Additive inverses are unique:

Suppose that x' and $\bar{x}$ are additive inverses of x. Thereupon, $x + x' = x + \bar{x}$. And consequently,

$$(x + x') + x' = (x + \bar{x}) + x'$$
$$(x + x') + x' \overset{Ax\ 1.}{=} x + (\bar{x} + x')$$
$$(x + x') + x' \overset{Ax\ 2.}{=} x + (x' + \bar{x})$$
$$(x + x') + x' \overset{Ax\ 1.}{=} (x + x') + \bar{x}$$
$$0 + x' \overset{Ax\ 4.}{=} 0 + \bar{x}$$
$$x' + 0 \overset{Ax\ 2.}{=} \bar{x} + 0$$
$$x' \overset{Ax\ 3.}{=} \bar{x} .$$

$$x' + x \overset{Ax.2}{=} \bar{x} + x$$
$$(x' + x) + x' = (\bar{x} + x) + x'$$
$$x' + (x + x') \overset{Ax.1}{=} \bar{x} + (x + x')$$
$$x' + 0 \overset{Ax.4}{=} \bar{x} + 0$$
$$x' \overset{Ax.3}{=} \bar{x}$$

Wherefore, we see that the additive inverse of x is indeed unique.

4.

Sketch:

$$0x = 0$$
$$(1+(-1))x = 0$$
$$x + (-1)x = 0$$
$$(-1)x = -x$$

Proof

~~First, we see that from part 1, $0x = 0$. By Axiom 4., this means that $(1+(-1))x = 0$. Thus, using Axiom 9, $1\cdot x + (-1)\cdot x = 0$.~~
~~Therefore, Axiom 7 tells us that $x + (-1)x = 0$.~~

$$0x \overset{\text{Part 1}}{=} 0$$

$(1+(-1))x \overset{\text{Ax. 4}}{=} 0$ (Existance of the Additive Inverse -1)

$x(1+(-1)) \overset{\text{Ax. 6}}{=} 0$ (Commutativity of Multiplication)

$x\cdot 1 + x\cdot(-1) \overset{\text{Ax. 9}}{=} 0$ (Left-Distributivity of Multiplication over Addition)

$1\cdot x + (-1)x \overset{\text{Ax. 6}}{=} 0$ (Commutativity of Multiplication)

$x + (-1)x \overset{\text{Ax. 7}}{=} 0$ (1 is the neutral element for multiplication)

$(-1)x + x \overset{\text{Ax. 2}}{=} 0$ (Commutativity of Addition)

$((-1)x + x) + (-x) \overset{\text{Ax. 4}}{=} -x$ (Existance of the Additive Inverse $-x$)

$(-1)x + (x+(-x)) \overset{\text{Ax. 1}}{=} -x$ (Associativity of Addition)

$(-1)x + 0 \overset{\text{Ax. 4}}{=} -x$ (Definition of Additive Inverses)

$(-1)x = -x$

2. $0 \neq 1$.

Ideas: Suppose for a contradiction that $0 = 1$.

Self Proof of Theorem 1.2

1. For all $x \in \mathbb{R}$, we have $0x = 0$.

Rough Ideas:

$$0x \overset{Ax\,4.}{=} (1+(-1))x \overset{Ax\,6}{=} x(1+(-1)) \overset{Ax\,9.}{=} 1\cdot x + (-1)\cdot x \overset{Ax\,7.}{=} x + (-1)x =$$

$$0x = (0+0)x = 0x + 0x$$

$$0x = x\cdot 0 = x\cdot(0+0) =$$

$$1\cdot x = x$$
$$(1+0)x = x$$
$$1\cdot x + 0\cdot x = x$$
$$x + 0\cdot x = x$$
$$0\cdot x = -x$$

Proof

$$1\cdot x \overset{Ax\,7.}{=} x$$
$$(1+0)\cdot x \overset{Ax\,3.}{=} x$$
$$x\cdot(1+0) \overset{Ax\,6.}{=} x$$
$$x\cdot 1 + x\cdot 0 \overset{Ax\,9.}{=} x$$
$$1\cdot x + 0\cdot x \overset{Ax\,6.}{=} x$$
$$x + 0x \overset{Ax\,7.}{=} x$$
$$(x + 0x) + (-x) \overset{Ax\,4.}{=} x + (-x)$$
$$x + (0x + (-x)) \overset{Ax\,1.}{=} x + (-x)$$
$$x + ((-x) + 0x) \overset{Ax\,2.}{=} x + (-x)$$
$$(x + (-x)) + 0x \overset{Ax\,1.}{=} x + (-x)$$
$$0 + 0x \overset{Ax\,4.}{=} 0$$
$$0x + 0 \overset{Ax\,2.}{=} 0$$
$$0x \overset{Ax\,3.}{=} 0 \quad \boxed{}$$

Exercises

1-1. Prove that $(-1)\cdot(-1) = 1$.

Ideas
$$(-1)(1+(-1)) = 0$$
$$-1 + (-1)(-1) = 0$$
$$(-1)(-1) = 1$$

Proof

$1+(-1) \overset{Ax.4}{=} 0$ (Existance of Additive Inverses)

$(-1)(1+(-1)) = (-1)(0)$

$(-1)(1) + (-1)(-1) \overset{Ax.9}{=} (-1)(0)$ (Distributivity of Multiplication over Addition)

$(1)(-1) + (-1)(-1) \overset{Ax.6}{=} (0)(-1)$ ((Commutativity of Multiplication)

$-1 + (-1)(-1) \underset{Thm 1-2}{\overset{Ax.7}{=}} 0$ (1 is the Multiplicative Identity and $0x = 0$))

$1 + (-1 + (-1)(-1)) = 1$

$(1 + (-1)) + (-1)(-1) \overset{Ax.1}{=} 1$ (Associativity of Addition)

$0 + (-1)(-1) \overset{Ax.4}{=} 1$ (Definition of Additive Inverses)

$(-1)(-1) + 0 \overset{Ax.2}{=} 1$ (Commutativity of Addition)

$(-1)(-1) \overset{Ax.3}{=} 1$ (0 is the Additive Identity)

$\boxed{2}$

1-2.

Proof

$$
\begin{aligned}
(x+y)z &\overset{Ax\cdot 6}{=} z(x+y) && \text{(Commutativity of Multiplication)} \\
&\overset{Ax\cdot 9}{=} zx + zy && \text{(Left-Distributivity of Multiplication over Addition)} \\
&\overset{Ax\cdot 6}{=} xz + yz && \text{(Commutativity of Multiplication)}
\end{aligned}
$$

$\boxed{2}$

1-5

Ideas $\quad xx^{-1} = 1 \quad yy^{-1} = 1$

$(xx^{-1})(yy^{-1}) = 1$

Proof

We first see that by Axiom 8, since $x, y \neq 0$, there exists the additive inverses x^{-1} and y^{-1} so that

$$xx^{-1} = 1 \quad \text{and} \quad yy^{-1} = 1.$$

Thus, notice that

$$
\begin{aligned}
(xx^{-1})(yy^{-1}) &= 1\cdot 1 \\
(xx^{-1})(yy^{-1}) &\overset{Ax\cdot 7}{=} 1 && \text{(1 is the Multiplicative Identity)} \\
((xx^{-1})\cdot y)\cdot y^{-1} &\overset{Ax\cdot 5}{=} 1 && \\
(x\cdot(x^{-1}y))\cdot y^{-1} &\overset{Ax\cdot 5}{=} 1 && \text{(Associativity of Multiplication)} \\
(x\cdot(yx^{-1}))\cdot y^{-1} &\overset{Ax\cdot 6}{=} 1 && \text{(Commutativity of Multiplication)} \\
((xy)\cdot x^{-1})\cdot y^{-1} &\overset{Ax\cdot 5}{=} 1 && \\
(xy)(x^{-1}y^{-1}) &\overset{Ax\cdot 5}{=} 1 && \text{(Associativity of Multiplication)}
\end{aligned}
$$

As $(xy)(x^{-1}y^{-1}) = 1$ and multiplicative inverses are unique by Exercise 1-3, $x^{-1}y^{-1}$ must be the multiplicative inverse of xy. In other words, $(xy)^{-1} = x^{-1}y^{-1}$. Consequently, xy is certainly nonzero because it has the multiplicative inverse $x^{-1}y^{-1}$ while 0 has no multiplicative inverse by Exercise 1-4.

$\boxed{2}$

1-7

Since the set $\{0,1\}$ is equipped with the usual multiplication and usual addition, except that $1+1 := 0$, Axioms 3 and 5 through 8 are always satisfied because these Axioms do not include the possibility of having $1+1$, which needs to be accounted for separately. Now, we consider the rest of the field Axioms individually when the sum $1+1$ is involved;

Axiom 1: Notice that when two of x, y, and z are 1, associativity still holds:

$$\begin{aligned}(1+1)+0 &= 0+0\\ &= 0\\ &= 1+1\\ &= 1+(1+0)\end{aligned}$$

$$\begin{aligned}(1+0)+1 &= 1+1\\ &= 1+(0+1)\end{aligned}$$

$$\begin{aligned}(0+1)+1 &= 1+1\\ &= 0+(1+1)\end{aligned}$$

Axiom 2: Clearly, $1+1=0=1+1$.

Axiom 4: For 1, its additive inverse is just itself because $1+1=0$. So, this Axiom indeed holds true.

Axiom 9:

$$\begin{aligned}1\cdot(1+1) &= 1\cdot 0\\ &= 0\\ &= 1+1\\ &= 1\cdot 1+1\cdot 1\end{aligned}$$

$$\begin{aligned}0\cdot(1+1) &= 0\cdot 0\\ &= 0\\ &= 0+0\\ &= 0\cdot 1+0\cdot 1\end{aligned}$$

In the rest of the cases where the sum $1+1$ is not present, these Axioms hold true instantly. Consequently, Axioms 1 to 9 hold true irregardless. Wherefore, the set $\{0,1\}$ with the usual multiplication and the usual addition, except that $1+1:=0$, is a field.

-4 (a)

First suppose that $x \cdot y = 0$. Then, the contrapositive of Exercise 1-5 tells us that $x = 0$ or $y = 0$. Conversely, assume that $x = 0$ or $y = 0$. In the former, $x \cdot y = 0 \cdot y \overset{Ax.2}{=} y \cdot 0 = 0$ by Theorem 1.2 (a). Even more immediately for the latter, $x \cdot y = x \cdot 0 = 0$ by the same theorem. Hence, we can conclude that $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$.

(b) We notice that the existence of multiplicative inverses is not guaranteed. Case in point: $x = 2$. Observe that $2 \cdot 0 = 0 \bmod 4$ and $2 \cdot 2 = 4 \bmod 4$, which both evaluate to 0. While $2 \cdot 1 = 2 \bmod 4$ and $2 \cdot 3 = 6 \bmod 4$, equating to 2. In addition, neither 0 nor 2 is a neutral element for multiplication, since $0 \cdot 1 = 0 \neq 1$ and $2 \cdot 1 = 2 \neq 1$. Consequently, we conclude that no element of $\{0, 1, 2, 3\}$ is the multiplicative inverse of 2. Hence, Axiom 8 is not satisfied because $1 + 2 = 3 \neq 1$ meaning that 2 [illegible] 1 is not the neutral element for addition. Wherefore, the set $\{0, 1, 2, 3\}$ with the sum and product of two elements being the remainder obtained when dividing the regular sum and product by 4 is not a field.

(c) Assume that the set $\{0, 1, \ldots, p-1\}$ with the sum $(+)$ and product $(\cdot)$ of two elements being the remainder obtained when dividing the regular sum $(+_{\mathbb{R}})$ and product $(\cdot_{\mathbb{R}})$ by p is a field. If p is a multiple of some natural $n > 1$ in this field—i.e. $p = kn$ for a natural k—then for any m in the field, $n \cdot m = nm \bmod p \neq 1$; lest there exists the natural number q with $nm = pq +_{\mathbb{R}} 1$, which would contradict the fact that nm is a multiple of $n > 1$, because $pq +_{\mathbb{R}} 1 = knq +_{\mathbb{R}} 1$ is not. Notice that 1 is the neutral element for multiplication, as $1 \cdot m = m \bmod p = m$ for all m in the field. Therefore, n does not have a multiplicative inverse in the field, violating Axiom 8 since $n > 1$ is not a neutral element for addition: $(p -_{\mathbb{R}} n) + n = [(kn -_{\mathbb{R}} n) +_{\mathbb{R}} n] \bmod p = kn \bmod p = 0$ is a counterexample. Conversely, …

Self Proof of Theorem 1.10

1. Assume x is positive, i.e. $x \in \mathbb{R}^+$. Immediately, it follows that $x - 0 \in \mathbb{R}^+$ as well, since $x - 0 = x \in \mathbb{R}^+$. It follows from the definition of our total order now, that $x > 0$. Consider x being negative now. The procedure is near identical. We know $-x \in \mathbb{R}^+$, and hence, $0 - x \in \mathbb{R}^+$. Consequently, it is certain now that $x < 0$.

   Conversely, suppose $x > 0$. Then, $x - 0 \in \mathbb{R}^+$. Similarly, when $x < 0$, $0 - x \in \mathbb{R}^+$. Clearly, $x \in \mathbb{R}^+$ and $-x \in \mathbb{R}^+$ in each respective case. (Corresponding to x being positive and x being negative)

   Wherefore, combining the above results, we have that the number x is positive iff $x > 0$ and negative iff $x < 0$ indeed. $\square$

2. If $x < y$, then it follows that $y - x \in \mathbb{R}^+$. Consequently, since $y - x = (y+z) - (x+z)$, we observe that $x + z < y + z$. In the case that $x = y$, $x + z = y + z$ clearly. Hence, in any case; if $x \leq y$, then $x + z \leq y + z$. $\square$

3. Consider the case where equality holds, i.e. $x = y$. Then, $xz = yz$ is trivial for any $z > 0$. When $x < y$, we know that $y - x \in \mathbb{R}^+$ by def. Hence, by Axiom 1.6 part 1., we know that $(y-x)z = yz - xz \in \mathbb{R}^+$ given that $z > 0$ (as by part 1 we know $z \in \mathbb{R}^+$ as a result). Clearly, $xz < yz$ follows suit. Wherefore, we can conclude that $xz \leq yz$ as long as $x \leq y$ and $z > 0$. $\square$

4. Again, if equality holds, the statement is trivial. Assume $x < y$ and $z < 0$ now. By part 1 of this theorem once more, z is negative, which means that $-z \in \mathbb{R}^+$. Utilising Axiom 1.6 part 1. (again), $(y - x)(-z) = xz - yz \in \mathbb{R}^+$ (since $y - x \in \mathbb{R}^+$ by definition). Just as expected, in either case; from $x \leq y$ and $z < 0$, we see that $yz \leq xz$. Thus, $yz < xz$. $\square$

5. Assume $0 < x \leq y$. By the transitivity of $\leq$ proven in Proposition 1.8 (part 3.), we notice that $0 \leq y$ (because $0 \leq x \leq y$). Additionally, observe that $y \neq 0$, lest $0 < 0$ in the case that $x = 0$ or $0 < x$ and $x < 0$ simultaneously — meaning x is both positive and negative. In both cases, part 2 of Axiom 1.6 is violated. So, we know that x and y are both nonzero. In other words, x^{-1} and y^{-1} both exist (Axiom 1.1 part 8.), and are in fact positive. Further note that $1 > 0$, lest for positive x (i.e. $x > 0$), $1 \cdot x = x < 0$ by part 4 of this theorem: Again, contradicting part 2. of Axiom 1.6. Therefore, $x^{-1} > 0$ (i.e. x^{-1} is positive). Otherwise, $xx^{-1} = 1 < 0$ by part 4 of this theorem once more. Which is in clear disagreement with the fact proven above that $1 > 0$. Consequently, we can now simply apply part 3 of this theorem, from which we see that (since $x \leq y$) $xy^{-1}x^{-1} \leq yy^{-1}x^{-1}$. Wherefore, we see that $y^{-1} \leq x^{-1}$. $\square$

uthor's
oof much
aner and
egant.

