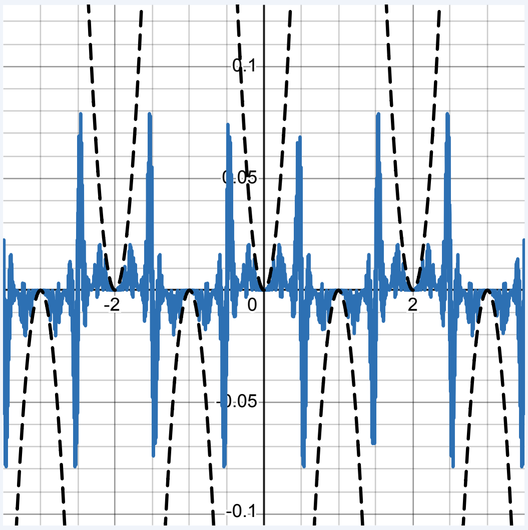


Figure 5.14: (Desmos)

Question 5.28. Is there a uniformly continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that is nowhere differentiable?

Question 5.29. Is there a uniformly continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that is differentiable at only one point?

Observation 5.30. It is possible that $f\colon X \to \mathbb{R}$ (where $X \subseteq \mathbb{R}$) is not differentiable at zero, but $|f|$ is.

Proof. Consider $f\colon \mathbb{R} \to \mathbb{R}$ with

$$f(x) := \begin{cases} -1 & \text{if } x \notin \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

Then, $|f|$ is the constant function 1.

Claim 5.31. Let $X, Y \subseteq \mathbb{R}$ be dense in $X \cup Y$. It is impossible for $f\colon X \cup Y \to \mathbb{R}$ to be differentiable on X and nowhere else.

Claim 5.32. Let $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$ be differentiable. Then, f' cannot contain a jump or removable discontinuity.

§5.8 (Self) Parametric derivatives

Some motivation: Consider $E \subseteq \mathbb{R}$ and $x, y\colon E \to \mathbb{R}$. In calculus, one comes across the (informal) notion that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

which is often just handwaved to be the chain rule. Yet, in the first place, we do not have definition for dy/dx — the derivative of one function with respect to *another function* is undefined. We are only afforded $x'(t)$ and $y'(t)$ from the usual definition of the derivative of real functions.

Of course, if x is invertible, then it is easy to recover dy/dx in a natural way: Since

$$\frac{dyx^{-1}x(t)}{dx(t)} = \left(yx^{-1}\right)'(x(t)) = \frac{y'(t)}{x'(t)}$$

from the usual definition, defining $dy/dx := y'/x'$ makes perfect sense.

However, in general there is no need for x to be locally invertible at any point, or for x and y to be differentiable. A simple example: the identity function on the reals $y(x) = x$ can be parametrised by $x(t) := y(t) := t^2 \sum_{n=0}^{\infty} a^n \cos(b^n \pi t)$ for $t \in \mathbb{R}$, where $a \in (0, 1)$ and b is a positive odd integer. See Figure 5.12.

But clearly, regardless of how we parametrise $y(x) = x$, its derivative dy/dx should always exist. After a number of revisions, this is what I came up with:

Definition 5.33. Let $E \subseteq \mathbb{R}$ and $x, y\colon E \to \mathbb{R}$. Then, y is *differentiable at* $s \in E$ *with respect to* x iff both following conditions are upheld.

- $x(s)$ is an interior point of $\left\{ x(t) : \left\| \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} - \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| < \eta \right\}$ for each $\eta > 0$.

- For some $d_x y(s) \in \mathbb{R}$ and all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{y(s)-y(t)}{x(s)-x(t)} - d_x y(s)\right| < \varepsilon \qquad \text{if} \qquad \left\| \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} - \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| < \delta.$$

The number $d_x y(s)$ is called the *derivative of y at s with respect to x*. When $d_x y(s)$ exists for every $s \in X$, we say that $d_x y$ is *the derivative of y with respect to x*.

Note 5.34. Sanity check ✓: When $x = \mathrm{id}_X$, we recover the usual definition of the derivative: $d_x y(s) = y'(s)$ for all interior points s of X.

Question 5.35. Suppose $x\colon E \to \mathbb{R}$ and $y\colon E \to \mathbb{R}$ are differentiable at s, where $E \subseteq \mathbb{R}$. Then, must $d_x y(s) = y'(s)/x'(s)$ if $x'(s) \neq 0$? If not, what additional conditions are sufficient for $d_x y(s) = y'(s)/x'(s)$?

Claim 5.36. Let $x\colon E \to \mathbb{R}$ and $y\colon E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$. When $d_x y(s)$ exists, neither $x'(s)$ nor $y'(s)$ need to exist.

§5.9 Theorems

Theorem 5.8 (Fermat). If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$ and has a local extremum at x, then $f'(x) = 0$.

Proof. Wlog, $f(x)$ is a local minimum. Let $\delta > 0$ such that $f(t) \geq f(x)$, for all $|t - x| < \delta$. Since $\frac{f(x)-f(t)}{x-t} \geq 0$ for $t < x$ and $\frac{f(x)-f(t)}{x-t} \leq 0$ for $t > x$, we have that $f'(x) \geq 0$ and $f'(x) \leq 0$. Hence, $f'(x) = 0$.

Theorem 5.9 (Cauchy's mean value theorem). If $f, g\colon [a, b] \to \mathbb{R}$ are both continuous everywhere and differentiable on (a, b), then

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$$

for some $x \in (a, b)$.

Note. Even if $f\colon [a, b] \to \mathbb{R}$ is differentiable at some point x, it does not have to be differentiable on some neighbourhood of x. A simple example: consider $f\colon [-1, 1] \to \mathbb{R}$ defined by

$$f(t) \coloneqq \begin{cases} 0 & \text{if } t = 0, \\ \lfloor t^{-1} \rfloor^{-2} & \text{if } t \in \mathbb{Q}, \\ -\lfloor t^{-1} \rfloor^{-2} & \text{if } t \notin \mathbb{Q}. \end{cases}$$

It is clear that f is differentiable at only $x = 0$, where $f'(x) = 0$. See Figure 5.15

for an illustration.