Rough Sketch of Exercise 1-11 (c).

$$\underbrace{||x|-|y||}_{\text{since } |x|-|y|<0.} = -(|x|-|y|) = |y|-|x| \leq |y-x| = |-(x-y)| = |x-y|$$

Exercises

1-12. Let $I, J \subseteq \mathbb{R}$ be intervals and $c, d \in I \cap J$. Thus, it follows that for any $c < x < d$, $x \in I$ and $x \in J$ by definition. In other words, $x \in I \cap J$. So, $I \cap J$ is indeed again an interval. $\square$

1-13. Assume that $a<b$ and $x, y \in [a,b]$. First, notice that $-x \leq -a$, $y \leq b$ and $-y \leq -a$, $x \leq b$ by definition. Now, we see that by combining the first and last two inequalities, we have that $y-x \leq b-a$ and $x-y \leq b-a$ (from part 2 of Theorem 1.10 and transitivity of our total order $\leq$). Therefore, when $x-y \geq 0$, $|x-y| = x-y \leq b-a$. And if $x-y < 0$, $|x-y| = -(x-y) = y-x \leq b-a$ again. Hence, in any case, we always have that $|x-y| \leq b-a$. $\square$

$a \leq x \leq y \leq b$; $a<b$ $\quad |x-y| \leq b-a \quad x-y \leq 0$

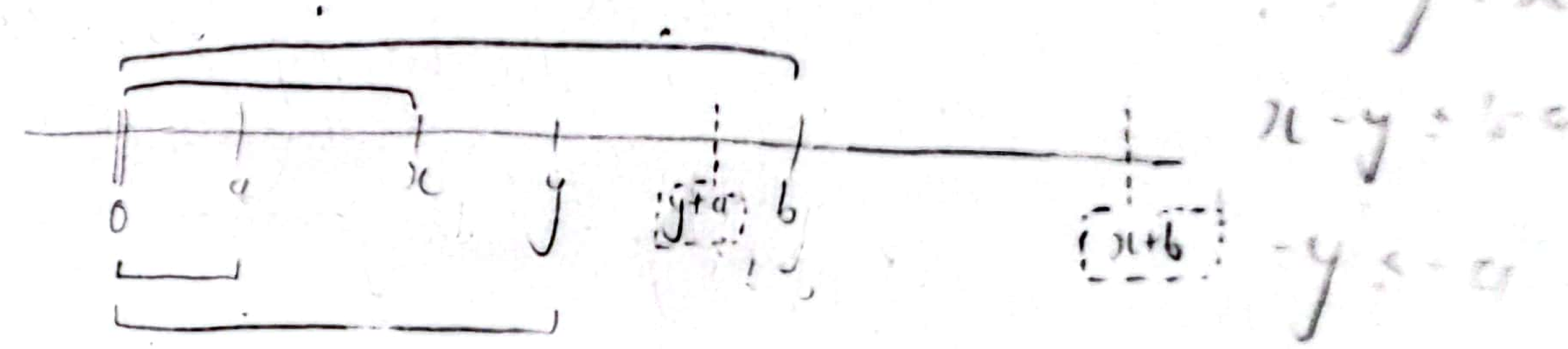
$|a|$

$\cancel{||b|-|a||} \leq \cancel{|b-a|} \leq |b|+|a| \quad y-x \leq b-a \quad y-x \geq 0$

$|b-a| \leq |b| + |-a| \quad y+a \leq x+b$

$\cancel{a-x \leq 0} \quad \cancel{y-b \leq 0} \quad -x \leq -a \quad y \leq b$

$0 \leq x-a \quad \cancel{0 \leq b-y}$



$y \geq x$

$x-y \leq b-a$

$-y \leq -a \quad x \leq b$

1-14. Suppose that there exists some natural p and subset $\mathbb{F}^+$ of $\mathbb{F} = \{0, 1, \cdots p-1\}$ that satisfies Axiom 1.6. Now, we notice that $1 +_{\mathbb{F}} (p-1) = p \bmod p = 0$. In other words, the additive inverse of 1 is $p-1$. Hence, by part 2 of Axiom 1.6, either $1 \in \mathbb{F}^+$ or $p-1 \in \mathbb{F}^+$ (but not both). If $1 \in \mathbb{F}^+$, we have the following. Utilizing the fact that $\mathbb{F}^+$ is closed under addition, we have that $\sum_{i=1}^{p-1} 1 = \underbrace{p-1}_{\in \mathbb{F}^+}$ — because $n +_{\mathbb{F}} 1 = (n +_{\mathbb{R}} 1) \bmod p = n +_{\mathbb{R}} 1$ for any $n \in \mathbb{F}$ less than $p-2$. So, $1 \in \mathbb{F}^+$ is not possible. Thus, it must now hold that $p-1 \in \mathbb{F}^+$. Similarly, from $\mathbb{F}^+$ being closed under multiplication, we see that $(p-1) \cdot_{\mathbb{F}} (p-1) = (p^2 - 2p +_{\mathbb{R}} 1) \bmod p = [p \cdot_{\mathbb{R}} (p-2) +_{\mathbb{R}} 1] \bmod p = 1$, which must be in $\mathbb{F}^+$. However, this again contradicts part 2 of Axiom 1.6 since $p-1 \in \mathbb{F}^+$ and $1 \in \mathbb{F}^+$ simultaneously. Wherefore, we conclude that none of the fields from Exercise 1-9c can satisfy Axiom 1.6. $\square$

Conversely, suppose $x > 0$. Then, $x - 0 \in \mathbb{R}^+$. Similarly, when $x < 0$, $0 - x \in \mathbb{R}^+$. Clearly, $x \in \mathbb{R}^+$ and $-x \in \mathbb{R}^+$ in each respective case.

Self Proof of Theorem 1.13

0. By part 2 of Axiom 1.6, we have 2 cases to consider: either $x \geq 0$ or $x < 0$. In the former, $|x| = x \geq 0$. While for the latter, x is negative which means that $-x$ is positive, and hence, $-x > 0$ by part 1 of Theorem 1.10. Thus, $|x| = -x > 0$. Irregardless, it is now clear that $|x| \geq 0$ for any $x \in \mathbb{R}$. $\square$

1. First assume $x = 0$. It clearly follows that $|x| = x = 0$. Conversely, now suppose that $|x| = 0$. By part 2 of Axiom 1.6, and part 1 of Theorem 1.10, precisely one of $x > 0$, $x = 0$ or $-x > 0$ holds true. In the case that $x > 0$, $|x| = x > 0$. Hence, x cannot be greater than 0 for $|x|$. Similarly, in the case that $-x > 0$, x is negative; And thus, utilising part 1 of Theorem 1.10, we know that $x < 0$. Consequently, $|x| = -x > 0$. Again, this means $|x| \neq 0$ and we can eliminate the possibility of $-x > 0$. In stark contrast, $x = 0$ is clearly one possible case since $|x| = 0$ here. Resultantly, $|x| = 0$ must mean that $x = 0$. Wherefore, $|x| = 0$ iff $x = 0$.

2. Consider x and y being both ~~positive~~. That is, $x \geq 0$ and $y > 0$. Immediately, $|xy| = xy = |x||y|$. * (since $xy > 0$ by part 3 of Thm 1.10) Now, if x and y are both negative, we know that $x < 0$ and $y < 0$. Utilising part 4 of Theorem 1.10, $xy > 0$. Additionally, notice that $xy = (-1)(-1)(xy) = (-1)x \cdot (-1)y = (-x)(-y)$. With this, we now see that, similar to the previous case, $|xy| = xy = (-x)(-y) = |x||y|$. Suppose without loss of generality that x is positive but y is negative. (i.e. $x > 0$, $y < 0$). Notice that $xy < 0$ and $-(xy) = x(-y)$. Consequently, $|xy| = -(xy) = x(-y) = |x||y|$. Finally, presume that x is any real number but $y = 0$, again, without loss of generality. It follows trivially that $|xy| = |0| = 0 = |x| \cdot 0 = |x||y|$. Wherefore, we have proven, by exhaustion, that $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

Me		Author	
	1. $x > 0$ & $y > 0$	(a) $x \geq 0$ & $y \geq 0$	[1. + 4.]
	2. $x < 0$ & $y < 0$	(b) $x \geq 0$ & $y < 0$	[3. + 4.]
(WLOG)	3. $x > 0$ & $y < 0$	(c) $x < 0$ & $y \geq 0$	[3. + 4.]
	4. x anything $y = 0$	(d) $x < 0$ & $y < 0$	[2.]

Exercises

1-17 ~~Suppose that s ∈ ℝ is the supremum of S.~~ Consider $S=\emptyset$. Observe that every $u \in \mathbb{R}$ is an upper bound of S (vacuously). But no $s \in \mathbb{R}$ is the supremum of S, since $s-1$ is ~~a~~ an upper bound of S such that $s>s-1$, for any $s \in \mathbb{R}$. Additionally, for any upper bounds s of S and $\varepsilon>0$, there (vacuously) does not exist an $x \in S$ so that $|s-x|<\varepsilon$. Thus, the statement holds true for S being empty. Now, we move onto the case of S being nonempty. If $S=\mathbb{R}$, then no upper bounds of S exists ~~in the first place~~, since for all $x \in \mathbb{R}$, there is the member $x+1$ of S with $x+1>x$, and hence $x \ngeq x+1$. Thereupon, the statement is again true, as S is not bounded above in the first place. When nonempty S is a proper subset of $\mathbb{R}$ that is bounded above, we first suppose $s \in \mathbb{R}$ is the supremum of S. Immediately, s is an upper bound of S, ~~meaning the set S is bounded above~~. Applying Proposition 1.21, we see that for every $\varepsilon>0$, there is an element $x \in S$ with $s-x<\varepsilon$. From s being an upper bound of S, it holds that $x \leq s$ and $s-x \geq 0$. Hence, $|s-x|<\varepsilon$. As desired, we have shown that $s \in \mathbb{R}$ being the supremum of S implies s is an upper bound of S and for all $\varepsilon>0$, there is an $x \in S$ so that $|s-x|<\varepsilon$. Conversely, assume s is an upper bound of S and for any $\varepsilon>0$, there exists a $x \in S$ such that $|s-x|<\varepsilon$ (where, again, S is a nonempty proper subset of $\mathbb{R}$ that is bounded above). Once more, by s being an upper bound of S, it follows that $s-x \geq 0$ certainly. That is, $|s-x|=s-x<\varepsilon$. Therefore, $s<x+\varepsilon \leq u+\varepsilon$, meaning $s<u+\varepsilon$ for any upper bounds u of S. It must be that $s \leq u$ or $s>u$. In the former, $s<u+\varepsilon$ is a consequence that is consistent with the fact mentioned above. In contrast, for $s>u$, $s-u>0$. As a result, let $\varepsilon=s-u>0$. We notice that $s-x \geq \varepsilon=s-u$ for all $x \in S$. Which clearly contradicts our prior assertion, that $s-x<\varepsilon$ for some $x \in S$. Consequently, the only possibility is that $s \leq u$, which holds for all upper bounds u of S. Accordingly, we have just proven that if s is an upper bound of nonempty $S \subsetneq \mathbb{R}$, then $s \in \mathbb{R}$ is the supremum of S. Combining the two facts we just proven, the biconditional statement is certainly true for any nonempty proper subset of $\mathbb{R}$. Recall that we have also shown the statement to be true for the special cases of $S=\emptyset$ and $S=\mathbb{R}$, at the beginning of the proof. Wherefore, for $S \subseteq \mathbb{R}$ bounded above, $s \in \mathbb{R}$ is the supremum of S iff s is an upper bound of S and for all $\varepsilon>0$ there is an $x \in S$ so that $|s-x|<\varepsilon$. □

1-18 (a) Let A and B be nonempty ~~prop~~ subsets of $\mathbb{R}$ that are bounded above. By Axiom 1.19, we know that $\sup(A)$ and $\sup(B)$ both exists. In addition, notice that $\sup(B) \geq a$, for any $a \in A$ (which is also an element of B). Hence, $\sup(B)$ is an upper bound of A. Consequently, $\sup(A) \leq \sup(B)$ clearly, by definition. □

(b) Near-identical to (a).

1-19 Let A be a nonempty subset of $\mathbb{R}$ that is bounded above and $A'=\{x \in \mathbb{R} \mid -x \in A\}$. We notice that for all $x \in A$, $\sup(A) \geq x$; where $\sup(A)$ exists by the Completeness Axiom. In other words, $-\sup(A)$ is a lower bound of A'. Consider any lower bounds ℓ of A'. We see that $-\ell \geq x$ for any $x \in A$, which tells us that $-\ell$ is also a upper bound of A. Hence, $\sup(A) \leq -\ell$. That is, $-\sup(A) \geq \ell$ for all lower bounds ℓ of A'. Consequently, $-\sup(A)=\inf(A')=\inf\{x \in \mathbb{R} \mid -x \in A\}$ indeed. □

Proof that $|x^n| = |x|^n$ for all natural n and real x.

Let the subset S of $\mathbb{N}$ contain only the naturals n with $|x^n| = |x|^n$ for any $x \in \mathbb{R}$. First, it is clear that $0 \in S$ since $|x^0| = 1 = |x|^0$. Now, assume $n \in S$. Then, we notice that $|x^{n+1}| = |x^n \cdot x|$, which is equivalent to $|x^n||x|$ by part 3 of theorem 1.13. Hence, we see that $|x^{n+1}| = |x^n||x| = |x|^n|x| = |x|^{n+1}$ as $n \in S$. Consequently, it follows that $n+1 \in S$. Wherefore, by induction $S = \mathbb{N}$.

□

Exercises

1-17 Ideas

$(\Rightarrow)$

Show: There exists $x \in S$ so that either

1. $\underbrace{s - x \geq 0}_{\text{Since } s \text{ is upper bound, so true for all } x \in S}$ and $\underbrace{s - x < \varepsilon}_{\text{Prop 1.21.}}$

when $S \neq \emptyset$.

2. $\underbrace{s - x < 0}_{\text{S is upper bound, so for all } x \neq s \text{ in } S,\ \cancel{x - s} < 0\ (\cancel{x} < s)}$ ~~x - s < 0~~ + $\underbrace{x - s < \varepsilon}_{\text{Since } x - s < 0 < \varepsilon,\ x - s < \varepsilon.}$

→ Not possible since $s < x$ contradicts s being (least) upper bound)

If $S = \emptyset$,

every $u \in \mathbb{R}$ is an upper bound of S

but no $s \in \mathbb{R}$ is the sup of S, lest $s = \sup(\emptyset)$ but $s > s - 1$.

Vacuously no such $x \in S$ exists.

$(\Leftarrow)$ (Assume) s is upper bound & for all $\varepsilon > 0$, there is an $x \in S$ so $|s - x| < \varepsilon$.

$0 \leq s - x < \varepsilon$ or $0 < x - s < \varepsilon$

$s < x$ X

Contradicts s being ~~(least)~~ upper bound by assumption.

$x \leq s$

$s < x + \varepsilon \leq u + \varepsilon$

$s < u + \varepsilon$ → Can't be true for any upper bound u of S.

(for all u) Either $s \leq u$ or $s > u$

$s < u + \varepsilon$, ~~$s + \varepsilon > u + \varepsilon$~~ → Let $\varepsilon > 0$ be $s - u$.

Then, $s - x \leq s - u$ for all $x \in S$. X

Contradiction.

$0 < s - u$ $\quad$ $0 \leq u - x$ $\quad$ $x \leq u$

$-x \leq -u$

$0 < s - x \leq s - u$

$s - x \leq s - u$

Show: $s \leq u$ for all upper bounds u of S.

Exercises

1-25. Assume, for the sake of contradiction, that $a, b \in \mathbb{R}$ are such that for all $\varepsilon > 0$, we have $a \leq b + \varepsilon$, but $a > b$. That is, $a - b > 0$. Now, fix $\varepsilon = 2(a-b)$. Thus, $a \leq b + \varepsilon$, meaning $a \leq 2a - b$. And accordingly, $b \leq a$. Simultaenously, $a \leq b$ when $\varepsilon = 0$. Consequently, $a = b$. However, this contradicts our original assumption that a is strictly greater than b. Wherefore, it must be that, actually, if $a, b \in \mathbb{R}$ and for any $\varepsilon > 0$ we have $a \leq b + \varepsilon$, then $a \leq b$. $\square$

$a - b$

$a \leq b$

$a \leq b + \varepsilon$

$a - b \leq \varepsilon$

$a > b$

$a - b > 0$

$a \leq b + \varepsilon$

$a \leq b + \frac{1}{2}(a-b)$

$a \leq \frac{1}{2}b + a$

$0 \leq \frac{1}{2}b$

$a \leq b + 2(a-b)$

$a \leq 2a - b$

$b \leq a$

but $a \leq b$

$a = b$ contradict $a > b$

Self Proof of Theorem 1.36

$\frac{n}{n}+a \geq b + \frac{l}{n}$

Ideas

$b \leq \frac{l+1}{n}$

$nb \leq l+1$

$\lfloor a \rfloor \leq a < q < b \leq \lceil b \rceil$

$\lfloor a \rfloor \leq a \leq \lceil a \rceil \quad \lfloor b \rfloor \leq b \leq \lceil b \rceil$

$\lceil a \rceil - \lfloor a \rfloor = 1$ or 0; test $\lceil a \rceil - \lfloor a \rfloor > 1$

$\lceil a \rceil - 1 > \lfloor a \rfloor$

Either $a \geq \lceil a \rceil - 1$ or $a < \lceil a \rceil - 1$

Contradict $\lfloor a \rfloor$ being the largest integer leq x.

Contradict $\lceil a \rceil$ being the smallest integer geq x.

Going on a tangent lol.

$a < \frac{n}{m} < b$

$\iff a < \frac{n}{m}$ & $\frac{n}{m} < b$

$\iff am < n$ & $n < bm$

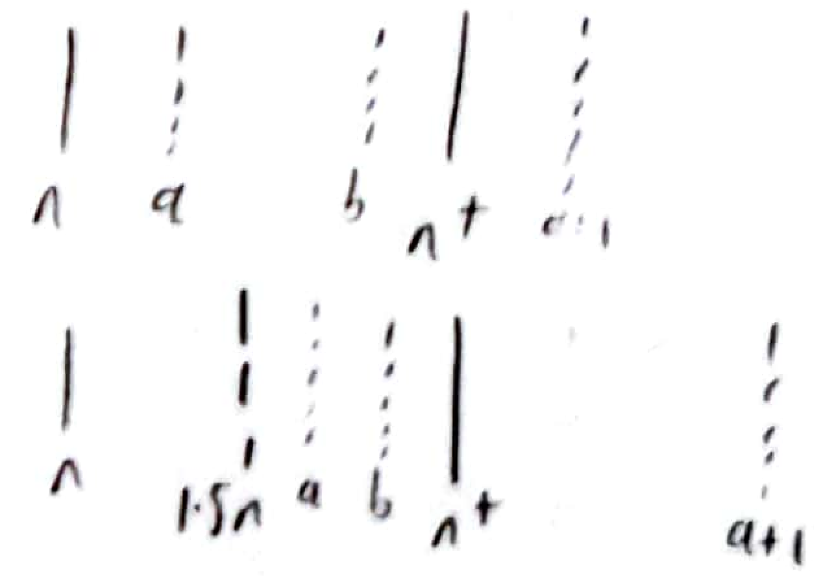
f_a f_b | F_a f_a | a | F_a | f_b | F_a f_b | b | F_a F_b

$\lfloor a \rfloor \leq \lfloor b \rfloor \quad \lceil a \rceil \leq \lceil b \rceil$

$b - a > 1$ trivial

$1 \geq b - a > 0$

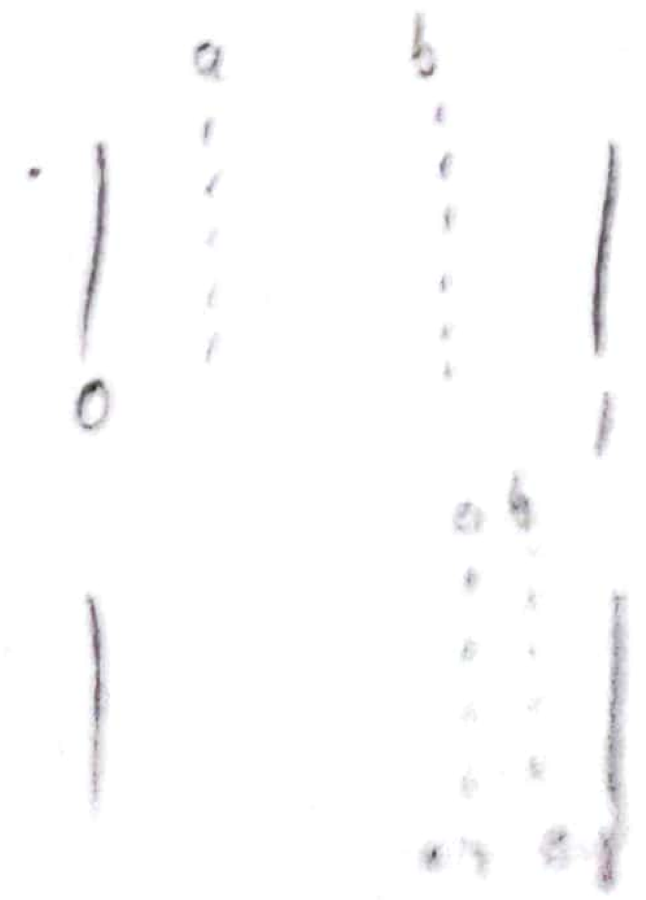
$a + 1 \geq b$



Note for tmr: Perhaps we have been looking at this with the 'wrong' idea. We might have to replicate (to some extent) what was done in Thm 1.32 instead of using the floor & ceiling functions.