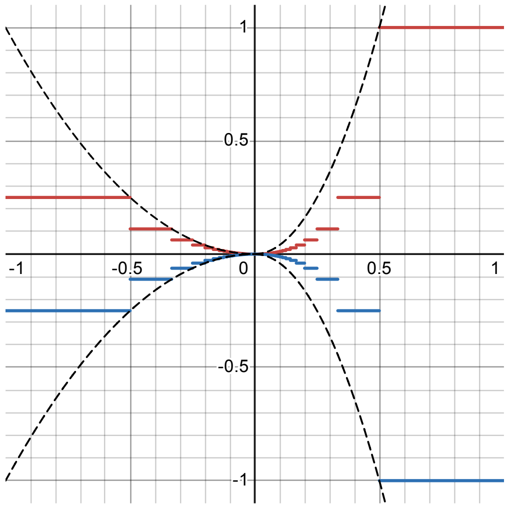


Figure 5.15: An illustration of f, which is bounded by $\pm x^2$ and $\pm 4x^2$ (Desmos).

Claim. Consider when $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$.

(a) If $f'(x) > 0$, then there exists $\delta > 0$, such that $f(u) < f(x) < f(v)$ for all $0 < x - u < \delta$ and $0 < v - x < \delta$.

(b) If $f'(x) = 0$, then x is a local extremum.

Proof.

(a) This is true. Consider when, for all $\delta > 0$, there is $0 < x - u < \delta$ with $f(x) - f(u) \leq 0$, or $0 < v - x < \delta$ with $f(x) - f(v) \geq 0$. Then, $\frac{f(x)-f(u)}{x-u}, \frac{f(x)-f(v)}{x-v} \leq 0$ so $f'(x) \leq 0$.

(b) No, this is false, for there exists stationary points of inflection. Consider $f(t) := t^3$ and $x = 0$. Then, $f'(0) = 0$.

Theorem 5.11. Suppose f is differentiable in (a, b).

(a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

(c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

Proof. Parts (a) and (c) are proven similarly to part (a) of the above claim. For (b), suppose wlog that $f(x) \neq f(y)$ for some $x, y \in (a, b)$. Then, by theorem 5.9, we have $f'(z) = \frac{f(x)-f(y)}{x-y} \neq 0$ for some $z \in (x, y)$.

Theorem 5.12 (Darboux). If $f\colon [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ with $f'(a) < \lambda < f'(b)$, then $f'(x) = \lambda$ for some $x \in (a, b)$.

Lemma. Let $X \subseteq Y$ and Z be metric spaces $f\colon X \times Y \to Z$. If $L := \lim_{x\to p} \lim_{y\to x} f(x, y)$ and $\lim_{y\to p} f(x, y)$ exist, then $\lim_{x\to p} \lim_{y\to p} f(x, y)$ exists and evaluates to L.

Proof. Let $\varepsilon > 0$, $L_x := \lim_{y \to x} f(x,y)$, and $K_x := \lim_{y \to p} f(x,y)$. Pick $\delta_p, \delta_x, \delta_x' > 0$ such that

- $d(L_x, L) < \varepsilon/3$ for all $x \in N_{\delta_p}(p)$.
- $d(f(x,y), L_x) < \varepsilon/3$ for each $y \in N_{\delta_x}(x)$.
- $d(f(x,y), K_x) < \varepsilon/3$ for every $y \in N_{\delta_x'}(p)$.

Now pick $x \in N_{\min\{\delta_p, \delta_x/2\}}(p)$ and $y \in N_{\min\{\delta_x', \delta_x/2\}}(p)$. We see that

$$d(K_x, L) \leq d(f(x,y), K_x) + d(f(x,y), L_x) + d(L_x, L) < \varepsilon.$$

Lemma (Improved ×). Let $X \subseteq Y$ and Z be metric spaces $f\colon X \times Y \to Z$. If $L := \lim_{x \to p} \lim_{y \to x} f(x,y)$ exists, then $\lim_{x \to p} \lim_{y \to p} f(x,y)$ exists and evaluates to L.

Proof. Let $\varepsilon > 0$ and $L_x := \lim_{y \to x} f(x,y)$. Pick $\delta_x > 0$ such that $d(f(x,y), L_x) < \varepsilon/2$ for each $y \in N_{\delta_x}(x)$. For a sequence $y_n \to p$, choose N such that $d(y_n, p) < \delta_x/2$ if $n \geq N$. For $x \in N_{\delta_x/2}(p)$, notice $d(x, y_n) < \delta_x$ so $d(f(x,y_n), f(x,y_m)) < \varepsilon$. Hence, K_x exists. The proof continues as above. ×

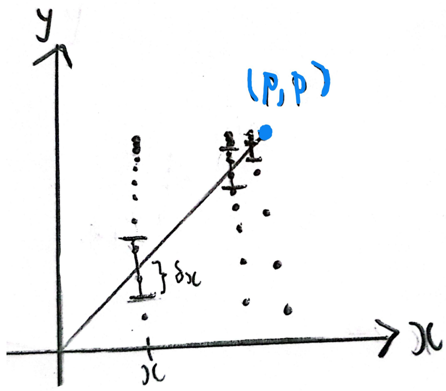


Figure 5.16: An illustration of how the improved lemma can fail.

Question. Given two metric spaces X and Y, is there any canonical metric on $X \times Y$? What about when $X \subseteq Y$?