$\lfloor b - a \rfloor$

$0 \leq \frac{\lceil b \rceil}{a+b} \leq 1$



(a) We see that $\frac{p}{2^n} = p\cdot(2^n)^{-1} = p\cdot 2\cdot(2^n)^{-1}\cdot 2^{-1}$. Utillising exercise 1-5, $(2^n)^{-1}\cdot 2^{-1} = (2^n\cdot 2)^{-1} = (2^{n+1})^{-1}$. So, $\frac{p}{2^n} = (2p)\cdot(2^{n+1})^{-1} = \frac{2p}{2^{n+1}}$. Therefore, every $\frac{p}{2^n}$ in D_n is also in D_{n+1}. Also, notice that $\frac{1}{2^{n+1}} \neq \frac{p}{2^n}$ for every integer p. Otherwise, $(2^{n+1})^{-1} = p\cdot(2^n)^{-1}$. And by exercise 1-5 again, $(2^n)^{-1}\cdot 2^{-1} = p\cdot(2^n)^{-1}$. As a result, $p = 2^{-1}$. Which would not be an integer; because $2^{-1} \neq 0$, and since $0 < 2^{-1} < 1$, we conclude with Proposition 1·26 that $2^{-1} \notin \mathbb{N}$ (and accordingly $-2^{-1} \notin \mathbb{N}$)

Key Idea

$\frac{p}{2^n} = \frac{2p}{2^{n+1}}$

(b) Clearly, every element of $\bigcup_{n=1}^{\infty} D_n$ must be a dyadic rational number (and so in D) by the provided definition of $\bigcup_{n=1}^{\infty} D_n$. That is, $\bigcup_{n=1}^{\infty} D_n \subseteq D$. Similarly, if $r \in D$, then there is certainly some integer p and natural n such that $r = \frac{p}{2^n}$. Accordingly, $r \in D_n$, and resultantly, $r \in \bigcup_{n=1}^{\infty} D_n$. In other words, $D \subseteq \bigcup_{n=1}^{\infty} D_n$. As such, $D = \bigcup_{n=1}^{\infty} D_n$.

□

(c)

$\frac{x}{n} < \varepsilon$

$\frac{x}{\varepsilon} < n$

$\frac{\left(\frac{m'}{n'}\right)}{\left(\frac{m}{n}\right)} < k$

$\frac{m'n}{mn'} < k$

$\frac{p}{q} > \frac{r}{s}$

$ps > qr$

$ps - qr > 0$

exists $n > ps - qr$ by Theorem 1·32

Want to find $\frac{z}{2^n}$ with

$\frac{p}{q} > \frac{z}{2^n} > \frac{r}{s}$

$p(2^n) > zq$ & $zs > r(2^n)$

We know

$\frac{p}{q}(2^n) > \frac{r}{s}(2^n)$

[Exercise 1-2(

$z = (m-1)(2$

$\min\left\{z \in \mathbb{Z} \,\middle|\, \frac{z}{2^n} > \frac{r}{s}\right\}$

$z = |r|2^n$

bounded below by $z = m(2^n)$

(Exercise 1-26)

$\max\left\{z \in \mathbb{Z} \,\middle|\, \frac{z}{2^n} < \frac{p}{q}\right\}$

bounded above by $z = |p|2^n$

1-29 (c)

Ideas

$\frac{r}{s} < \frac{z}{2^n} < \frac{p}{q}$

Ass. for contradiction

$$\{z \in \mathbb{Z} \mid \frac{z}{2^n} > \frac{p}{q}\} \cap \{z \in \mathbb{Z} \mid \frac{z}{2^n} < \frac{r}{s}\} = \emptyset$$

$\frac{p}{q} > \frac{r}{s}$

$\frac{p}{q} - \frac{r}{s} > 0$

By Theorem 1.32, $n \geq \frac{p}{q} - \frac{r}{s} > 0$

$\frac{u}{2^n} < \frac{p}{q}$ & $\frac{r}{s} < \frac{l}{2^n}$

$2^n \geq \frac{p}{q} - \frac{r}{s} > 0$

$\frac{1}{(\frac{p}{q} - \frac{r}{s})} \geq \frac{1}{2^n}$

$2^n + \frac{r}{s} \geq \frac{p}{q}$

$2^n + \frac{l}{2^n} > \frac{p}{q} > \frac{u}{2^n}$

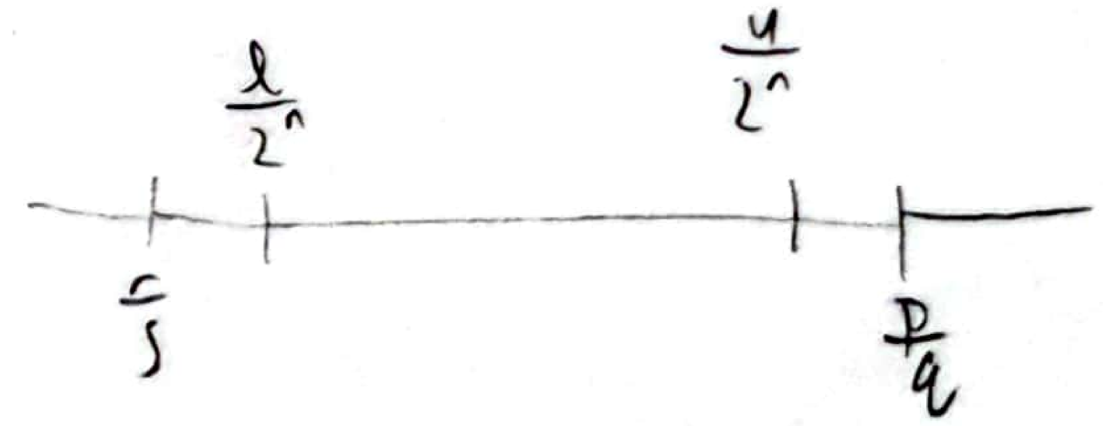
$2^n + \frac{l}{2^n} > \frac{u}{2^n}$

$2^n > \frac{u-l}{2^n}$

$\frac{p}{q} - \frac{r}{s} > \frac{u}{2^n} - \frac{l}{2^n} = \frac{u-l}{2^n} > 0$

$\frac{u}{2^n} > \frac{l}{2^n}$

$\Rightarrow \frac{p}{q} > \frac{u}{2^n} > \frac{l}{2^n} > \frac{r}{s}$



Ideas

-30 (a) x rational $\Rightarrow$ trivial

x irrational $\Rightarrow$ Assume for the sake of cont. that $x \neq \sup\{r \in \mathbb{Q} \mid r \le x\}$. $\Rightarrow$ there is some $\overbrace{\text{real number } y}^{y = \sup \cdots < x}$ with $r \le y < x$ for every r, by the Completeness Ax
Since x is certainly an upper bound of $\{r \in \mathbb{Q} \mid r \le x\}$.

By Thm 1.36, there is some rational number p with $y < p < b$.
Contradicts $y = \sup \cdots$ since $p \in \{r \in \mathbb{Q} \mid r \le x\}$.

(b)

~~Let $\frac{z}{n} = \frac{z'}{n'}$ be rational numbers (where z and z' are integers, while n and n' are naturals).~~ Oops thought Thm 1.39 has been introduced. $\Downarrow$

$f(1)$ well-defined since $f(1) := \tilde{1}$, $-n \neq 1$ for any $n \in \mathbb{N}$ — meaning $f(-n) := \tilde{-}\tilde{n}$ is not applicable, and lastly, $f(\frac{1}{1}) = \frac{\tilde{1}}{\tilde{1}} = \tilde{1}$.
Which means, $f(1)$ is always equal to $\tilde{1}$, and nothing else. Assume $f(n)$ is single-valued. Again, the mapping $f(-n) := \tilde{-}\tilde{n}$ cannot be applied.
Similarly, when $\frac{m}{d} = n$, $m = nd$. So, $f(\frac{nd}{d}) = f(n)$. Therefore, $f(n+1)$ is only equivalent to the unique number $f(n) \tilde{+} \tilde{1}$.
By induction, $f(n)$ is unique — and thus f is not self contradictory — for every natural number n. As a result, $f(-n) := \tilde{-} f(n)$
must be too, for the same reasons. Consequently, $f(\frac{n}{d}) := \frac{f(n)}{f(d)}$, because $f(n)$ and $f(d)$. (hence the additive inverse of $f(d)$)
are unique. Wherefore, we can conclude that f is non-self-contradictory.

$f(x) \in \tilde{\mathbb{Q}}$ for each $x \in \mathbb{Q}$ is trivial. $nd' < n'd \overset{\text{show}}{\Rightarrow} f(nd') <$

Suppose that $x < z$. In other words, $\overbrace{\frac{n}{d} < \frac{n'}{d'}}$. First notice that $f(n) < f(n')$ as long as $n < n'$.

Proof of above statement: Classical double induction I don't wanna formally write out :P

Thus, f preserves order for natural numbers. It also does for all integers, since $f(-n) = \tilde{-} f(n)$ and additive inverses are, again, uniq

$$f(nm) \overset{\text{show}}{=} f(n) f(m) \qquad f(nm) = (\widetilde{nm})$$
$$= \tilde{n}\tilde{m}$$

1-30(a)

Assume, for the sake of contradiction, that for some $x \in \mathbb{R}$ we have that $x \neq \sup\{r \in \mathbb{Q} \mid r \leq x\}$. Clearly, x is an upper bound of $\{r \in \mathbb{Q} \mid r \leq x\}$, i.e. $x > \sup\{r \in \mathbb{Q} \mid r \leq x\}$. By Theorem 1-36, there is some rational number p with $\sup\{r \in \mathbb{Q} \mid r \leq x\} < p < x$. However, now $p \in \{r \in \mathbb{Q} \mid r \leq x$ and $p > \sup\{r \in \mathbb{Q} \mid r \leq x\}$ is true simultaenously, a contradiction. Therefore, it must be true that $x = \sup\{r \in \mathbb{Q} \mid r \leq x\}$ as long as $x \in \mathbb{R}$

$x > \sup\{r \in \mathbb{Q} \mid r \le x\}$. By Theorem 1.36, there is some rational number p with $\sup\{r \in \mathbb{Q} \mid r \le x\} < p < x$. However, now $p \in \{r \in \mathbb{Q} \mid r \le x\}$

Self - Proof of Proposition 1.49

Quick Sketch / Ideation

Assume there exists some rational number r such that $r^2 = 2$. i.e. $r = \frac{p}{q}$ for some integer p and natural q. If $p = nm$ and $q = nk$ for some integers n, m, k ($n \neq 1$) then $r = \frac{m}{k}$. Hence suppose wlog that no such n exists.

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

Show $p = 2k_1$ for some $k_1 \in \mathbb{N}$.

$p \neq 2k \Rightarrow p^2 \neq 2k'$

$p^2 = 2k \Rightarrow p = 2k'$

Prove for all $p \in \mathbb{N}$, this holds. $S := \{p \in \mathbb{N} \mid (\exists k \in \mathbb{N})\, p^2 = 2k \Rightarrow (\exists k' \in \mathbb{N})\, p = 2k'\}$

$p = 1$: No such k exists. $k \in S$

Assume $p \in S$:

$(p+1)^2 = 2k$

$T := \{k \in \mathbb{N} \mid (p+1)^2 = 2k \Rightarrow (\exists k' \in \mathbb{N})(p = 2k')\}$

$1 \in T$

Suppose $k \in T$

$(p+1)^2 = 2(k+1)$

$p^2 + 2p + 1 = 2k + 2$

$p^2 + 2p = 2k + 1$

$p^2 = 2(k-p) + 1$

Then $2(2k_1^2) = 2q^2$

$2k_1^2 = q^2$

$2k_1^2 = 2k_2^2$

Cont.

for the sake of contradiction, that for some $x \in \mathbb{R}$ we have that $x \neq \sup\{r \in \mathbb{Q} \mid r \le x\}$ ... is some rational number p with $\sup\{r \in \mathbb{Q} \mid r \le x\} < p < x$. However, now $p \in$ ...

... $x = \sup\{r \in \mathbb{Q} \mid r \le x\}$ as long as $x \in \mathbb{R}$.

Self Proof of Theorem 1.5

☆ Rough Ideas (i.e. not writing in out in proper phrasing) Just rough work to brainstorm + check ideas

(ii)

Lemma 1: $a^n a^m = a^{n+m}$ for each $n, m \in \mathbb{Z}$

Case 1. $n, m \in \mathbb{N}$ trivial

Case 2 $n, m \in \mathbb{Z} \setminus \mathbb{N}$

~~$\frac{1}{\prod_{i=1}^{-n} a} \cdot \frac{1}{\prod_{i=1}^{-m} a} = \prod_{i=1}^{-n} \frac{1}{a} \cdot \prod_{i=1}^{-m} \frac{1}{a}$ by induction~~

~~$= \prod_{i=1}^{-n-m} \frac{1}{a}$~~

~~$= \frac{1}{\prod_{i=1}^{-(n+m)} a}$~~

~~$=$~~

Show: $(a^p)^{\frac{1}{q}} (a^r)^{\frac{1}{s}} = a^{\frac{ps+qr}{qs}}$
Eventually

$a^{ps} a^{qr} = a^{ps+qr}$

$$a^n a^m = (a^{-n})^{-1} \cdot (a^{-m})^{-1}$$
$$= (a^{-n} a^{-m})^{-1} \text{ by } 1\text{-}5 \ (a=0 \text{ trivial})$$
$$= (a^{-(n+m)})^{-1} \text{ by case } 1$$
$$= a^{n+m}$$

$2 \cdot 2^{-1} \cdot 2^{-1}$

Case 3 $n \in \mathbb{N}$ but $m \in \mathbb{Z} \setminus \mathbb{N}$

$$a^n a^m = \left(\prod_{i=1}^{n} a\right) \cdot \left(\prod_{i=1}^{-m} a^{-1}\right)$$ by induction with 1-5 (Fact 2) (Can show $\left(\prod_{i=1}^{k} a\right)^{-1} = \prod_{i=1}^{k} a^{-1}$, $(a^k)^{-1} = (a^{-1})^k$)

If $-m > n$,

$$= \left(\prod_{i=1}^{n} a\right) \cdot \left(\prod_{i=1}^{n} a^{-1}\right) \cdot \left(\prod_{i=n+1}^{-m} a^{-1}\right)$$
$$= \left(\prod_{i=1}^{n} a\right) \cdot \left(\prod_{i=1}^{n} a\right)^{-1} \cdot \left(\prod_{i=1}^{-m-n} a\right)^{-1}$$ by Fact 2 + Reindexing products

$= a^{n+m}$

(i)

$$(ab)^x = (ab)^{\frac{p}{q}}$$
$$= \left[(ab)^p\right]^{\frac{1}{q}}$$

Wlog suppose $p \in \mathbb{N}$. By induction, we can prove that $(ab)^p = a^p b^p$. Hence,

$$(ab)^x = (a^p b^p)^{\frac{1}{q}}$$
$$= (a^p b^p)^{\frac{1}{q}}$$

(Thoughts in process)

Show $(a^p b^p)^{\frac{1}{q}} = (a^p)^{\frac{1}{q}} (b^p)^{\frac{1}{q}}$

~~$I := (a^p)^{\frac{1}{q}}$~~ ~~$J := (b^p)^{\frac{1}{q}}$~~

$I^q := a^p$ $\quad J^q := b^p$

Now, $(a^p b^p)^{\frac{1}{q}} = (I^q J^q)^{\frac{1}{q}}$ for some unique I and J whose existence is guaranteed by Theorem 1.47.

$$= (IJ)^{\frac{q}{q}}$$ induct
$$= IJ$$
$$= a^{\frac{p}{q}} b^{\frac{p}{q}}$$

So, $(ab)^x = a^x b^x$ given $x \in \mathbb{Q}$.

◇

If $n \ge m$, repeat similar procedure. By Lemma 1,

$$a^{ps} a^{qr} = a^{ps+qr}$$
$$(a^{ps} a^{qr})^{\frac{1}{qs}} = (a^{ps+qr})^{\frac{1}{qs}}$$
$$a^{\frac{p}{q}} a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}} \text{ from (i)}.$$

◇

Self - Proof of Theorem 1.5 / Rough ideas, again

wlog suppose $p \in \mathbb{N}$

(iii)

Show: $\frac{a^x}{a^y} = a^{x-y}$

by defn. $\Rightarrow \underline{a^{\frac{p}{q}} a^{-\frac{r}{s}}} = a^{\frac{p}{q} - \frac{r}{s}}$

$a^{\frac{p}{q} - \frac{r}{s}} =$ by (ii)

$\frac{a^{\frac{p}{q}}}{a^{\frac{r}{s}}} = \cdots = a^{\frac{p}{q} - \frac{r}{s}}$ ◇

(iv)

Show: $(a^x)^y = a^{xy}$

$(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$

$(a^{\frac{p}{q}})^r = a^{\frac{pr}{q}} = (a^{pr})^{\frac{1}{q}}$

$\prod_{i=1}^{r} a^{\frac{p}{q}}$

$((a^p)^{-q})^r = (a^{pr})^{\frac{1}{q}}$

$((a^p)^{-q})^{rq} = a^{pr}$

Fact 3

$(a^n)^m = a^{nm}$ by induction

If $r > 0$, i.e. $r \in \mathbb{N}$

$((a^p)^{-q})^{rq} = ((a^p)^{-q \cdot q})^r$ by fact 3

$= (a^p)^r$

If $r \leq 0$, i.e. $r \in \mathbb{Z} - \mathbb{N}$,

$((a^p)^{-q})^{rq} = ((a^p)^{-q})^{-r}$

⋮

If $p \in \mathbb{Z} - \mathbb{N}$,

$(a^p)^r = [(a^{-p})^{-1}]^r$

$= (a^{-p})^{-r}$ fact 2

$= a^{pr}$ fact 2

⋮

(v) $\left(\frac{a}{b}\right)^x = a^x (b^{-1})^x$ by (i)

$= a^x (b^x)^{-1}$ by (iv)

$= \frac{a^x}{b^x}$

◇

Well we can rewrite this properly if we wanted

Theorem 1.47, Self Proof

Ideas:

$n=1$: $r=a$

Assume that for every nonnegative a there exists a nonnegative r so that $r^n=a$.

Show $r^n = s^{n+1}$

$\{r\in\mathbb{R}^+_0 \mid r^n \geq a\}$, $i^n > a$ or $i^n \leq a$

lower bounded by $r=0$
Not upper bounded
inf exists, say i

$\frac{1}{m_1} > a$ $\quad a > \frac{1}{m_2}$ $\quad (m_2 > m_1)$

~~Top $:= \{k\in\mathbb{Z} \mid (\frac{k}{m_1})^n > a\}$~~

~~Unbounded from above~~

lower bound of $k=0$

~~Bot $:= \{k\in\mathbb{Z} \mid (\frac{k}{m_2})^n < a\}$~~

~~upper bound of $k = m_2(a+1)$~~

~~Unbounded from below~~

$(a+1)^n = \sum_{i=0}^{n}\binom{n}{i}a^i$

$= 1 + a + na^2 + \frac{n(n-1)}{2}a^3 + \dots > a$

$\{x\in\mathbb{R}^+_0 \mid x^n \leq a\}$
bounded below by 0
bounded above by $a+1$
sup exists, say some r

correct step / idea

$r^n \leq a$ or $r^n > a$

Wlog, suppose $r^n < a$.

Hint: Find q with
$r^n < (r+q)^n < r^n+\varepsilon < a$

ε can be $\frac{a-r^n}{2}$ for example

$\underbrace{}_{\frac{a+r^n}{2}}$

$(r + \frac{a-r^n}{a})^n$, $r>0$, $1-\frac{r^n}{a}$

$[r+(a-r^n)a]^n$, $r<0$

$\sum_{i=0}^{n}\binom{n}{i}r^{n-i}q^i$

$= r^n + \underbrace{\dots + q^n}_{<\varepsilon\ (=\frac{a-r^n}{2})}$

$x^2 = 2$ $\quad x^2 \leq x^2+y^2 \leq (x+y)^2 = x^2+y^2+2xy$

$S := \{r\in\mathbb{R}^+_0 \mid r^n \leq 2\}$

$x^2 \leq 2$ $\quad x \leq r$

$x^2 \leq rx$ & $xr \leq r^2$

$\Rightarrow x^2 \leq r^2 \leq 2$ for all $r\in S$

Assume $x^2 < 2$,

$\frac{x^2+2}{2} < 2$

$\frac{1}{2}x^2 + 1 < 2$

$(x+\ \ )^2 < x^2 + \frac{2-x^2}{2} < 2$

find ε with
$(x+\varepsilon)^2 < 2$

$y^2 \leq \frac{2-x^2}{2}$. If $y^2 < \frac{2-x^2}{2}$,

$(2-2y^2)^2$
$= 4 - 8y^2 + 4y^4$

$r^2 = 2$

Let $\sup\{x \in \mathbb{R} \mid x^2 \leq 2\} = r$.

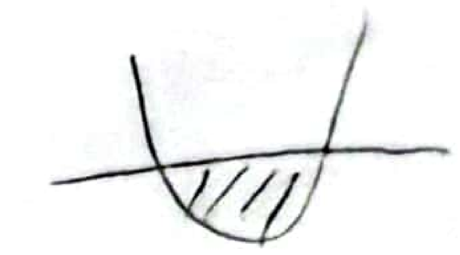
Suppose ftsoc that $r^2 < 2$:

$(r+\varepsilon)^2 < r^2 + \delta < 2$

Find such a ε.

$(r+\varepsilon)^2 = r^2 + \varepsilon^2 + 2\varepsilon r$

$\varepsilon^2 + 2\varepsilon r \leq \delta$



$\varepsilon^2 + 2\varepsilon r - \delta \overset{\leq}{=} 0$

$\varepsilon = \dfrac{-2r \pm \sqrt{4r^2 + 4}}{2}$

$= -r \pm \sqrt{r^2 + \delta}$

ignoring negative branch

$\varepsilon^2 = r^2 +$

Self Proof of Theorem 1.47

Ideas

$S := \{ x \in \mathbb{R}_0^+ \mid x^n \leq a \}$ $0 \in S$, upper bounded by $a+1$ by the binom thm

Call $\sup S =: r$

Suppose ftsoc that $r^n < a$,

$m > a - r^n > 0$ for some natural m

$1 > \frac{a-r^n}{m} > 0$

$M := \max \{ \binom{n}{k} r^{n-k} \mid k \leq n \} + 1$

$T := \{ x \in \mathbb{R}_0^+ \mid x^n \leq a - r^n \}$

$\delta \in T$

$\left(\frac{\delta}{Mn}\right)^k < \left(\frac{\delta}{Mn}\right)^k$

$\binom{n}{k} r^{n-k} \delta^k$ $\qquad \binom{n}{k} r^{n-k} \left(\frac{\delta}{Mn}\right)^k$

$\binom{n}{k} r^{n-k} \left(\frac{1}{M}\right)^k < \binom{n}{k} r^{n-k} \left(\frac{1}{M}\right) < 1$

$\binom{n}{k} r^{n-k} \left(\frac{1}{Mn}\right)^k < \frac{1}{n}$

$\binom{n}{k} r^{n-k} \left(\frac{\delta}{Mn}\right)^k < \frac{a-r^n}{n}$

$$\sum_{k=0}^{n} \binom{n}{k} r^{n-k} \left(\frac{\delta}{Mn}\right)^k < r^n + a - r^n = a$$

$a - r^n > \frac{a-r^n}{m} > \left(\frac{a-r^n}{m}\right)^n$

If $r > a$, repeat procedure

Proof

Let $r := \sup\{x \in \mathbb{R}_0^+ \mid x^n \leq a\}$ which must exist as this set contains 0 and is bounded above by $a+1$. Suppose, for the sake of contradiction, that $r^n < a$. By theorem 1.32, there exists some natural number $\bar{m}$ with $\bar{m} > a - r^n > 0$ so $1 > \frac{a-r^n}{\bar{m}} > 0$. As such, $a - r^n > \frac{a-r^n}{\bar{m}} \geq \left(\frac{a-r^n}{\bar{m}}\right)^k$ for any natural $k \geq 1$. Define $\delta := \frac{a-r^n}{\bar{m}}$ and $M := \max\{\binom{n}{k} r^{n-k} \mid k \leq n\} + 1$. Now notice that given any natural $1 \leq k \leq n$, $\binom{n}{k} r^{n-k} \left(\frac{1}{M}\right)^k < 1$ and hence $\binom{n}{k} r^{n-k} \left(\frac{1}{Mn}\right)^k < \frac{1}{n}$. Thus, $\binom{n}{k} r^{n-k} \left(\frac{\delta}{Mn}\right)^k < \frac{\delta^k}{n} < \frac{a-r^n}{n}$. Consequently,

$$
\begin{aligned}
\left(r + \frac{\delta}{Mn}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \left(\frac{\delta}{Mn}\right)^k \\
&= r^n + \sum_{k=1}^{n} r^{n-k} \left(\frac{\delta}{Mn}\right)^k \\
&< r^n + n \cdot \frac{a-r^n}{n} \\
&= r^n + a - r^n \\
&= a
\end{aligned}
$$

even though $r + \frac{\delta}{Mn} > r$. Therefore, contradicting the construction of r. By repeating a similar procedure, we can show the impossibility of $r^n > a$. Wherefore, it then must be that $r^n = a$. $\square$

$T := \{ x \in \mathbb{R}_0^+ \mid x^n \leq a - r^n \}$

Chapter 1.5

Exercises

1-35 The base case of $n=1$ is trivial since $\sum_{j=s}^{s+1} a_j = a_s + a_{s+1} = a_{1+s-1} + a_{2+s-1} = \sum_{k=1}^{1+1} a_{k+s-1}$. Now assume this is true for n. Then notice $\sum_{j=s}^{s+n+1} a_j = a_{s+n+1} + \sum_{j=s}^{s+n} a_j = a_{n+2+s-1} + \sum_{k=1}^{n+1} a_{k+s-1} = \sum_{k=1}^{n+2} a_{k+s-1}$. Which proves the $(n+1)$th case. By induction, this holds for every $n \in \mathbb{N}$. $\square$

1-38

Ideas

$a^n < b^n$ is certain for $n \in \mathbb{N}$ — Trivial for $a=0$.

Show $a^{\frac{n}{m}} < b^{\frac{n}{m}}$ — Lemma 1-38A

$(a^n)^{\frac{1}{m}} < (b^n)^{\frac{1}{m}}$ lest $(a^n)^{\frac{1}{m}} = (b^n)^{\frac{1}{m}}$ or $(b^n)^{\frac{1}{m}} < (a^n)^{\frac{1}{m}}$

$a^n = b^n$ — Cont. w/ L1-38A

$b^n < a^n$ by L1-38A — Cont.

Proof

Lemma 1-38A For any $n \in \mathbb{N}$, $a^n < b^n$.

Proof: When $n=1$, $a<b$ by definition. Suppose $a^n < b^n$. Then $a^{n+1} < ab^n < b^{n+1}$ so that $a^{n+1} < b^{n+1}$. By induction, this indeed holds for every $n \in \mathbb{N}$. ◇

It must follow that $a^{\frac{n}{m}} < b^{\frac{n}{m}}$ for each $n, m \in \mathbb{Q}$, lest $(a^n)^{\frac{1}{m}} = (b^n)^{\frac{1}{m}}$ or $(b^n)^{\frac{1}{m}} < (a^n)^{\frac{1}{m}}$. In the former, $a^n = b^n$ but this contradicts $a^n < b^n$ by Lemma 1-38A. Similarly in the latter, $b^n < a^n$ follows by Lemma 1-38A, again contradicting $a^n < b^n$. Thus, $a^{\frac{n}{m}} < b^{\frac{n}{m}}$ is the only possibility by trichotomy.

$\square$

Ideas

$S := \{x \in \mathbb{R}_0^+ \mid x^n \le a\}$ $0 \in S$, upper bounded by $a+1$ by the binom thm

Call sup $S =: r$

Suppose there short $r^n < a$, $\left(\frac{\delta}{Mn}\right)^k < \left(\frac{\delta}{n}\right)^k$

$\{x \in \mathbb{R}^+ \mid x^n \le a - r^n\}$

1-39 (a) We know $x = \frac{n}{m}$ for some $n \in \mathbb{N}$ and $m \in \mathbb{N}$ as $x > 1$ so $n > m$. By induction we can show that $a^n > a^m$. From there, it is clear that $a^x > a$. $\square$

(b)

⋮

(d)

1-42

Ideas

$|x_1| = |x_1| \Rightarrow n = 1$ ✓

Assume true for n. Apply Δ

Proof

When $n=1$, this inequality trivially holds as $\left|\sum_{i=1}^{1} x_i\right| = |x_1| = \sum_{i=1}^{1} |x_i|$. Thus assume this is true for n. We notice that $\left|\sum_{i=1}^{n+1} x_i\right| = \left|x_{n+1} + \sum_{i=1}^{n} x_i\right| \le |x_{n+1}| + \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n+1} |x_i|$ by the triangular inequality and our assumption.

Consequently, this holds for any $n \in \mathbb{N}$ by induction. $\square$

1-43

Ideas

Show $ab \le \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4}$

$2ab \le a^2 + b^2$

$0 \le a^2 - 2ab + b^2$

$0 \le (a-b)^2 = (b-a)^2$

Proof

First assume without loss of generality that $a > b \ge 0$, because if $a = b = 0$ the result holds instantly. Thus, $a - b > 0$ and $(a-b)^2 > 0$. Which means $a^2 + b^2 > 2ab$. Then $ab < \left(\frac{a+b}{2}\right)^2$ and $\sqrt{ab} < \frac{a+b}{2}$ follows by our power laws. $\square$

1-44 (a) Notice $\sqrt{a^2+b^2} = \sqrt{|a|^2 + |b|^2} \le \sqrt{|a|^2 + 2|a||b| + |b|^2}$ follows from 1-38. So, $\sqrt{a^2+b^2} \le \sqrt{(|a|+|b|)^2} = \big||a| + |b|\big| = |a| + |b|$ indeed. $\square$

(b) If $n=1$, $\sqrt{\sum_{j=1}^{1} a_j^2} = \sqrt{a_1^2} = |a_1| = \sum_{j=1}^{1} |a_j|$, making the statement true. Assume this is true for $n \in \mathbb{N}$.

Now, $\sqrt{\sum_{j=1}^{n+1} a_j^2} = \sqrt{a_{n+1}^2 + \sum_{j=1}^{n} a_j^2} \le |a_{n+1}| + \left|\sqrt{\sum_{j=1}^{n} a_j^2}\right|$ by base case $\le |a_{n+1}| + \sum_{j=1}^{n} |a_j|$ by our above assumption.

Clearly, the inequality is true for $n+1$, and must be true for every $n \in \mathbb{N}$ by induction. $\square$

1-45

Ideas

$a < \frac{p}{q} < b$

$aq < p \quad p < bq$

$bq - aq > 1$

$q > \frac{1}{b-a}, \quad q+1 > \frac{1}{b-a}$

$a < c\sqrt{2} < b$

$\frac{a}{c} < \sqrt{2} < \frac{b}{c}$

$\rightsquigarrow ac < \sqrt{2} < bc$

$bc - ac > \sqrt{2}$

$c > \frac{\sqrt{2}}{b-a}$

$b - a > d\sqrt{2}$

Proof 1

By Theorem 1.36, there exists rational numbers p and q with $a < p < q < b$. Now, Theorem 1.32 tells us there exists some natural n for which $n > \frac{\sqrt{2}}{q-p}$ so $a < p + \frac{1}{n}\sqrt{2} < b$. Where $p + \frac{1}{n}\sqrt{2}$ must be irrational lest $[(p + \frac{1}{n}\sqrt{2}) - p]\cdot n = \sqrt{2}$ is rational. $\square$

Proof 2 (Taking hint of Mimicking Thm 1.36's Proof)

By Theorem 1.32, there is an $n \in \mathbb{N}$ so that $0 < \frac{\sqrt{2}}{b-a} < n$. Hence, we obtain $\frac{\sqrt{2}}{n} < b - a$. Now let $u := \min\{m \in \mathbb{Z} \mid \frac{m}{n} \geq b\}$ and similarly let $l := \max\{m \in \mathbb{Z} \mid \frac{m}{n} \leq a\}$. Then $\frac{u}{n} - \frac{l}{n} \geq b - a > \frac{\sqrt{2}}{n}$, which means $\frac{l + \sqrt{2}}{n} < \frac{u}{n}$. By definition of l and u we infer that $a < \frac{l+\sqrt{2}}{n} < b$. Clearly, it must be irrational for the same reason. $\square$

$\sum_{j=S}$ ... $j=S$

Self-Proof of Proposition 2.4

Assume that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers with limits L and M. Without loss of generality, consider $L > M$. As $\frac{L-M}{2} > 0$, there exists a corresponding $N \in \mathbb{N}$ so $|a_N - L| < \frac{L-M}{2}$ and $|a_N - M| < \frac{L-M}{2}$. Thus, $a_N < M + \frac{L-M}{2} = L - \frac{L-M}{2} < a_N$, a contradiction.

Therefore, it must be that, instead, $L = M$. $\square$

Self-Proof of Theorem 2.11

Ideas

Let $L := \lim_{n\to\infty} a_n$, $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so...

If $N > K$, $a_N = b_N$ so ...

When $K > N$, $a_K = b_K$ so ...

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

By symmetry the biconditional holds.

Proof

Suppose $\{a_n\}_{n=1}^{\infty}$ converges to L and let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ so for all natural $n \geq N$, $|a_n - L| < \varepsilon$. There are now 2 cases to consider:

Case 1 $N > K$, then simply notice $a_N = b_N$ so $|b_N - L| < \varepsilon$.

Case 2 $K > N$, thus $a_K = b_K$ hence $|b_K - L| < \varepsilon$.