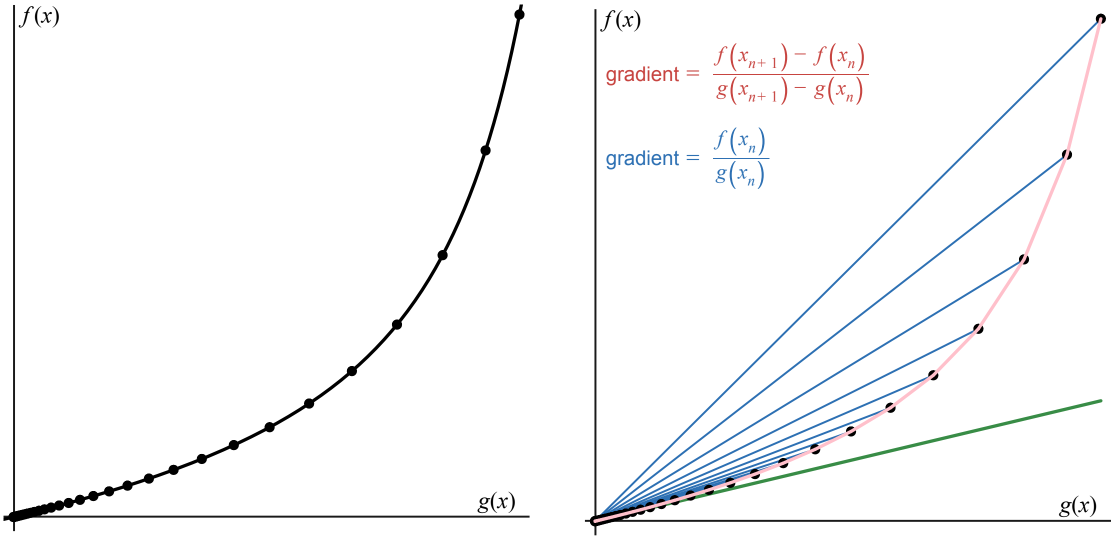
Theorem 5.13 (L'Hospital's Rule). Suppose $f, g\colon X \to \mathbb{R}$ are differentiable on $(a,b) \subseteq X \subseteq \mathbb{R}$, and $g'(x) \neq 0$ for all $x \in (a,b)$, where $a, b \in [-\infty, \infty]$. Assume that $f'(x)/g'(x) \to A$ as $x \to a$. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, or if $\lim_{x \to a} g(x) = \pm\infty$, then $f(x)/g(x) \to A$ as $x \to a$.

L'Hospital is pronounced as *loh-pee-tahl*.

Proof. Wlog, (for each x) there is $\delta > 0$ for which $g(t) \neq g(x)$ if $t \in N_\delta(x)$. So,

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \lim_{t\to x} \frac{f(x) - f(t)}{g(x) - g(t)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

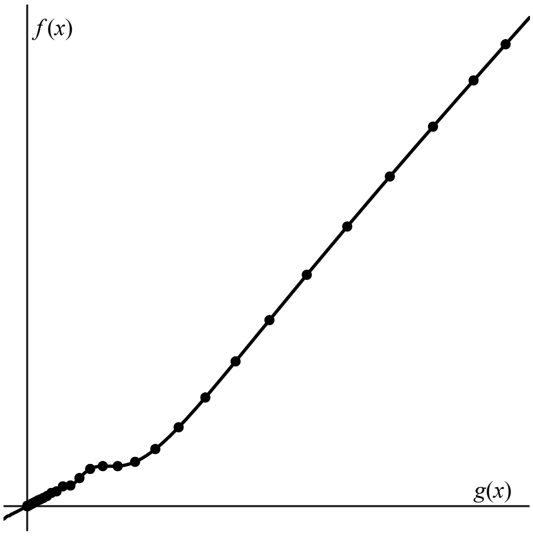
by the above lemma, when $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.



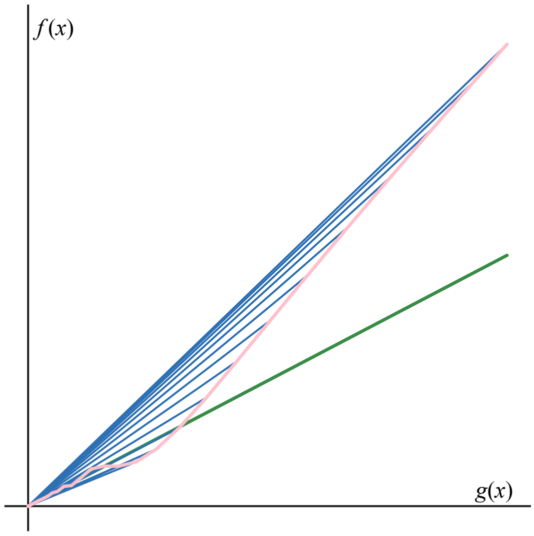
(a) The graph of $f(x) = e^{x^2(1.7-x)^{-1}} + x - 1$ against $g(x) = x$ and some selected points $(g(x_n), f(x_n))$.

(b) The line segments joining the origin and the selected $(g(x_n), f(x_n))$; the line segments joining $(g(x_n), f(x_n))$ and $(g(x_{n+1}), f(x_{n+1}))$.

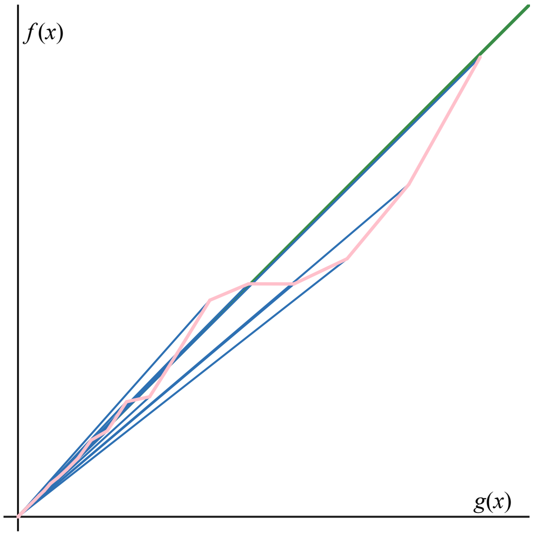
Figure 5.17: Notice that the pink line segments and blue line segments both converge to the green tangent; the gradients $\frac{f(x_{n+1})-f(x_n)}{g(x_{n+1})-g(x_n)}$ and $\frac{f(x_n)}{g(x_n)}$ converge to the same limit 1 (Desmos).