Therefore, $\{b_n\}_{n=1}^{\infty}$ converges, in fact, to L. By symmetry of the above arguments the biconditional holds. $\square$

theorem 2.11

Self-Attempt of Example 2.12

Ideas

$$\frac{3n-1}{2n+5} = \frac{3}{2} - \frac{17}{2(2n+5)}$$

Claim, for any $\varepsilon > 0$, $\left|\frac{3}{2} - \frac{17}{2(2n+5)} - \frac{3}{2}\right| < \varepsilon$

$$\frac{17}{2(2n+5)} < \varepsilon$$

$$\frac{17}{2\varepsilon} < 2n+5$$

$$\frac{17}{4\varepsilon} - \frac{5}{2} < n$$

Proof

Let $\varepsilon > 0$. There exists natural $N > \frac{17}{4\varepsilon} - \frac{5}{2}$ so for any natural $n \geq N$, $\left|\frac{3n-1}{2n+5} - \frac{3}{2}\right| = \left|\left[\frac{3}{2} - \frac{17}{2(2n+5)}\right] - \frac{3}{2}\right| = \frac{17}{2(2n+5)} < \varepsilon$. Thus, $\lim_{n\to\infty}\left(\frac{3n-1}{2n+5}\right) = \frac{3}{2}$.

Exercises

2-4 Ideas

let $\varepsilon > 0$,

$|a_n - b_n| < \frac{1}{n} < \varepsilon$, $|a_n - L| < \varepsilon$ for all n greater than some N

$\geq ||a_n| - |b_n||$ $\quad |a_n| = |a_n - L + L| < |a_n - L| + L$

$-\frac{1}{n} < a_n - b_n < \frac{1}{n}$

$b_n - \frac{1}{n} < a_n < b_n + \frac{1}{n}$

$a_n - L < \varepsilon$ $\quad 0 < L - a_n < \varepsilon$

$a_n - \varepsilon < L$ $\quad a_n < L$

$-L < -a_n + \varepsilon$ $\quad a_n - \varepsilon < L - \varepsilon < L$

$|b_n - a_n|$

$a_n - \varepsilon < L$

$a_n - L < \varepsilon$

$|b_n - L| < |b_n - (a_n - \varepsilon)|$

$= |b_n - a_n| + \varepsilon$

$|b_m - a_m| < \frac{1}{m} < \varepsilon$

Proof: Let $\varepsilon > 0$ and suppose $\{a_n\}_{n=1}^{\infty}$ converges to some limit L. Then there exists some natural N so that $|a_N - b_N| = |b_N - a_N| < \frac{1}{N} < \frac{\varepsilon}{2}$ and for all natural $n \geq N$, $|a_n - L| < \frac{\varepsilon}{2}$, which means that $L > a_n - \frac{\varepsilon}{2}$. Now, notice that given $n \geq N$, $|b_n - L| < |b_n - (a_n - \frac{\varepsilon}{2})| = |b_n - a_n| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$

$1-M$ $\quad$ $L-M$ $\quad 1 - \frac{L-M}{} < a_N$, a contradiction.

2-5

Ideas

$\Rightarrow$ trivial

Suppose $\lim_{n\to\infty} |a_n - L| = 0$, i.e. $|a_n - L|$

Proof

When $\lim_{n\to\infty} a_n = L$, then for any $\varepsilon > 0$, there exists natural $N \in \mathbb{N}$ so for each natural $n \geq N$, $|a_n - L + 0| = |a_n - L| < \varepsilon$ by definition.

Similarly, if $\lim_{n\to\infty} |a_n - L| = 0$, then for all $\varepsilon > 0$, there is some natural $N \in \mathbb{N}$ such that given a natural number $n \geq N$, $|a_n - L| = |a_n - L - 0| < \varepsilon$

$\square$

2-6 (c)

Ideas $\frac{2n+4}{5n^2-11} = \frac{\frac{2}{n} + \frac{4}{n^2}}{5 - \frac{11}{n^2}}$ as long as $n \neq 0$

$\left|\frac{2n+4}{5n^2-11} - 0\right| < \varepsilon$

$2n+4 < \varepsilon(5n^2-11)$ $\quad$ or $\quad$ $2n+4 < \varepsilon(11-5n^2)$

$0 < 5\varepsilon n^2 - 2n - 11\varepsilon - 4$ $\quad$ $5\varepsilon n^2 + 2n - 11\varepsilon + 4 < 0$

roots: $n = \frac{2 \pm \sqrt{4 - 4(5\varepsilon)(-11\varepsilon - 4)}}{10\varepsilon}$

$= \frac{1}{5\varepsilon} \pm \frac{1}{5\varepsilon}\sqrt{55\varepsilon^2 + 20\varepsilon + 1}$ $\quad$ $1 + 5\varepsilon(11\varepsilon + 4) = 1 + 55\varepsilon^2 + 20\varepsilon$

$n > \frac{1}{5\varepsilon} + \frac{1}{5\varepsilon}\sqrt{55\varepsilon^2 + 20\varepsilon + 1}$

Proof

Let $\varepsilon > 0$ and choose N to be any natural number larger than $\frac{1}{5\varepsilon} + \frac{1}{5\varepsilon}\sqrt{55\varepsilon^2 + 20\varepsilon + 1}$. Then for each $n \geq N$, we can simplify $\left|\frac{2n+4}{5n^2-11} - 0\right| < \varepsilon$

$\square$

(d) Ideas $|\sqrt{n+1} - \sqrt{n}| = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon$

$\sqrt{\frac{1}{4\varepsilon^2}} - \sqrt{\frac{1}{4\varepsilon^2} - 1}$ $\quad$ $\frac{1}{2\sqrt{\frac{1}{4\varepsilon^2} - 1}}$ $\quad$ $\frac{1}{\varepsilon} < \sqrt{n+1} + \sqrt{n}$

$= \frac{1}{2\varepsilon} - \frac{\sqrt{1 - 4\varepsilon^2}}{2\varepsilon}$ $\quad$ $\frac{1}{\left(\frac{2\sqrt{1-4\varepsilon^2}}{2\varepsilon}\right)}$ $\quad$ $\frac{1}{\varepsilon} < 2\sqrt{n+1}$

$= \frac{1}{2\varepsilon} - \frac{\sqrt{\frac{1}{4} - \varepsilon^2}}{\varepsilon}$ $\quad$ $\frac{1}{2\varepsilon} < \sqrt{n+1}$

$(1-2\varepsilon)(1+2\varepsilon)$ $\quad$ $\frac{1}{4\varepsilon^2} - 1 < n$

$\frac{1}{\frac{1-2\varepsilon}{\varepsilon}}$ $\quad$ $\frac{\varepsilon}{1-2\varepsilon}$

Proof Let $\varepsilon > 0$, and $N \in \mathbb{N}$ be such that $N > \left(\frac{1-\varepsilon^2}{2\varepsilon}\right)^2$

Then for every natural $n \geq N$, $|\sqrt{n+1} - \sqrt{n}| < \varepsilon$.

$\sqrt{n+1} - \sqrt{n} < \varepsilon$

$\Leftrightarrow \sqrt{n+1} < \varepsilon + \sqrt{n}$

$\Leftrightarrow \cancel{n}+1 < \varepsilon^2 + 2\varepsilon\sqrt{n} + \cancel{n}$

$\Leftrightarrow 1 - \varepsilon^2 < 2\varepsilon\sqrt{n}$

$\Leftrightarrow \frac{1-\varepsilon^2}{2\varepsilon} < \sqrt{n}$

$\Leftrightarrow \left(\frac{1-\varepsilon^2}{2\varepsilon}\right)^2 < n$

n/c IN so for each natural n

that given a natural number

Self - Proof of Theorem 2.14

4. Ideas

$$\left|\frac{1}{b_n}-\frac{1}{L_b}\right| = \left|\frac{L_b-b_n}{b_nL_b}\right| = |b_n-L_b|\frac{1}{|b_n||L_b|} < c\varepsilon\cdot\frac{1}{|b_n||L_b|}$$

$L-\varepsilon < b_n < L+\varepsilon$

Show $c\varepsilon\cdot\frac{1}{|b_n||L_b|} < \varepsilon$

$|b_n| > |L-\varepsilon|$ or $|b_n| > |L+\varepsilon|$

ε

$|b_n| > \min(|L-\varepsilon|, |L+\varepsilon|)$

$\frac{c}{|b_n||L_b|} < 1$

$\frac{1}{c} > \frac{1}{|b_n||L_b|}$

$c < |b_n||L_b|$

$c < |b_n||L_b|$

$\frac{c\varepsilon}{|b_n||L_b|} < \varepsilon$

Proof

Similarly as in 2, it suffices to prove $\left\{\frac{1}{b_n}\right\}_{n=1}^{\infty}$ converges to $\frac{1}{L_b}$ because of result 3. Let $\varepsilon > 0$ and the constant $c := |L_b| \min(|L_b-\varepsilon|, |L_b+\varepsilon|)$. Then there exists some natural N for which when $n \geq N$, $|b_n - L_b| < c\varepsilon$. Now, we can see that

$$\left|\frac{1}{b_n}-\frac{1}{L_b}\right| = \left|\frac{L_b-b_n}{b_nL_b}\right| = |b_n-L_b|\frac{1}{|b_n||L_b|} < c\varepsilon\cdot\frac{1}{|b_n||L_b|} < \varepsilon.$$

Self-Proof of Theorem 2.13

Let $\varepsilon > 0$ and N be any natural number larger than $\frac{1}{\varepsilon}$. Then, given a natural $n \geq N$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$ since $n > \frac{1}{\varepsilon}$. $\square$

Self-Proof of Theorem 2.14

1. Ideas $|a_n - L_a + b_n - L_b| \leq |a_n - L_a| + |b_n - L_b| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$

Proof

Let $L_a := \lim_{n\to\infty} a_n$ and $L_b := \lim_{n\to\infty} b_n$, $\varepsilon > 0$ and N be a natural number so $|a_n - L_a| < \frac{1}{2}\varepsilon$ and $|b_n - L_b| < \frac{1}{2}\varepsilon$ so long as $n \geq N$. It follows that $|a_n + b_n - (L_a + L_b)| \leq |a_n - L_a| + |b_n - L_b| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$, given $n \geq N$. Therefore, the sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges, in fact, to $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ ✓ Say $\square$

2. Because of the above result, it suffices to show $\{-b_n\}_{n=1}^{\infty}$ converges to $-L_b$. Notice that $|-b_n - (-L_b)| = |-b_n + L_b| = |b_n - L_b| < \frac{1}{2}\varepsilon$.
(choice of N and n is as in 1.)

3. Ideas

$$|a_n \cdot b_n - L_a \cdot L_b| = |a_n \cdot b_n + L_a \cdot L_b - a_n \cdot L_b - b_n \cdot L_a + \overbrace{(a_n \cdot L_b + b_n \cdot L_a - 2 L_a \cdot L_b)}^{L_b(a_n - L_a) + L_a(b_n - L_b)}|$$

Let $\varepsilon > 0$, $|a_n - L_a| < \frac{\varepsilon}{4\underbrace{\max(|L_a|, |L_b|)}_{M}}$

$$\leq |a_n - L_a||b_n - L_b| + |L_b||a_n - L_a| + |L_a||b_n - L_b|$$

If $M \geq 1$

$$< \frac{\varepsilon}{4M} \cdot \frac{\varepsilon}{4M} + |L_b| \cdot \frac{\varepsilon}{4M} + |L_a| \cdot \frac{\varepsilon}{4M}$$

$$< \frac{\varepsilon^2}{16M} + \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M}$$ (wlog we can suppose $\varepsilon < 1$) so $\varepsilon^2 < \varepsilon$

$$< \frac{\varepsilon}{16M} + \frac{\varepsilon}{2}$$

Oops forgot to cancel the M away

$$= \frac{9}{16M}\varepsilon$$

$$< \varepsilon$$

Proof

Let $\varepsilon > 0$, we can suppose without loss of generality that $\varepsilon < 1$, and define $M := 1 + \max(|L_a|, |L_b|)$. → This might be 0! Thought about this but forgot about it oops. There exists some natural N so $|a_n - L_a|$ and $|b_n - L_b|$ are each less than both $\frac{\varepsilon}{4}$ and $\frac{\varepsilon}{4M}$ given $n \geq N$. Now notice $|a_n \cdot b_n - L_a \cdot L_b| = |a_n \cdot b_n - a_n \cdot L_b - b_n \cdot L_a + L_a \cdot L_b + (a_n \cdot L_b + b_n \cdot L_a - 2 L_a \cdot L_b)|$ $< |a_n - L_a||b_n - L_b| + |L_b||a_n - L_a| + |L_a||b_n - L_b| < \frac{\varepsilon}{4} \cdot \frac{\varepsilon}{4} + |L_b| \cdot \frac{\varepsilon}{4M} + |L_a| \cdot \frac{\varepsilon}{4M} \leq \frac{\varepsilon}{16} + \frac{\varepsilon}{2} < \varepsilon$ if $M \geq 1$. And if $M < 1$, we know $|L_a|, |L_b| < 1$ such that $|a_n \cdot b_n - L_a \cdot L_b| < \frac{\varepsilon}{4} \cdot \frac{\varepsilon}{4} + |L_b| \cdot \frac{\varepsilon}{4} + |L_a| \cdot \frac{\varepsilon}{4} < \varepsilon$. Either ways, since $|a_n \cdot b_n - L_a \cdot L_b| < \varepsilon$, the sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges, in fact, to $L_a L_b$.

Exercises

2-12

Ideas

$||a_n| - |L|| \le |a_n - L| < \varepsilon$

Proof

Let $\varepsilon > 0$, and N be some natural number with $|a_n - L| < \varepsilon$ for any $n \ge N$. We see that $||a_n| - |L|| \le |a_n - L| < \varepsilon$, where $L := \lim_{n\to\infty} a_n$, informing us that $\lim_{n\to\infty} |a_n| = |\lim_{n\to\infty} a_n|$ indeed. $\square$

2-13

Ideas

$|\sqrt{a_n} - \sqrt{L}| = \left|\frac{a_n - L}{\sqrt{a_n}+\sqrt{L}}\right| < c\varepsilon \cdot \frac{1}{\sqrt{a_n}+\sqrt{L}} < \varepsilon$

$\sqrt{L-\varepsilon} < \sqrt{a_n} < \sqrt{L+\varepsilon}$

$c := \sqrt{L-\varepsilon} + \sqrt{L}$

$c < \sqrt{a_n} + \sqrt{L}$

$\frac{1}{\sqrt{a_n}+\sqrt{L}} < \frac{1}{c}$

Proof $L := \lim a_n$

Let $\varepsilon > 0$, $c := \sqrt{L-\varepsilon} + \sqrt{L}$ so there is some natural N with $|a_n - L| < c\varepsilon$ when $n \ge N$. We now see that $|\sqrt{a_n} - \sqrt{L}| = \frac{|a_n - L|}{\sqrt{a_n}+\sqrt{L}} < c\varepsilon \cdot \frac{1}{\sqrt{a_n}+L} < \varepsilon$ indeed. $\square$

(If $\sqrt{a_n} = \sqrt{L} = 0$, then $|\sqrt{a_n} - \sqrt{L}| = 0 < \varepsilon$, thus this case isn't an issue)

2-14

Ideas

$\left|\frac{q^n}{n!}\right| < \varepsilon$

$\frac{q^n}{n!} < \varepsilon$

$(0<) \frac{q^n}{\varepsilon} < n!$

Exists natural $n > q$

$n^n > q^n$

$(2n)! > q^n$

Proof W.l.o.g, let $0 < \varepsilon < 1$, n be any natural number larger than $\frac{q}{\varepsilon}$, so $(2n)! \ge n^n > \frac{q^n}{\varepsilon^n} > \frac{q^n}{\varepsilon}$. Given any natural $m \ge 2n$, $\frac{q^m}{\varepsilon} < m!$ such that $\left|\frac{q^m}{m!} - 0\right| = \frac{q^m}{m!} < \varepsilon$. $\square$

2-15

Conjecture: $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Ideas

Show: $\frac{n!}{n^n} < \varepsilon$

Know: $n^n > n!$ for each natural $n > 1$

Claim: $\frac{n!}{n^n} < \frac{2}{n(n-1)}$ when $n > 1$

$n=2$: $\frac{1}{2} < 1$ $n=3$: $\frac{2}{9} < \frac{1}{3}$

Step: $\frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!\ ???}{n^{n+1}} < \frac{1}{n} \cdot \frac{2}{n(n-1)} = \frac{2}{n^3 - n^2} < \frac{2}{n(n+1)}$ Oopsie careless mistake

$\frac{2}{n(n+1)} = \frac{2}{n^2+n}$

$n^3 - n^2 > n^2 + n$

$n^3 - n > 2n^2$

$n(n^2-1) > 2n^2$

$n^2(n-1) > n(n+1)$

$n(n-1) > n+1$

If $n \ge 3$, $n^2 \ge 3n > n+2$ as $2n \ge 6 > 2$, $3n > n+2$

$n^2 - 1 > n+1$

$\frac{1}{n(n+1)} > \frac{1}{n^2(n-1)}$

$n=3$: ~~$6 > 4$~~

Step: ~~Show $n(n+1) > n+2$~~

Claim: $\frac{n!}{n^n} \le \frac{1}{n}$ for any $n \in \mathbb{N}$

$n! \le n^{n-1}$

$n=1$: $1! = 1 = 1^{1-1}$

Step: $(n+1)! = (n+1)(n!)$

$\le (n+1)(n^{n-1})$

$\le (n+1)(n+1)^{n-1}$

$= (n+1)^{(n+1)-1}$

Self - Proof of Theorem 2·17

Ideas

$k \in \mathbb{N} \Rightarrow$ Induction

$k=0$ trivial

$k<0 \Rightarrow$ Induction

Proof

When $k=0$ and $k=1$, the result is immediate. Hence, assume that $\lim_{n\to\infty} a_n^k = (\lim_{n\to\infty} a_n)^k$. Then, by the limit laws, $\lim_{n\to\infty} a_n^{k+1} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} a_n^k$ $= (\lim_{n\to\infty} a_n)^{k+1}$ by assumption. So, this is true of any $k \in \mathbb{N}$. Now suppose, for the sake of argument, that none of the terms or the limit is zero. If $k=1$, $\lim_{n\to\infty} a_n^{-1} = (\lim_{n\to\infty} a_n)^{-1}$ follows easily from the 4th limit law. Consider when the result is true of $k \in \mathbb{N}$, then $\lim_{n\to\infty} a_n^{-(k+1)} = (\lim_{n\to\infty} a_n)^{-k} \cdot (\lim_{n\to\infty} a_n)^{-1}$ $= (\lim_{n\to\infty} a_n)^{-(k+1)}$ holds.

$\square$

Self - Proof of The Squeeze Theorem

Ideas

$|a_n - L| < \varepsilon \quad |c_n - L| < \varepsilon$

$a_n - L \le b_n - L \le c_n - L$

$-(a_n - L) \ge -(c_n - L) = L - c_n$

$|b_n - L| > |c_n - L| \Rightarrow |b_n - L| \le |a_n - L|$?

$b_n - L > c_n - L$ / $L - b_n > L - c_n$ / $b_n - L > L - c_n$

$b_n > c_n$

N.A.

$(0>) c_n - L > b_n - L$

$(\ge a_n - L)$

$-(b_n - L) \le -(a_n - L)$

$|b_n - L| \le |a_n - L| < \varepsilon$

N.A

$L - b_n > c_n - L > 0$

$(0>) -(c_n - L) > b_n - L \ (\ge a_n - L)$

$|b_n - L| \le |a_n - L| < \varepsilon$

$c_n - L > L - c_n$

$c_n > L$

$c_n - L > 0$

2-15

By a simple inductive argument, we show that for every $n \in \mathbb{N}$, $n! \leq n^{n-1}$ so $\frac{n!}{n^n} \leq \frac{1}{n}$. By the Squeeze Theorem, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$. $\square$

2-16 (b)

Idea

Claim: $\sum_{i=1}^{n} (-1)^i = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

$n=1$: $S = -1$ ✓

Step: $\sum_{i=1}^{2m+1} (-1)^i = 0 + (-1) = -1$, $\sum_{i=1}^{2m+2} (-1)^i = -1 + (-1)^{2m+2} = 0$ ✓

$\frac{1}{N(N+1)} < \frac{2}{2N^2} = \frac{1}{N^2} < \varepsilon$ $\quad \frac{1}{\varepsilon} < N^2$ $\quad \frac{1}{\sqrt{\varepsilon}} < N$ $\quad 0 < \frac{1}{\sqrt{\varepsilon}} < N$ $\quad \frac{1}{\varepsilon} < N^2$ $\quad \frac{1}{N^2} < \varepsilon$

Proof

Let $p_k = 1$ and $a_k = (-1)^k$. Then, a simple inductive argument shows that $\sum_{i=1}^{n} (-1)^i = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$. Now, given any $\varepsilon > 0$, choose any $N > \frac{1}{\varepsilon}$. We see that for $n \geq N$, $\left| \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ when n is odd, else this quotient is just $0 < \varepsilon$ if n odd. Therefore, $\left\{ \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} \right\}_{n=1}^{\infty}$ converges, in fact to 0.

If $\{a_n\}_{n=1}^{\infty}$ converges, part (a) tells us that $\lim_{n\to\infty} a_n = 0$. But for $\varepsilon = \frac{1}{2}$, and any $n \in \mathbb{N}$, $|(-1)^n - 0| = |-1|^n = 1 \not< \frac{1}{2} = \varepsilon$, a contradiction. Thus, $\{a_n\}_{n=1}^{\infty}$ must not converge, inspite of $\left\{ \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} \right\}_{n=1}^{\infty}$ converging. $\square$

(a)

Ideas

$\frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} - a < \varepsilon$ / $a - \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} < \varepsilon$

$\frac{a \sum_{k=1}^{n} p_k}{\sum_{k=1}^{n} p_k} = \frac{\sum_{k=1}^{n} a p_k}{\cdots}$

$\left| \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} - a \right| = \left| \frac{\sum_{k=1}^{n} p_k (a_k - a)}{\sum_{k=1}^{n} p_k} \right| \leq \frac{\sum_{k=1}^{n} p_k |a_k - a|}{\sum_{k=1}^{n} p_k}$ $\quad p_n > 0$!!!

$= \underbrace{\frac{\sum_{k=1}^{N-1} p_k |a_k - a|}{\sum_{k=1}^{n} p_k}}_{<\varepsilon} + \underbrace{\frac{\sum_{k=N}^{n} p_k |a_k - a|}{\cdots}}_{<\varepsilon}$

exists N' so for all $n > N'$: $\frac{1}{\sum_{k=1}^{n} p_k} < \frac{\varepsilon}{\sum_{k=1}^{N-1} \cdots}$

for large enough n

Proof

Let $\varepsilon > 0$ and N_1 and N_2 be any natural numbers with $|a_n - a| < \frac{1}{2}\varepsilon$ for $n \geq N_1$, and $\frac{1}{\sum_{k=1}^{m} p_k} < \frac{\frac{1}{2}\varepsilon}{\sum_{k=1}^{N_1-1} p_k |a_k - a|}$ given $m \geq N_2$. Now, notice that for $n \geq \max\{N_1, N_2\}$,

$$\left| \frac{\sum_{k=1}^{n} p_k a_k}{\sum_{k=1}^{n} p_k} - a \right| = \left| \frac{\sum_{k=1}^{n} p_k (a_k - a)}{\sum_{k=1}^{n} p_k} \right| \leq \frac{\sum_{k=1}^{n} p_k |a_k - a|}{\sum_{k=1}^{n} p_k} = \frac{\sum_{k=1}^{N_1-1} p_k |a_k - a|}{\sum_{k=1}^{n} p_k} + \frac{\sum_{k=N_1}^{n} p_k |a_k - a|}{\sum_{k=1}^{n} p_k} < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$\square$

$||a_n| - |L|| \leq |a_n - L|$ ... informs us that $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right|$ indeed

Self-Proof of Theorem 2.27

Ideas

$|a_n - L| < \varepsilon$ for $n \geq$ some N

$L - \varepsilon < a_n < L + \varepsilon$

$-2\varepsilon < a_n - a_m < 2\varepsilon$

$|a_n - a_m| < 2\varepsilon$

$|a_n - a_m| < \varepsilon$

$\varepsilon > |a_n - a_m| = |(a_n - L) - (a_m - L)| \geq ||a_n - L| - |a_m - L||$

If a Cauchy sequence does not converge, then for any $L \in \mathbb{R}$, there exists $\varepsilon \in \mathbb{R}^+$ with so for all $N \in \mathbb{N}$ there exists $n \geq N$ $|a_n - L| \geq \varepsilon$

$-\varepsilon < a_n - a_m < \varepsilon$ OR $-\varepsilon < a_m - a_n < \varepsilon$

$a_m - \varepsilon < a_n < a_m + \varepsilon$ ~~$a_n - \varepsilon < a_m < a_n + \varepsilon$~~

$a_m + \varepsilon > a_n > a_m - \varepsilon$

$a_{N_m} - 10^{-m} \leq \inf S_m \leq a_{N_m} + 10^{-m}$ ($\leq a_{N_m}$)

$\ldots \leq L \leq \ldots$

Let N_m be the least natural with $|a_n - a_m| < 10^{-m}$ so long as $n, m \geq N$, and $S_m := \{a_n \in \mathbb{R} \mid n \geq N_m\}$. $\inf S_m$ and $\sup S_m$ exists since $a_{N_m} - \varepsilon$, $a_{N_m} + \varepsilon$ are ...

Let $\{u_m\}_{n=1}^{\infty}$ and $\{l_m\}_{m=1}^{\infty}$ be defined by $u_m := \sup S_m$ and $l_m := \inf S_m$.

$L := \sup\{\inf S_m \mid m \in \mathbb{N}\}$ exists as $a_{N_1} + 10$ is an upper bound.

$n \geq N_m \Rightarrow L - 10^{-m} \leq a_n \leq L + 10^{-m}$?

$L \leq a_n + 10^{-m}$

$\inf S_m \leq a_n$

Given any $k \geq N_m$, $a_k \leq a_n + 10^{-m}$

$N_M \geq N_m$: $a_n + 10^{-m}$ is an upper bound of S_M

$\Rightarrow \inf S_M \leq a_n + 10^{-m}$

$N_M < N_m$:

Given $k \geq N_M$,

$a_k < a_n < a_n + 10^{-m}$ OR $a_k \geq a_n$

$\Rightarrow \inf S_M \leq a_n + 10^{-m}$

$\Rightarrow \inf S_M \leq a_n < a_n + 10^{-m}$

$\Rightarrow L \leq a_n + 10^{-m}$

$L - 10^{-m} \leq a_n$

$\Longrightarrow L - 10^{-m} \leq a_n \leq L + 10^{-m}$

$-\varepsilon < a_n - L < \varepsilon$

1 3 6 5 2 4

L

Fix an arbitrary $n \geq N_m$

As $|a_n - a_k| < 10^{-m}$, $a_n - 10^{-m} < a_k < a_n + 10^{-m}$ for all $k \geq N_m$

Thus, $a_n - 10^{-m}$ is a lower bound of S_m, i.e.

$a_n - 10^{-m} \leq \inf S_m \leq L$

$a_n \leq L + 10^{-m}$

S_m

$n \geq k$ as $n < k$ tells us $k > N_m$, then previous case applies

$-10^{-m} < a_k - a_n < 10^{-m}$ or $-10^{-m} < a_n - a_k < 10^{-m}$

$a_n - 10^{-m} < a_k < a_n + 10^{-m}$ ... $-a_n - 10^{-m} < -a_k < -a_n + 10^{-m}$

$a_n + 10^{-m} > a_k > a_n - 10^{-m}$

$|a_n - a_m| < 2\varepsilon$

+ [illegible] then for any $L \in \mathbb{R}$, there exists $\varepsilon \in \mathbb{R}^+$ so for all $N \in \mathbb{N}$ there exists $n \geq N$ with $|a_n - L| \geq \varepsilon$

Self-Proof of Theorem 2.27

First, assume $\{a_n\}_{n=1}^{\infty}$ converges to some limit L. Let $\varepsilon > 0$ and N be any natural with $|a_n - L| < \frac{1}{2}\varepsilon$ when $n \geq N$. Then $L - \frac{1}{2}\varepsilon < a_n < L + \frac{1}{2}\varepsilon$ and $L - \frac{1}{2}\varepsilon < a_m < L + \frac{1}{2}\varepsilon$ so that $-\varepsilon < a_n - a_m < \varepsilon$. Hence, $|a_n - a_m| < \varepsilon$ provided $n, m \geq N$. Informing us this is a Cauchy sequence as expected.

Conversely, suppose $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Define N_m to be any natural with $|a_n - a_m| < 10^{-m}$ as long as $n, m \geq N$, and $S_m := \{a_n \in \mathbb{R} \mid n \geq N_m\}$, lastly $L := \sup\{\inf S_m \mid m \in \mathbb{N}\}$. Fix any two naturals n, m with $n \geq N_m$. Now for every natural k, M with $k \geq M$ we have

Case 1 $N_M \geq N_m$: Then, $a_k \leq a_n + 10^{-m}$ as $k, n \geq N_m$. Accordingly, $a_n + 10^{-m}$ is an upper bound of S_M so $\inf S_M \leq a_n + 10^{-m}$.

Case 2 $N_M < N_m$:

(A) $a_k \geq a_n$, we consider $n > k$ because otherwise $k > N_m$ and case 1 applies. As such, $\inf S_M \leq a_n \leq a_n + 10^{-m}$.

(B) $a_k < a_n$, again $\inf S_M \leq a_n + 10^{-m}$.

In any situation, we see that $\inf S_M \leq a_n + 10^{-m}$ must hold. Therefore, $L \leq a_n + 10^{-m}$. Next, let k be any natural with $k \geq N_m$. It follows that $a_n - 10^{-m} < a_k < a_n + 10^{-m}$. As a result, $a_n - 10^{-m}$ is a lower bound of S_m, that is: $a_n - 10^{-m} \leq \inf S_m \leq L$. Consequently, $L - 10^{-m} \leq a_n \leq L + 10^{-m}$. Wherefore, given $\varepsilon > 0$, select any $10^{-m} < \varepsilon$ such that $|a_n - L| \leq 10^{-m} < \varepsilon$. Which shows $\{a_n\}_{n=1}^{\infty}$ converges to L.

$\square$

$|a_n - a_m| < \varepsilon$

$|a_n - L| < \varepsilon$ for $n \geq$ some N

$\varepsilon > |a_n - a_m| = |(a_n - L) - (a_m - L)| \geq ||a_n - L| - |a_m - L||$

... there exists $n \geq N$

Exercises

Up to first m decimals of $\sqrt{2}$

2-17 Ideas

$$\left| \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} - \frac{\lfloor 10^m \sqrt{2} \rfloor}{10^m} \right| \leq \left| \sqrt{2} - \frac{\lfloor 10^m \sqrt{2} \rfloor}{10^m} \right| = \qquad \left| \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} - \frac{p}{q} \right|$$

$10^m \varepsilon > 1$ $\quad$ $\varepsilon > \frac{1}{10^m}$

$k\varepsilon > 1$
$k > \frac{1}{\varepsilon}$
$\lceil k \rceil > \frac{1}{\varepsilon}$
$10^{\lceil k \rceil} > \frac{1}{\varepsilon}$
$= 10^k$

for $m \geq k$

$10^m \sqrt{2} \geq \lfloor 10^m \sqrt{2} \rfloor$
$1 > 10^m \sqrt{2} - \lfloor 10^m \sqrt{2} \rfloor \geq 0$
$10^m \sqrt{2} - \lfloor 10^m \sqrt{2} \rfloor < 10^m \varepsilon \xrightarrow{+}$ Convergence to $\sqrt{2}$

Proof

Let $\varepsilon > 0$ and N a natural number large than $\frac{1}{\varepsilon}$ so $\varepsilon > 10^{-n}$ if $n \geq N$. Notice that $0 < 10^m \sqrt{2} - \lfloor 10^m \sqrt{2} \rfloor < 1 < 10^m \varepsilon$, and as such, when $m, n \geq N$

$$\left| \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} - \frac{\lfloor 10^m \sqrt{2} \rfloor}{10^m} \right| \leq \left| \sqrt{2} - \frac{\lfloor 10^m \sqrt{2} \rfloor}{10^m} \right| < \varepsilon.$$

Therefore, we have proven $\left\{ \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} \right\}_{n=1}^{\infty}$ converges, in fact, to $\sqrt{2}$. $\square$

2-22

(a) Since $(-1)^n$ is always either -1 or 1, $\left| \frac{(-1)^n}{n} \right| = \frac{1}{n}$. From here, it is easy to show the convergence of $\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$ to 0.

(b) Let $\varepsilon > 0$ and N any natural number. Then $|(N + \lceil \varepsilon \rceil) - N| = \lceil \varepsilon \rceil \not< \varepsilon$, thus proving it is not Cauchy, and hence diverges.

(c) Let $\varepsilon > 0$, and $N \in \mathbb{N}$. We see that $|\sqrt{(N + \lceil \varepsilon \rceil)^2} - \sqrt{N^2}| = \lceil \varepsilon \rceil \not< \varepsilon$ again, as $N \geq 1$. Again, this sequence must diverge.

(d) In this case, proving convergence is actually easier, with the simple fact that $\frac{2n+1}{3n-2} = \frac{2}{3} + \frac{7}{3(3n-2)}$ by the definition of convergence. Though doing it via showing it's Cauchy still work

$\square$

$\lfloor 10^m \sqrt{2} \rfloor$ $\left| \frac{\lfloor 10^n \sqrt{2} \rfloor}{} - \frac{p}{q} \right|$

2-23 (a) In this case, it is clear that $a_n - b_n = 0$, which is an obviously convergent sequence. However, none of $\{a_n\}_{n=1}^\infty$ or $\{b_n\}_{n=1}^\infty$ converges, as seen in Example 2.29.

(b) This follows easily from (a) by setting $a_n = b_n = (-1)^n$ for any $n \in \mathbb{N}$.

(c) Again, set $a_n = b_n = (-1)^n$. Then $a_n \cdot b_n = 1$ clearly converges to 1 while again Example 2.29 tells us neither $\{a_n\}_{n=1}^\infty$ nor $\{b_n\}_{n=1}^\infty$ is convergent.

(d) follows similarly.

2-24 No, since every (real-valued) Cauchy sequence is a sequence of real numbers, Proposition 2.4 immediately informs us limits must be unique.