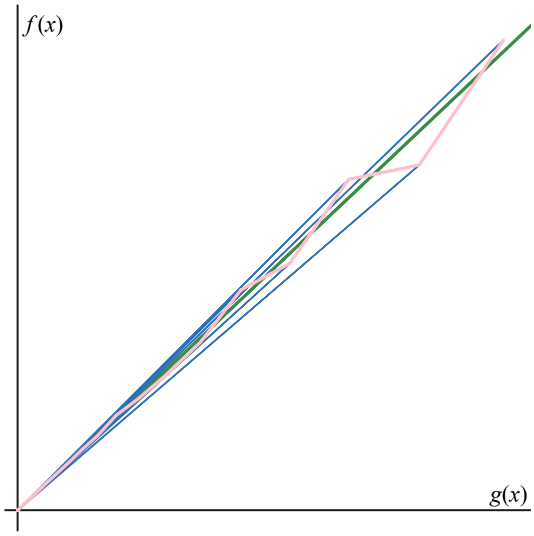
(a) $f(x) = x^2 \sin(1/x) + x$ against $g(x) = x$.



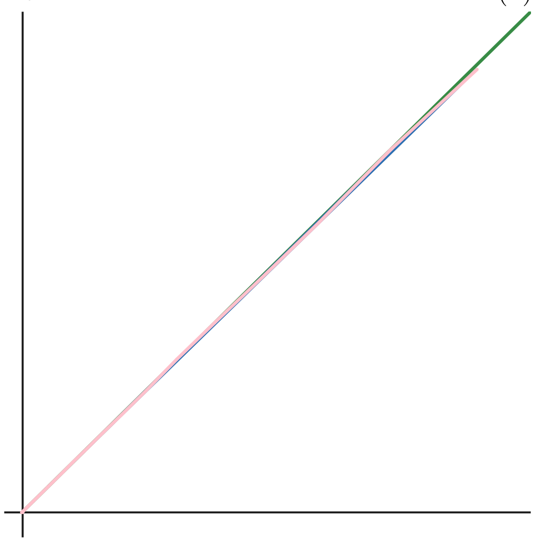
(b) $n \geq 1$



(c) $n \geq 10$

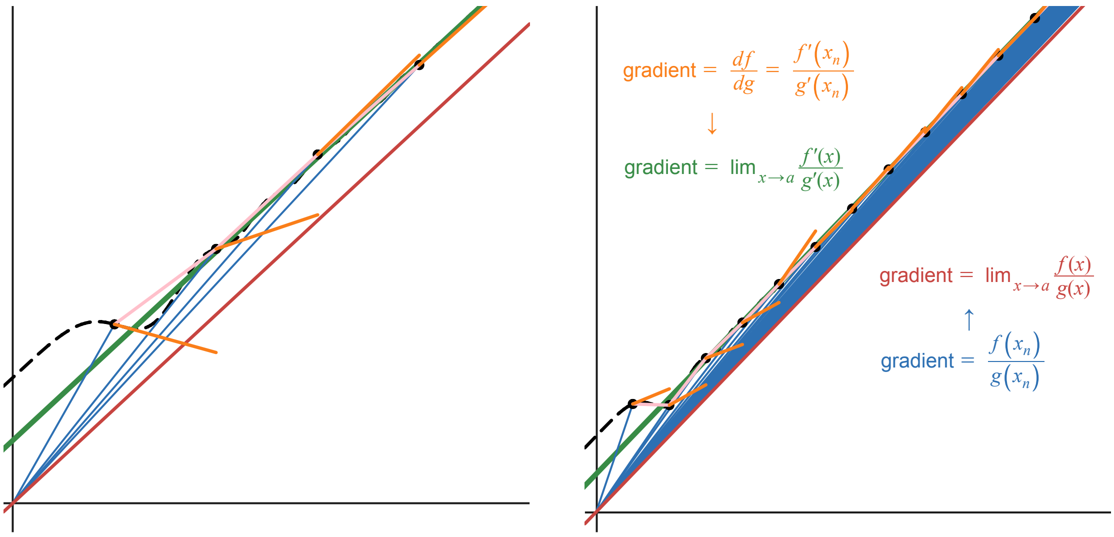


(d) $n \geq 20$



(e) $n \geq 30$

Figure 5.18: Notice the same behaviour as in 5.17. (Desmos)



(a) (b)

Figure 5.19: The case of $\lim_{x\to a} g(x) = \infty$, with $f(x) = g(x) + 1 + \sin(g(x)^2)/g(x)^2$ (Desmos).

Theorem 5.14 (Taylor). Suppose n is a positive integer and $f\colon [a,b] \to \mathbb{R}$, such that $f^{(n-1)}\colon [a,b] \to \mathbb{R}$ is continuous and $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$ and define

$$P(t) := \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n,$$

for some $x \in (\alpha,\beta) \cup (\beta,\alpha)$.

§5.10 Hw 9

Exercise 5.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ and suppose that

$$|f(x) - f(y)| \le (x-y)^2$$

for all real x and y. Prove that f is constant.

Proof. Since $\left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|$, we notice that $f'(x) = 0$ for all $x \in \mathbb{R}$. Hence, f is a constant function, by theorem 5.11.