2-25

Idea)

Let T be a nonempty set of reals bounded above by $U \in \mathbb{R}$. i.e. $x \in T \Rightarrow x \le U$

Show exists supremum s of T, so $s \le U$ and $x \in T \Rightarrow x \le s (\le U)$

$l_n := \max\{\lfloor 10^{n-1} x \rfloor \mid x \in T\}$ ($\lfloor U \rfloor$ is a upper bound)

Define the sequence $\{a_n\}_{n=1}^\infty$ by $a_n := \frac{l_n}{10^{n-1}}$

$x \ge y (\ge \lfloor y \rfloor) \Leftrightarrow \lfloor x \rfloor \ge \lfloor y \rfloor$

$l_n = \lfloor 10^n x \rfloor$ $\quad 10^n x \ge 10^n y$

$\lfloor 10^n x \rfloor \ge \lfloor 10^n y \rfloor$

$x \ge y$

$\lfloor 10^m x \rfloor \ge \lfloor 10^m y \rfloor$

$\left| M - \frac{l_n}{10^n} \right|$

$= \left| M - \frac{\lfloor 10^n M \rfloor}{10^n} \right|$ $\quad 10^n M \ge \lfloor 10^n x \rfloor$ $\quad \lfloor 10^n x \rfloor \ge \lfloor 10^n M \rfloor$ by maxi

$10^n M \ge \lfloor 10^n M \rfloor$ $\quad 10^n M < \lfloor 10^n M \rfloor + 1$

$M \ge \frac{\lfloor 10^n M \rfloor}{10^n}$ $\quad M < \frac{\lfloor 10^n M \rfloor}{10^n} + 10^{-n}$ $\quad \lfloor 10^n M \rfloor \ge \lfloor 10^n x \rfloor$

$M - \frac{\lfloor 10^n M \rfloor}{10^n} < 10^{-n} < \varepsilon$

$n, m \ge k+1$

$\left| \frac{l_n}{10^{n-1}} - \frac{l_m}{10^{m-1}} \right| =$

$k > \frac{1}{\varepsilon}$

$10^k > \frac{1}{\varepsilon}$

$\varepsilon > 10^{-k}$

$(10^k \varepsilon > 1)$

Show $10^k \left| \frac{l_n}{10^{n-1}} - \frac{l_m}{10^{m-1}} \right| < 1$

* Check Thm 1.32

Suppose $n \ge m$

$\lfloor 10^{n-1} x \rfloor \ge \lfloor 10^{m-1} x \rfloor$ $\quad \varepsilon = \frac{1}{2}(x - L)$

$10^{m-1} x \ge \lfloor 10^{m-1} x \rfloor$

$10^{n-1} x \ge 10^{n-m} \lfloor 10^{m-1} x \rfloor$

$\lfloor 10^{n-1} x \rfloor \ge \lfloor 10^{n-m} \lfloor 10^{m-1} x \rfloor \rfloor$

$= 10^{n-m} \lfloor 10^{m-1} x \rfloor$ $\quad L - x > 0$

as $10^{n-m} \lfloor 10^{m-1} x \rfloor$ is an integer itself

$0 \le 10^{m-1} x - \lfloor 10^{m-1} x \rfloor < 1$

$10^{n-1} x < 10^{n-m} \lfloor 10^{m-1} x \rfloor + 10^{n-m}$

$\lfloor 10^{n-1} x \rfloor < \lfloor 10^{n-m} \lfloor 10^{m-1} x \rfloor + 10^{n-m} \rfloor$

$= 10^{n-m} \lfloor 10^{m-1} x \rfloor + 10$

$\frac{\lfloor 10^{n-1} x \rfloor}{10^{n-1}} - \frac{\lfloor 10^{m-1} x \rfloor}{10^{m-1}} < 1$

$\frac{l_n}{10^{n-1}} - \frac{l_m}{10^{m-1}} <$

10

'Supremumality' of L

$x \le u$

$x \ge \frac{\lfloor 10^n x \rfloor}{10^n}$ for x s.t. $l_n = \lfloor 10^n x \rfloor$

$L > u$ Show $x > u$ for some $x \in T$

$\left| u - \frac{\lfloor 10^n x \rfloor}{10^n} \right| = u - \frac{\lfloor 10^n x \rfloor}{10^n}$

$L - \varepsilon < \frac{\lfloor 10^n x \rfloor}{10^n} < L + \varepsilon$

$u - \varepsilon < \frac{\lfloor 10^n x \rfloor}{10^n}$

$\underbrace{-\varepsilon < 0 \le u - \frac{\lfloor 10^n x \rfloor}{10^n}}_{\text{upper bound}} < \varepsilon$

N:

2-17 $\left| \frac{\lfloor 10^n \sqrt{2} \rfloor}{} - \frac{\lfloor 10^m \sqrt{2} \rfloor}{10^m} \right| \leq \left| \sqrt{2} - \frac{\lfloor 10^m \sqrt{2} \rfloor}{10^m} \right| =$ $\left| \frac{\lfloor \quad \rfloor}{10^n} - q \right|$

2-25 Proof
(No Hint)

Let A be a nonempty subset of $\mathbb{R}$ bounded above by some $u \in \mathbb{R}$, and $l_n := \max\{\lfloor 10^n x \rfloor \mid x \in T\}$ which exists as $\lfloor 10^n u \rfloor$ is a upper bound (Proposition 1.30 can be proved without completeness, by the well-ordering on $\mathbb{N}$). Define $\varepsilon > 0$, N be any natural with $N > \frac{1}{\varepsilon}$ such that $\varepsilon > 10^{-N}$ (again without adding completeness in Thm 1.32) and lastly, $n, m \geq N$, where we can suppose without loss of generality that $n \geq m$. By construction, there exists some $x, x' \in A$ with $l_n = \lfloor 10^n x \rfloor$ and $l_m = \lfloor 10^m x' \rfloor$. From maximality, $\lfloor 10^m x' \rfloor \geq \lfloor 10^m x \rfloor$ and $\lfloor 10^n x \rfloor \geq \lfloor 10^n x' \rfloor$. Thus, $x \geq x'$ and $\lfloor 10^m x \rfloor \geq \lfloor 10^m x' \rfloor$. In other words, $l_m = \lfloor 10^m x \rfloor$. Hence,

$$10^m x - \lfloor 10^m x \rfloor \geq 0 \qquad \& \qquad 10^m x - \lfloor 10^m x \rfloor < 1$$

$$\lfloor 10^n x \rfloor \geq \lfloor 10^{n-m} \lfloor 10^m x \rfloor \rfloor \qquad \lfloor 10^n x \rfloor < \lfloor 10^{n-m} \lfloor 10^m x \rfloor + 10^{n-m} \rfloor$$

$$\lfloor 10^n x \rfloor \geq 10^{n-m} \lfloor 10^m x \rfloor \text{ as this is an integer itself} \qquad \lfloor 10^n x \rfloor < 10^{n-m} \lfloor 10^m x \rfloor + 10^{n-m} \text{ since it itself is an integer}$$

$$10^k \left(\frac{l_n}{10^n} - \frac{l_m}{10^m} \right) \geq 0 \qquad 10^N \left(\frac{l_n}{10^n} - \frac{l_m}{10^m} \right) < 10^{N-m} \leq 1 \text{ as } m \geq N$$

Therefore, it holds that $\left| \frac{l_n}{10^n} - \frac{l_m}{10^m} \right| < 10^{-N} < \varepsilon$. As such, the sequence $\{\frac{l_n}{10^n}\}_{n=1}^{\infty}$ converges, as it is Cauchy, say to some limit $L \in \mathbb{R}$. ~~Moreover, notice that this sequence is nondecreasing since we have shown~~ $\cancel{\frac{l_n}{10^n} \geq \frac{l_m}{10^m}}$ ~~for~~ $\cancel{n \geq m}$ Assume, for the sake of contradiction, that there exists $y \in A$ with $y > L$. Then there is also $k \in \mathbb{N}$ with $10^k(y - L) > 1$. Because $0 \leq 10^k L - \lfloor 10^k L \rfloor < 1$, we know $L < \frac{\lfloor 10^k L \rfloor}{10^k} + 10^{-k}$. Thence,

$$\lfloor 10^k y \rfloor > \lfloor 10^k L \rfloor + 1$$

$$\frac{\lfloor 10^k y \rfloor}{10^k} > \frac{\lfloor 10^k L \rfloor}{10^k} + 10^{-k} > L.$$

Now select $\varepsilon := \frac{1}{2}\left(\frac{\lfloor 10^k y \rfloor}{10^k} - L\right)$; for any $N \in \mathbb{N}$, $N + k \geq N$ so $\frac{l_{N+k}}{10^{N+k}} \geq \frac{\lfloor 10^{N+k} y \rfloor}{10^{N+k}} > L$ by maximality. Accordingly, $\left| \frac{l_{N+k}}{10^{N+k}} - L \right| > \frac{\lfloor 10^k L \rfloor}{10^k} - L > \varepsilon$, contradicting L being the limit. Accordingly, it must be true that L is an upper bound of A. Suppose an upper bound $u < L$ of A exists. Then, let $\varepsilon > 0$ and N be a natural such that $\left| \frac{l_n}{10^n} - L \right| < \varepsilon$ holds as long as $n \geq N$. We see that $u - \frac{l_n}{10^n} \geq 0$, and $\frac{l_n}{10^n} > L - \varepsilon > u - \varepsilon$ so $u - \frac{l_n}{10^n} < \varepsilon$. That is, $\left| u - \frac{l_n}{10^n} \right| < \varepsilon$, informing us u is also a limit of $\{\frac{l_n}{10^n}\}_{n=1}^{\infty}$ even though $L \neq u$, a contradiction again. Consequently, L is certainly the supremum of A. $\square$

and $\lfloor 10^m x \rfloor \ge \lfloor 10^m x' \rfloor$. Thus, $x \ge x'$ and $\lfloor 10^m x \rfloor \ge \lfloor 10^m x' \rfloor$. In other words, $l_m = \lfloor 10^m x \rfloor$. Hence,

Self-Proof of Proposition 2.34

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence converging to L, $\varepsilon > 0$ and N be any natural number with $|a_n - L| < \varepsilon$ for $n \ge N$. Then take $u := \max\{L+\varepsilon, a_1, a_2, \dots, a_{N-1}\}$ and $l := \min\{L-\varepsilon, a_1, a_2, \dots, a_{N-1}\}$.
It is clear that u and l are upper and lower bounds of $\{a_n\}_{n=1}^{\infty}$ respectively.

$\square$

Self-Proof of Theorem 2.37 The Monotone Sequence Theorem

Actually we can just use Proposition 1.21

1. Let $L := \sup\{a_n \mid n \in \mathbb{N}\}$ and $\varepsilon > 0$, as usual. There must exist some a_N with $L - a_N < \varepsilon$, lest $L - \varepsilon \ge a_N$ for all natural N, but this contradicts L being supremal. Since the sequence is nondecreasing, so for each $n \ge N$, $|L - a_n| \le |L - a_N| < \varepsilon$. Indeed, L is the limit of $\{a_n\}_{n=1}^{\infty}$.

2. Follows similarly to 1.'s proof.

$\square$

Self-Proof of Proposition 2.40

Let $\varepsilon > 0$ and N be a natural number for which $|a_m - L| < \varepsilon$ so long as $m \ge N$. Notice $n_m \ge m$ as $\{n_m\}_{m=1}^{\infty}$ is strictly increasing. As such, for $m \ge N$, we can be certain that $|a_{n_m} - L| < L$. Consequently, $\lim_{n\to\infty} a_n = \lim_{m\to\infty} a_{n_m} = L$.

$\square$

Self-Proof of Theorem 2.41 The Bolzano-Weierstrass Theorem

Ideas

$\inf I_n \le \inf I_m$ $n < m$

Let $I_n = \{a_k \mid k \ge n\}$, $L = \sup\{\inf I_n \mid n \in \mathbb{N}\}$ $l_n :=$ the least element of $\{k \ge n \mid a_k \in (L - \frac{1}{2}10^{-n}, L + \frac{1}{2}10^{-n})\} \ne \emptyset$

$a_k \ge \inf I_n$

Since $L - \frac{1}{2}10^{-n} < L$, exists $L \ge \inf I_m > L - \frac{1}{2}10^{-n}$ wlog $m \ge n$ since if $n > m$, $\inf I_n \ge \inf I_m$

$a_k > L - \frac{1}{2}10^{-n}$ for $k \ge m$

If $a_k \ge L + \frac{1}{2}10^{-n}$ for each $k \ge m$, $\inf I_m \ge L + \frac{1}{2}10^{-n}$ so $L \ge L + \frac{1}{2}10^{-n}$.

A contradiction.

$b_n := a_{l_n}$

Proof

Let $I_n := \{a_k \mid k \geq n\}$, $L := \sup\{\inf I_n \mid n \in \mathbb{N}\}$ and ℓ_n be the least element of $\{k \geq n \mid a_k \in (L - 10^{-n}, L + 10^{-n})\}$. Since $L > L - 10^{-n}$, there must exist $\inf I_m > L - 10^{-n}$ so $a_k > L - 10^{-n}$ given $k \geq m$, where we can suppose wlog that $m \geq n$. Otherwise, $\inf I_n \geq \inf I_m$ when $n > m$ anyways. If $a_k \geq L + 10^{-n}$ for each $k \geq m$, $\inf I_m \geq L + 10^{-n} > L$, a contradiction. Thus, there must exist $k \geq m \geq n$ with $a_k \in (L - 10^{-n}, L + 10^{-n})$. Therefore, ℓ_n must exist and using the recursion theorem, we can define the strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ by $n_1 = \ell_1$, $n_{k+1} = \ell_{n_k+1}$ $(> n_k)$. As such, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ clearly converges to L: for $\varepsilon > 0$, there is some $M \in \mathbb{N}$ with $\varepsilon > 10^{-M}$. When $m \geq M$, $|a_{n_m} - L| < 10^{-m} < 10^{-M} < \varepsilon$, as desired. $\square$

Exercises

2-28 (a) Take $n_k = 2k$ and $n_k = 2k+1$. Note how they converge to two different values, contradicting Proposition 2.40 if $\{(-1)^n\}_{n=1}^{\infty}$ converges.

(b) Take $n_k = 2k+1$ so the corresponding subsequence converges to 0. If $\{(1+(-1)^n)n\}_{n=1}^{\infty}$ converges, its limit is 0 by Proposition 2.40. But given $\varepsilon = 1$ for example $|(1+(-1)^{2N})2N| = 4N > 1$ for all $N \in \mathbb{N}$, a contradiction.

2-29

In the proof of Proposition 2.34, we simply needed to find any upper and lower bounds for a convergent sequence $\{a_n\}_{n=1}^{\infty}$. Our procedure allowed us to find an upper and a lower bound for any choice of ε, including $\varepsilon = 1$ for instance. Even if we suddenly choose another value for ε, the previous bounds are still..., well, bounds. In this case, the choice of ε does not affect our argument since it does not involve ε, directly or indirectly.

In a general convergence proof, we are showing that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $|a_n - L| < \varepsilon$ for $n \geq N$. Hence, if we choose a specific value of ε, like 1, and prove such an N exists for it, it does not complete the proof since there are still (uncountably infinite many) other choices of ε which we do not know if the same holds. $\square$

2-30 See self-proof $\square$

It follows that u and l are upper and lower bounds of $\{a_n\}_{n=1}^{\infty}$ respectively. $\ldots, a_{N-1}\}$ and $l := \min\{L-\varepsilon, a_1, a_2, \ldots, a_{N-1}\}$.

2-33

Proof. The set $G_\varepsilon := \{n \in \mathbb{N} \mid |x_n - x| \geq \varepsilon\}$ must be infinite for some $\varepsilon > 0$ because if not, G_ε would be finite while $L_\varepsilon := \{n \in \mathbb{N} \mid |x_n - x| < \varepsilon\}$ would be infinite, for all $\varepsilon > 0$. Thus, for every $\varepsilon > 0$, there exists $N := \min\{n \in L_\varepsilon \mid n > m \text{ for all } m \in G_\varepsilon\}$ so given $k \geq N$, $k \in L_\varepsilon$ meaning $|x_k - x| < \varepsilon$. Hence implying $\lim_{k\to\infty} x_k = x$, a contradiction. Indeed, G_ε must be infinite for some $\varepsilon > 0$. Considering this, we now easily form a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ recursively, by $n_1 := \min G_\varepsilon$ and $n_{k+1} := \min\{n \in G_\varepsilon \mid n > n_k\}$. Now, we have the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ which must obey $|x_{n_k} - x| \geq \varepsilon$ for all $k \in \mathbb{N}$ by construction.

$\square$

2-35

Proof.
$G_\varepsilon := \{n \in \mathbb{N} \mid |x_n - L| \geq \varepsilon\}$ must be finite given $\varepsilon > 0$, lest it is infinite for some $\varepsilon > 0$ which would mean all subsequences of the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ defined recursively by $n_1 = \min G_\varepsilon$ and $n_{k+1} = \min\{n \in G_\varepsilon \mid n > n_k\}$ would not converge to L; Now, given $\varepsilon > 0$, there exists $N := \min\{n \in \mathbb{N} \mid |x_n - L| < \varepsilon \text{ and } n > m \text{ for all } m \in G_\varepsilon\}$ with $|x_n - L| < \varepsilon$ if $n \geq N$. So, $\{x_n\}_{n=1}^{\infty}$ indeed converges, in fact, to L.

$\square$

2-36

(a) When $n = 1$, $b^{n+1} - a^{n+1} = b^2 - a^2 = (b-a)(b+a) = (b-a)\sum_{j=0}^{n} a^j b^{n-j}$, as expected. Assume $b^{k+1} - a^{k+1} = (b-a)\sum_{j=0}^{k} a^j b^{k-j}$ for any $k \leq n$. Now, we see that
$b^{(n+1)+1} - a^{(n+1)+1} = b(b^{n+1} - a^{n+1}) + a(b^{n+1} - a^{n+1}) + a^{n+1}b - ab^{n+1} = (a+b)(b-a)\sum_{j=0}^{n} a^j b^{n-j} - ab(b-a)\sum_{j=0}^{n-1} a^j b^{n-j-1} = (b-a)\sum_{j=0}^{n} a^j b^{n+1-j} + (b-a)a^{n+1}$
$= (b-a)\sum_{j=0}^{n+1} a^j b^{n+1-j}$. Therefore, by strong induction, $b^{n+1} - a^{n+1} = (b-a)\sum_{j=0}^{n} a^j b^{n-j}$.

(b) From (a), $\dfrac{b^{n+1} - a^{n+1}}{b-a} = \sum_{j=0}^{n} a^j b^{n-j}$, which must be strictly less than $(n+1)b^n$ because $0 \leq a < b$ tells us $a^j b^{n-j} < b^n$.