Exercise 5.2. Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in

(a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof. This preceding claim suffices to prove that f is strictly increasing. Pick a sequence $y_n \to f(x)$ and let $x_n = g(y_n)$. Since $x_n \to x$ by continuity,

$$g'(f(x)) = \lim_{n\to\infty} \frac{g(f(x)) - g(y_n)}{f(x) - y_n} = \lim_{n\to\infty} \frac{x - x_n}{f(x) - f(x_n)} = \frac{1}{f'(x)}.$$

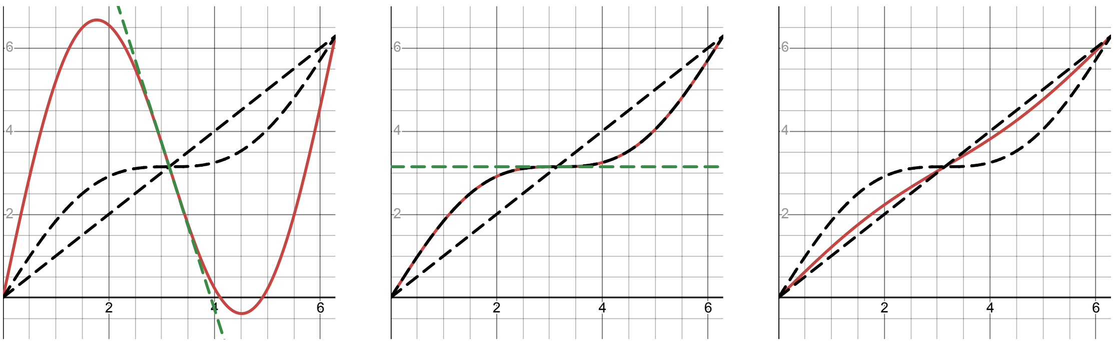
Remark. Notice that

$$g'(f(x)) = \lim_{y\to f(x)} \frac{g(f(x)) - g(y)}{f(x) - y} = \lim_{t\to x} \frac{x - t}{f(x) - f(t)} = \frac{1}{f'(x)}$$

is also a viable alternative, courtesy of continuity.

Exercise 5.3. Suppose $g\colon \mathbb{R} \to \mathbb{R}$ has a bounded derivative (say $|g'| < M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is injective if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)

Proof. Pick $\varepsilon < 1/M$. Then, for $g'(x) \le 0$, we have $f'(x) = 1 + \varepsilon g'(x) \ge 1 + g'(x)/M > 0$. i.e. f is strictly increasing.



(a) ε is too big. (b) ε is just small enough. (c) ε is very small.

Figure 5.20: An example for exercise 5.3 given by $g(x) := \sin(x)$. The black curves indicate the region in which f is injective and the green line is the tangent to g at $x = \pi$ (Desmos) (mp4 animation).

Exercise 5.5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) := f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to \infty$.

Proof. Pick $M \in \mathbb{R}$ such that $|f'(x)| < \varepsilon$, for each $x > M$. By the mean value theorem, there exists $c \in (x, x+1)$ with $|f(x+1) - f(x)| = |f'(c)| < \varepsilon$.

Exercise 5.6. Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically increasing.

Put

$$g(x) \coloneqq \frac{f(x)}{x} \qquad (x > 0).$$

and prove that g is monotonically increasing.

Proof. Let $x > 0$. By the mean value theorem, $f'(x) \geq f'(c) = \frac{f(x)}{x}$ for some $c \in (0, x)$. Hence, $g'(x) \geq 0$.

Remark. The above holds for $f(0) \leq 0$, since we have that $f'(c) \geq \frac{f(x)}{x}$. Furthermore, Cauchy's mean value theorem yields the following generalisation. Let $f, g \colon [0, \infty] \to \mathbb{R}$ be continuous everywhere and differentiable on $(0, \infty)$, such that $f(0) \leq 0$ and $g(0) \geq 0$; both f' and g' are monotonically increasing. Then, $h \colon (0, \infty) \to \mathbb{R}$ with $h(x) \coloneqq f(x)/g(x)$ is monotonically increasing.

Note. Let $f, g \colon (0, \infty) \to \mathbb{R}$ with $g(x) \coloneqq f(x)/x$. For g to be monotonically increasing, it is necessary that $f(0, \delta) \subseteq (-\infty, 0]$ for some $\delta > 0$.

Proof. Suppose wlog that there is a sequence $x_n \to 0^+$ and some $\varepsilon > 0$, with $f(x_n) > \varepsilon$. Then, $\lim_{n\to\infty} g(x_n) \geq \lim_{n\to\infty} \varepsilon/x_n = \infty$. So, g is not monotonically increasing.

Exercise 5.7. Suppose that $f'(x)$ and $g'(x) \neq 0$ exist, with $f(x) = g(x) = 0$. Prove that

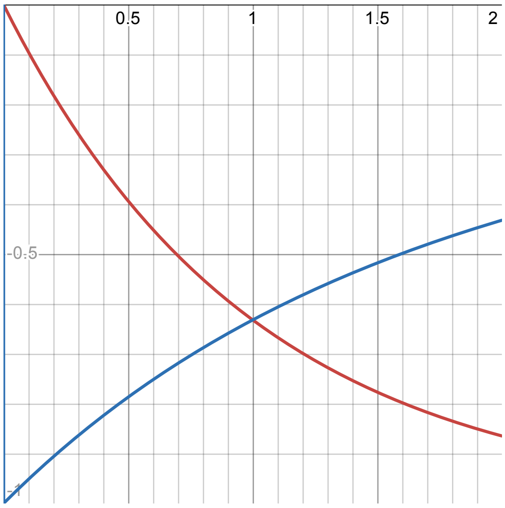
$$\lim_{t\to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

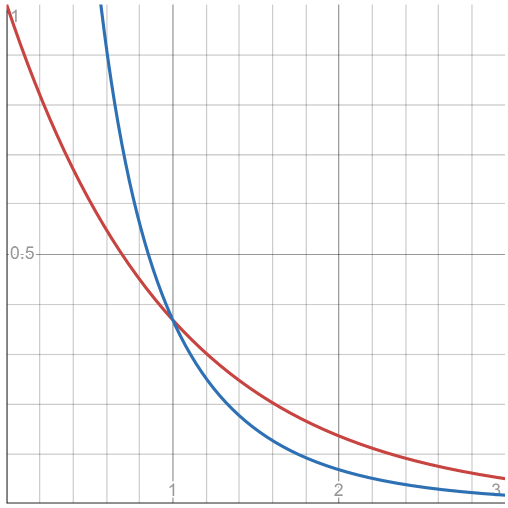
Proof. Observe that

$$\frac{f'(x)}{g'(x)} = \lim_{t\to x} \frac{\frac{f(t)}{t-x}}{\frac{g(t)}{t-x}} = \lim_{t\to x} \frac{f(t)}{g(t)}.$$

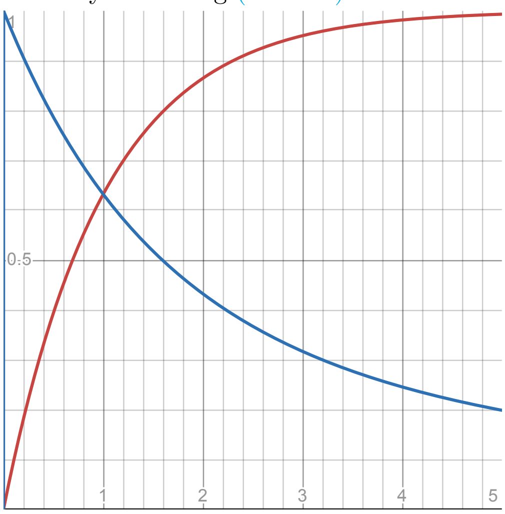
(We are allowed to combine the limits into one because $g'(x) = 0$ implies x is not a limit point of the zeros of g.)



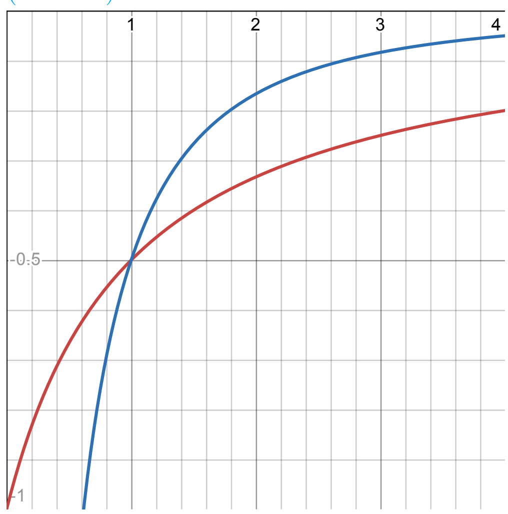
(a) All conditions are satisfied, so g is monotonically increasing (Desmos).



(b) When $f(0) \neq 0$ and g is decreasing (Desmos).

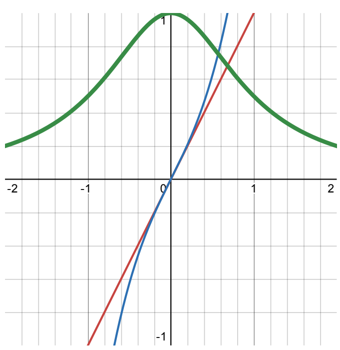


(c) When f' and g are both decreasing (Desmos).

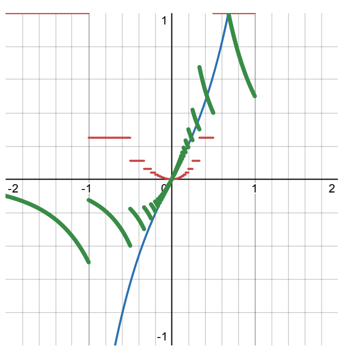


(d) When f' is decreasing, but g is increasing (Desmos).

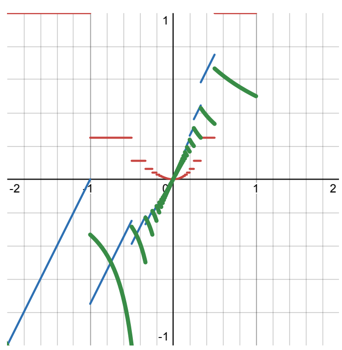
Figure 5.21: Some examples for exercise 5.6. The red curve denotes f and the blue curve denotes g.



(a) (Desmos)



(b) (Desmos)



(c) (Desmos)

Figure 5.22: Some examples illustrating exercise 5.7. Red denotes f, blue denotes g, and green denotes f/g.

§5.11 Other exercises

Exercise 5.8 (uniform differentiability). Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon,$$

whenever $0 < |t - x| < \delta$, and $a \leq x \leq b$, and $a \leq t \leq b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Claim. Let $X \subseteq \mathbb{R}$ be compact and $f' \colon X \to \mathbb{R}$ be continuous. Then, f is uniformly differentiable.
Hint by Eric: Apply the Mean Value Theorem.

Proof. Let $\varepsilon > 0$. By compactness, f is uniformly continuous: Pick $\delta > 0$, such that $|f'(x) - f'(t)| < \varepsilon$ if $0 < |x - t| < \delta$. By the Mean Value Theorem,

$$\left| \frac{f(x) - f(t)}{x - t} - f'(x) \right| = |f'(y) - f'(x)| < \varepsilon$$

for some y.

Note. The converse is trivial: the triangle inequality implies that any uniformly differentiable function has a uniformly continuous derivative.

Question. Let $X \subseteq \mathbb{R}$ and $f \colon X \to \mathbb{R}$ be differentiable. If $|f'|$ is bounded, must f be uniformly continuous?

Claim. Let $X \subseteq \mathbb{R}$. A function $f \colon X \to \mathbb{R}$ is uniformly continuous iff the difference quotient

$$\frac{f(x) - f(y)}{x - y}$$

is bounded, over all $x, y \in X$.

Proof. No, this is false. Consider $f \colon [0, 1] \to \mathbb{R}$ with $f(x) := \sqrt{x}$, which is uniformly continuous. Then, $\lim_{x \to 0} f'(x) = \infty$.

Claim. The difference quotient $\mathcal{M} \colon X^2 \to \mathbb{R}$ of a function $f \colon X \to \mathbb{R}$ is given by

$$\mathcal{M}(x, y) := \frac{f(x) - f(y)}{x - y}.$$

There exists f and some $x \in X$, such that

$$f'(x) \neq \lim_{(t,u)\to(x,x)} \mathcal{M}(t,u).$$

Question. Is there anything special about the class of functions $f\colon X \to \mathbb{R}$ with

$$f'(x) \neq \lim_{(t,u)\to(x,x)} \mathcal{M}(t,u)$$

for all $x \in X$ (where $X \subseteq \mathbb{R}$)?

Exercise 5.14. Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. Prove that f is convex iff f' is monotonically increasing. Assume next that f'' exists, and prove that f is convex iff $f'' \geq 0$.

Proof. If f is convex, then f' is monotonically increasing by exercise 4.23. Conversely, when f' is monotonically increasing and $y < t < x$,

$$\frac{f(t)-f(y)}{t-y} \leq \frac{f(x)-f(y)}{x-y}$$

by the mean value theorem. Simplifying,

$$f(t) \leq \frac{t-y}{x-y} f(x) + \left(1 - \frac{t-y}{x-y}\right) f(y).$$

Convexity follows from letting $t = \lambda x + (1-\lambda)y$. Finally, the equivalence of a differentiable function being monotonically increasing and its derivative being nonnegative makes the final equivalence trivial.

Exercise 5.15. Suppose $a \in \mathbb{R}$ and $f\colon (a,\infty) \to \mathbb{R}$ is twice-differentiable, and M_0, M_1, M_2 are the suprema of $|f|$, $|f'|$ and $|f''|$, respectively. Prove that

$$M_1^2 \leq 4M_0M_2.$$

Exercise 5.16. Suppose f is twice-differentiable on $(0,\infty)$, such that f'' is bounded on $(0,\infty)$ and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Exercise 5.17. Suppose $f\colon [-1,1] \to \mathbb{R}$ is three times differentiable, such that

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$. Note that equality holds for $(x^3+x^2)/2$.

Exercise 5.19. Suppose $f\colon (-1,1) \to \mathbb{R}$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n <$

1, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n := \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.

(b) If $0 < \alpha_n < \beta_n$ and $(\beta_n/(\beta_n - \alpha_n))$ is bounded, then $\lim D_n = f'(0)$.

(c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Exercise 5.21. Let E be a closed subset of $\mathbb{R}$. We saw in exercise 4.22 that there is a real continuous function f on $\mathbb{R}$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbb{R}$, or one which is n times differentiable, or even is infinitely differentiable on $\mathbb{R}$?

Exercise 5.26. Suppose f is differentiable on $[a, b]$, and $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Exercise 5.27.

(a) Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$ and $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function $f\colon [a, b] \to \mathbb{R}$ such that $f(a) = c$, and $\alpha \leq f \leq \beta$, and

$$f' = \phi(x, f(x)) \qquad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1), (x, y_2) \in R$.

(b) Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

Proof.

(a) Let f and g be two solutions. Then,

$$\left|(f-g)'(x)\right| \leq A|(f-g)(x)|$$

implies that $f = g$, by exercise 5.26.

(b) The set of zeros $Z(f)$ of any solution f is closed. So pick $x_0 \notin Z(f)$ and $v := \max\{z \in Z(f) \mid z < x_0\}$. Then, $f(x) = (x-v)^2/4$ for all $v \leq x \leq x_0$. By continuity, $f(x) = (x-v)^2/4$ for every $x \geq v$. So, the general solution is

$$f(x) := \begin{cases} 0 & \text{if } u \leq x \leq v, \\ (x-u)^2/4 & \text{if } x \leq u, \\ (x-v)^2/4 & \text{if } x \geq v. \end{cases}$$

where $-\infty \leq u \leq 0$ and $0 \leq v \leq \infty$.

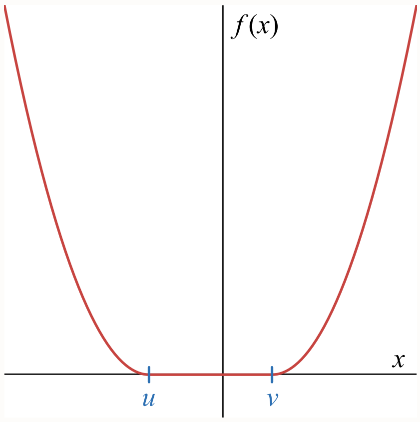


Figure 5.23: The graph of $f(x)$ against x (Desmos).

Exercise 5.28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi(x, y_1, \ldots, y_k), \qquad y_j(a) = c_j \qquad (j = 1, \ldots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \mathbf{\Phi}(x, \mathbf{y}), \qquad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ ranges over a k-cell, $\mathbf{\Phi}$ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \ldots, \phi_k$ and $\mathbf{c}$ is the vector $c_1, \ldots, c_k$. Use Exercise 26, for vector-valued functions.

Chapter 6

Miscellaneous

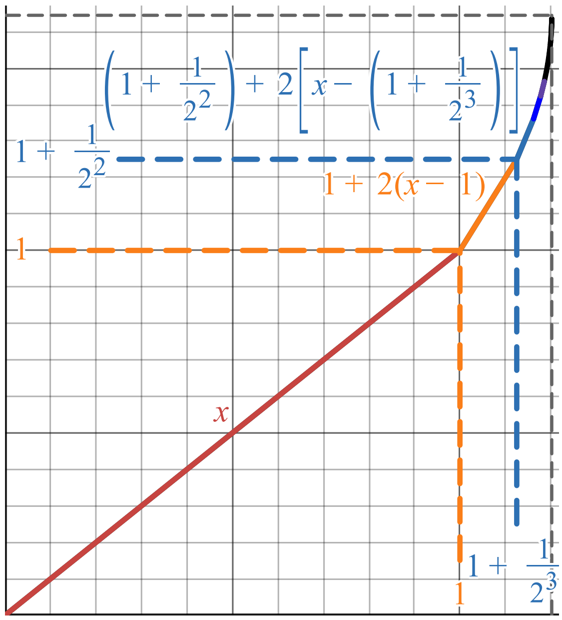


Figure 6.1: Here's an illustration of the function that I tried to use to answer my question in chapter 4. But I realised it obviously doesn't work... after I had spent the time to draw out the graph and all. For the curious, it is the function $f\colon [0, \sum 1/n^3] \to \mathbb{R}$ defined by $f(x) \coloneqq \sum_{i=1}^{n-1} 1/i^2 + n\left(x - \sum_{i=1}^{n-1} 1/i^3\right)$, for $x \in \left[\sum_{i=1}^{n-1} 1/i^3, \sum_{i=1}^{n} 1/i^3\right]$, and $f(\sum 1/n^3) \coloneqq \sum 1/n^2$ (Desmos).

Modified to allow any length and gradient for the line segments: (Desmos)

Chapter 7

(Self-Chapter) A rabbit hole?

Note. This chapter is still very much a work in progress. This is the core question of this chapter:

Question 7.1. Let E be a subset of a metric space X and Y be a metric space. Then, what conditions are strong enough for all bounded functions $f: E \to Y$ to have a continuous extension to X?

From Figure 4.5 and the associated collection of self-exercises, we see that 'holes' in Y might be the main issue in preventing continuous extensions from always existing. So, let us try to "patch" it up with new points p.

The idea is to define each point p based on its distance from every point in Y, hence uniquely identifying the point p. That is, $p = \{(y, d(p, y)) \mid y \in Y\}$. Take $Y = \mathbb{R} - \{0\}$ for example: $p_0 = \{(y, y) \mid y \in Y\} = \mathrm{id}_Y$ is the new point representing the hole $0 \notin Y$. All other points x are represented by $p_x : Y \to \mathbb{R}_0^+$ where $p_x := |x - y|$ for all $y \in Y$. So, the 'patched version' of Y is $Y^{\otimes} := \{p_x \mid x \in Y\} \cup \{p_0\} \cong \mathbb{R}$.

Notice that the metric $|\cdot|$ on $\mathbb{R} \cong Y^{\otimes}$ was what enabled us to define $Y^{\otimes}$ easily. In general, however, we lack this luxury. We may attempt to circumvent this issue by broadly defining $Y^{\otimes}$ as the set of all functions Y mapping into $\mathbb{R}_0^+$. Immediately, we sense that this definition is inappropriate: there can exist $p_1 \neq p_2$ such that $p_1(x) = p_2(x) = 0$ for some x. But the latter implies, informally, that $d(p_1, x) = d(p_2, x) = 0$ so $p_1 = p_2$.

An easy fix is found by imposing this condition:

$$\text{If } p(x) = 0 \text{ for some } x, \text{ then } p(y) = d(x, y) \text{ for all } y \in X.$$

Yet,

Definition 7.2. For a subset E of a metric space X, its *exterior* $\mathrm{Ext}_X(E)$ is the maximal connected subset of $X - E$ (which exists by Hausdorff's Maximal Principle).

Definition 7.3. × A metric space X is *perforated* iff $\operatorname{Ext}_Y(X) \neq X^{\complement}$ for some metric space Y which X can be (isometrically) embedded into.

Definition 7.4. × A metric space X is perforated iff it contains a disconnected subset.

Question 7.5. Let $E \subseteq X$ and Y be metric spaces. If Y is not perforated, then do all bounded functions $f \colon E \to Y$ have a continuous extension to X?

Definition 7.6. The $\mathbb{L}$-*patch point* of a metric space X, that lies a distance r from x on the line segment joining x and y, is as the function $p_{x,y,r} \colon X \to \mathbb{R}_0^+$, such that $p_{x,y,r}(y) \coloneqq d(x,y) - r$ and

$$p_{x,y,r}(z) \coloneqq \sqrt{r^2 + d(x,z)^2 + \frac{d(y,z)^2 - d(x,z)^2 - [r + d(x,y) - d]^2}{r[r + d(x,y) - d]}}.$$

Definition 7.7. The *line patching* $\mathbb{L}_X$ of a metric space X, is defined as the set of functions $p_{x,y,r} \colon X \to \mathbb{R}_0^+$ for $x, y \in X$ and $r \geq 0$. That is,

$$\mathbb{L}_X \coloneqq \{p_{x,y,r} \mid x, y \in X \text{ and } r \geq 0\}.$$

Question 7.8. Let $E \subseteq X$ and Y be metric spaces. Does a bounded function $f \colon E \to Y$ such that $f[\operatorname{Fr}_E(E)] \subseteq \operatorname{Int}(Y)$ always have a continuous extension to X?

Chapter 8

Bibliography

(a) Cover page is modified from https://tex.stackexchange.com/a/85989.

(b) Diagram on cover page: https://tex.stackexchange.com/a/525667.

(c) Figure 3.5 is modified from https://tex.stackexchange.com/a/333261.