(c) Ideas

$$\frac{\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1}}{\frac{1}{n} - \frac{1}{n+1}} < (n+1)\left(1+\frac{1}{n}\right)^n$$

$$\frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$$

$$\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1} < \frac{1}{n}\left(1+\frac{1}{n}\right)^n$$

$$\left(1+\frac{1}{n}\right)^{n+1} - \frac{1}{n}\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$1 + \frac{1}{n} - \frac{1}{n} = 1 \checkmark$$

$$\left(1+\frac{1}{1}\right)^1 = 2 < 10$$

$$\left(1+\frac{1}{n}\right)^n < 10$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} < \left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)^n$$

$$\left|\left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n\right| < \left|\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n}\right)^n - \left(1+\frac{1}{n}\right)^n\right| = \left|\frac{1}{n+1}\left(1+\frac{1}{n}\right)^n\right|$$

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1} - 10^{n+1}}{\frac{1}{n+1} - 9} < (n+1)(10^n)$$

$$\frac{1}{n+1} - \frac{9n+9}{n+1} = \frac{-9n+8}{n+1}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} - 10^{n+1} < (-9n+8)(10^n)$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} < (-9n+18)(10^n)$$

Self-Proof of Theorem 2·37 The Monotone Sequence Theorem Actually we can

... with $L - a_N < \varepsilon$, lest $L - \varepsilon \geq a_N$ for all natural N, but this contradicts L

2-34

Core argument : For every $x \in \mathbb{R}$, such an $\varepsilon_x > 0$ must exist lest $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Details : Assume for the sake of contradiction, that there exists some $x \in \mathbb{R}$ so for any $\varepsilon_x > 0$, $\{n \in \mathbb{N} \mid |x_n - x| < \varepsilon_x\}$ is infinite. Now define the strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$, using the Recursion Theorem, by $n_1 = \min\{n \in \mathbb{N} \mid |x_n - x| < 1\}$ and $n_{k+1} = \min\{n > n_k \mid |x_n - x| < \frac{1}{k+1}\}$. So, for each $k \in \mathbb{N}$ we have $|x_{n_k} - x| < \frac{1}{k}$ by construction. Hence, $\lim_{k \to \infty} |x_{n_k} - x| = 0$ by Squeeze's Theorem. Which implies $\lim_{k \to \infty} x_{n_k} = x$. Thus, contradicting the nonexistence of convergent subsequences of $\{x_n\}_{n=1}^{\infty}$. $\square$

2-36 (c) Proof (b) tells us that

$$\frac{(1+\frac{1}{n})^{n+1} - (1+\frac{1}{n+1})^{n+1}}{\frac{1}{n} - \frac{1}{n+1}} < (n+1)(1+\tfrac{1}{n})^n$$

$$(1+\tfrac{1}{n})^{n+1} - (1+\tfrac{1}{n+1})^{n+1} < \tfrac{1}{n}(1+\tfrac{1}{n})^n$$

$$(1+\tfrac{1}{n}-\tfrac{1}{n})(1+\tfrac{1}{n})^n < (1+\tfrac{1}{n+1})^{n+1}$$

$$(1+\tfrac{1}{n})^n < (1+\tfrac{1}{n+1})^{n+1},$$

so $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ is indeed increasing. It is also bounded above by 10 : When $n=1$ or $n=2$, $(1+\frac{1}{1})^1 = 2 < 10$ and $(1+\frac{1}{2})^2 = \frac{9}{4} < 10$.

Assuming $(1+\frac{1}{n})^n < 10$, if contrary to our expectations, $(1+\frac{1}{n+1})^{n+1} \not< 10$, then

$$\frac{(1+\frac{1}{n+1})^{n+1} - 10^{n+1}}{\frac{1}{n+1} - 9} < (n+1)10^n$$

$$(1+\tfrac{1}{n+1})^{n+1} - 10^{n+1} < (-9n+8)10^n$$

$$(1+\tfrac{1}{n+1})^{n+1} < (-9n+18)10^n.$$

But for $n \geq 2$, $-9n + 18 \leq 0$ even though $(1+\frac{1}{n+1})^{n+1} > 0$ is a fact. Hence, it must be that $(1+\frac{1}{n+1})^{n+1} < 10$ instead. By induction, the sequence $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ is certainly bounded above by 10. By the Monotone Sequence Theorem, $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ must converge. $\square$

Thm! 2-37. Joined this, we now easily ... which must obey $|x_{n_k} - x| \geq \varepsilon$

Idea: with an upper bound $u \in \mathbb{R}$ — defined recursively

Let $A \neq \emptyset$, H_n be $\{x \in A \mid \text{for all } y \in A,\ y - x < \frac{1}{n}\}$ and C be the choice function on A. Define the sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 := C(A)$ and

$$a_{n+1} := \begin{cases} C(\{x \in A \mid x > a_n \ \& \text{ for all } y \in A,\ y - x < \frac{1}{n+1}\}) & \text{if it exists,} \\ a_n & \text{otherwise.} \end{cases}$$

$\exists e \in A$ $\{x \in A \mid (\forall y \in A)(y - x < \frac{1}{n})\}$
~~If max(A) exists, $\neq \emptyset$ trivially.~~
~~If for all $x \in A$ there exists $y \in A$ with $y > x$,~~
If $\forall x \in A$, $\exists y \in A$ so $y - x \geq \frac{1}{n}$, A unbounded from above.
Lor: $\varepsilon + \lceil u - e \rceil (n)(\frac{1}{n})$, induct over $n \lceil u - e \rceil$

By construction it must be nondecreasing & upper bounded by u. So it converges to some limit L by MST.

$a_n \leq L$ for all $n \in \mathbb{N}$ is clear from the sequence being nondecreasing

So, for $x \in A$, if x is not the maximum of A, i.e. there exists $y \in A$ with $y > x$, $y - x > 0$ $y - x > \frac{1}{n}$

$L - z > 0$ $L - z > \frac{1}{n}$ $y - a_n < \frac{1}{n} < y - x$

$L - a_n \leq a_{n+1} - a_n < \frac{1}{n} < L - z$ $x < a_n$

When x is the maximum of A, i.e. $x \geq y$ for all $y \in A$, $z < a_n$ If $a_n = a_{n-1}$, $\{\cdots \frac{1}{n}\} = \emptyset \Rightarrow a_n$ is maximum in A.

let $\varepsilon > 0$, $\varepsilon > \frac{1}{N}$ for some $N \in \mathbb{N}$, s.t. $|x - a_n| < \frac{1}{n} < \varepsilon$ for each $n \geq N$. As such, $L = x$.

Proof

Let A be any nonempty subset of $\mathbb{R}$ bounded above by some $u \in \mathbb{R}$, and define the sequence $\{a_n\}_{n=1}^{\infty}$ recursively by having $a_1 := C(\{x \in A \mid \text{for all } y \in A,\ y - x < 1\})$, $a_{n+1} = C(\{x \in A \mid x > a_n \ \& \text{ for any } y \in A,\ y - x < \frac{1}{n+1}\})$ if it exists, and $a_{n+1} = a_n$ otherwise. Where C is a choice function on A. We see that $\{a_n\}_{n=1}^{\infty}$ must be nondecreasing and upper bounded by u. Hence, the Monotone Sequence Theorem informs us that $\{a_n\}_{n=1}^{\infty}$ converges to some limit $L \in \mathbb{R}$.

We claim that this must be the supremum of A. Suppose $x \in A$, there are 2 cases to evaluate:

Case 1 x is not the maximum of A, thus there is some $y \in A$ with $y > x$. Select any $n \in \mathbb{N}$ with $y - x > \frac{1}{n}$. Since $y - a_n < \frac{1}{n} < y - x$, $a_n > x$. Now, $L > x$ too because $L \geq a_n$ by virtue of $\{a_n\}_{n=1}^{\infty}$ being nondecreasing.

Case 2 x is the maximum of A. Then for every $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $\varepsilon > \frac{1}{N}$. Given $n \geq N$, $|x - a_n| < \frac{1}{n} < \varepsilon$. Therefore $L := \lim_{n \to \infty} a_n = x$.

Indeed, L is an upper bound of A. If $z < L$, select $n \in \mathbb{N}$ such that $L - z > \frac{1}{n}$. So, $L - a_n \leq a_{n+1} - a_n < \frac{1}{n} < L - z$ and $a_n > z$ follows. Consequently, L is supremal in A as expected.

2-33

Proof The set $G_\varepsilon := \{n \in \mathbb{N} \mid |x_n - x| \geq \varepsilon\}$ must be infinite for some $\varepsilon > 0$ because if not, G_ε would be finite while $L_\varepsilon := \{n \in \mathbb{N} \mid |x_n - x| < \varepsilon\}$ ... there exists $N := \min\{n \in L_\varepsilon \mid n > m \text{ for all } m \in G_\varepsilon\}$ so given $k \geq N$, $k \in L_\varepsilon$ meaning $|x_k - x| < \varepsilon$. Hence implying $\lim_{k \to \infty} x_k = x$, a contradiction. ... form a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ recursively, by $n_1 := \min G_\varepsilon$ and

2-38 Ideas

Construct $\{a_n\}_{n=1}^{\infty}$ as in 2-37. Exists $\bar{x} \in A$. $l_n := a_n$ is a lower bound for $\{x \in A \mid x \geq a_n\}$

BWT: Exists $\{a_{n_k}\}_{k=1}^{\infty}$ with limit L. Given $\varepsilon > 0$, exists $K \in \mathbb{N}$ so for every $k \geq K$, $|a_{n_k} - L| < \varepsilon$

$n_K \in \mathbb{N}$ so for every $m \geq n_K$, $a_{n_K} \leq a_m \leq a_{n_{K+m}}$ $\quad (n_{K+m} \geq K+m > m)$

$\varepsilon > L - a_{n_K} \geq L - a_m \geq L - a_{n_{K+m}} \geq 0$

Indeed, $\lim_{n \to \infty} a_n = L$

Proof Construct $\{a_n\}_{n=1}^{\infty}$ as in 2-37. Since this is bounded above by u and bounded below by a_1, by the Bolzano-Weierstrass Theorem, it has a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ that converges to some $L \in \mathbb{R}$. That is, given $\varepsilon > 0$, there is $K \in \mathbb{N}$ so for every $k \geq K$, $|a_{n_k} - L| < \varepsilon$. Since $n_m \geq m$, $\varepsilon > L - a_{n_K} \geq L - a_m \geq L - a_{n_m} \geq 0$ for $m \geq n_K$ as the sequence is nondecreasing. Consequently, $\{a_n\}_{n=1}^{\infty}$ also converges to L.

The rest of the proof follows as before. $\square$

2-39 Ideas

There must exist some upper bound u and lower bound l of $\{a_n\}_{n=1}^{\infty}$. Exists $N \in \mathbb{N}$ so when $n, m \geq N$, $|a_n - a_m| < 1$ $\quad u := \max\{a_1, a_2, a_3, \dots a_N + 1\}$, $l := \min\{a_1, a_2, a_3, \dots, a_N - 1\}$.

By BWT: Exists $\{a_{n_k}\}_{k=1}^{\infty}$ that converges to some limit L. Given $\bar{\varepsilon} > 0$, exists $K \in \mathbb{N}$ with $|a_{n_k} - L| < \bar{\varepsilon}$ for $k \geq K$. Let $\bar{\varepsilon} = 2\varepsilon$

$m \geq n_K$: $|a_m - L| = |a_m - a_k + a_k - L|$

$\leq |a_m - a_k| + |a_k - L|$

$\varepsilon \left\{ \begin{array}{l} \bar{\varepsilon} \left\{ \begin{array}{l} -L \\ -a_{n_k} \end{array} \right. \\ \bar{\varepsilon} \left\{ \right. \end{array} \right.$

Proof Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence of real numbers. It must be bounded below and above by some u and l. For example, there exists $N \in \mathbb{N}$ so when $n, m \geq N$, $|a_n - a_m| < 1$ thus $u := \max\{a_1, a_2, a_3, \dots, a_N + 1\}$ and $l := \max\{a_1, a_2, a_3, \dots, a_N - 1\}$ is one possible construction. By the Bolzano-Weierstrass Theorem, there exists some subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ that converges to some limit L. Given $\varepsilon > 0$, there hence exists $K_1, K_2 \in \mathbb{N}$ with $|a_{n_{k_1}} - L| < \frac{1}{2}\varepsilon$ and $|a_{m_2} - a_{m_2'}| < \frac{1}{2}\varepsilon$ for $k_1 \geq K_1$ and $m_2, m_2' \geq K_2$. So, when $m \geq \max\{K_1, K_2\}$, $|a_m - L| = |a_m - a_{m+1} + a_{m+1} - L| \leq |a_m - a_{m+1}| + |a_{m+1} - L| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. Subsequently, $\{a_n\}_{n=1}^{\infty}$ is seen to indeed converge, in fact, to $L \in \mathbb{R}$. $\square$

2-40 Ideas

Let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence that is upper bounded by u. a_0 is a lower bound since the sequence is nondecreasing. By BWT: exists $\{a_{n_k}\}_{k=1}^{\infty}$ that converges to L.

Given $\varepsilon > 0$, exists $K \in \mathbb{N}$ with $|a_{n_k} - L| < \varepsilon$ for $k \geq K$. For $m \geq n_K$, $\varepsilon > L - a_{n_K} \geq L - a_m \geq L - a_{n_m} \geq 0$ as $n_m \geq m$.

nondecreasing → Limit $L \geq a_m$ for all $m \in \mathbb{N}$

Basically what we did in 2-38

$a_{n_K} \leq a_m \leq a_{n_m}$

Repeat for pt 2. of MCT

2-41

(a) Proof Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences bounded above by u_1 and u_2 respectively. For $n \in \mathbb{N}$, $a_n \leq u_1$ and $b_n \leq u_2$ so $a_n + b_n \leq u_1 + u_2$. which means $\{a_n + b_n\}_{n=1}^{\infty}$ is bounded above by $u_1 + u_2$. Similarly, it must be bounded below.

(b) Proof Let u_1 be an upper bound and l_1 a lower bound of $\{a_n\}_{n=1}^{\infty}$, u_2 an upper bound and l_2 a lower bound of $\{b_n\}_{n=1}^{\infty}$. Fix $a := \max\{|u_1|, |l_1|\}$ and $b := \max\{|u_2|, |l_2|\}$. They respectively bound $\{|a_n|\}_{n=1}^{\infty}$ and $\{|b_n|\}_{n=1}^{\infty}$ from above. So for each $n \in \mathbb{N}$, $a_n b_n \leq |a_n b_n| \leq ab$, meaning ab bounds $\{a_n b_n\}_{n=1}^{\infty}$ from above. Again, showing $\{a_n b_n\}_{n=1}^{\infty}$ is bounded from below follows similarly.

(c) Proof Given any $n \in \mathbb{N}$, $\frac{1}{\delta} > \frac{1}{b_n} > 0$. Thus, $\{\frac{1}{b_n}\}_{n=1}^{\infty}$ is bounded above and below by $\frac{1}{\delta}$ and 0 respectively. Therefore, $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ is a bounded sequence by (b). $\square$

2-42 Ideas

Show: Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $|a_n + \frac{1}{2}(x - a_n^2) - \sqrt{x}| < \varepsilon$ for $n+1 \geq N$
$n \geq N$

$a_{n+1} = \frac{1}{2}(-a_n^2 + 2a_n + x)$ ✓

When $n=0$, $a_{0+1} = 0 + \frac{1}{2}(x - 0^2) = \frac{1}{2}x \underset{\geq 0 = a_0}{\in} [0, \sqrt{x}] \subseteq [0,1]$

Assume $\sqrt{x} \geq a_{k+1} \geq a_k \geq 0$ for $k \leq n$,

$a_{n+2} = a_{n+1} + \frac{1}{2}(x - a_{n+1}^2)$

$\left(= a_n + \frac{1}{2}(x - a_n^2) + \frac{1}{2}\left[x - \left(a_n + \frac{1}{2}(x - a_n^2)\right)^2\right]\right)$

$a_{n+1}^2 \leq x \Rightarrow \frac{1}{2}(x - a_{n+1}^2) \geq 0$
$\Rightarrow a_{n+2} \geq a_{n+1} \ (\geq 0)$ ✓

$\frac{1}{4}x^2 \leq x$ ✓

$-\frac{1}{4}x^2 + x = -\frac{1}{4}(x^2 - 4x) = -\frac{1}{4}(x-2)^2 + 1 \in [0, \frac{3}{4}]$

$x \in [0,1]$ $\quad x - 2 \in [-2,-1]$
$(x-2)^2 \in [1,4]$
$-\frac{1}{4}(x-2)^2 \in [-1, -\frac{1}{4}]$

$\frac{1}{4}x^2 \leq x$
$\frac{1}{2}x \leq \sqrt{x}$ ✓

$\frac{1}{4}a_n^4 - a_n^3 + (1 - \frac{1}{2}x)a_n^2 + x a + \frac{1}{4}x^2$

$a_{n+1} \leq 1 - \sqrt{x - 2\sqrt{x} + 1}$

$x - 2\sqrt{x} + 1 \leq (1 - a_{n+1})^2$

$x \leq 1$
$x^2 \leq x$
$x \leq \sqrt{x} \leq 1$
$x + 1$

Show: $\frac{1}{2}(x - a_{n+1}^2) < \sqrt{x} - a_{n+1}$

$-\frac{1}{2}a_{n+1}^2 + a_{n+1} + \frac{1}{2}x - \sqrt{x} < 0$

$a_{n+1}^2 - 2a_{n+1} - x + 2\sqrt{x} > 0$

$(-2)^2 - 4(1)(-x + 2\sqrt{x})$
$= 4[1 - (2\sqrt{x} - x)]$
$= 4[x + 1 - 2\sqrt{x}]$

$\frac{2 \pm \sqrt{4 - 4(1)(-x + 2\sqrt{x})}}{2}$
$= 1 \pm \sqrt{1 - (-x + 2\sqrt{x})}$
$= 1 \pm \sqrt{x - 2\sqrt{x} + 1}$

$x + 1 \geq 2\sqrt{x}$
$x^2 + 2x + 1 \geq 2x$

Example 2.43

Ideas

$x^1 = x = x + (1-1)(x-1)$

$x^2 = [(x-1)+1]x > x + (x-1)$

$x^3 = (x-1)x^2 + x^2$

$= (x-1)x^2 + [(x-1)+1]x$

$> (x-1) + (x-1) + x$

$= 2(x-1) + x$

Assume $x^n \geq x + (n-1)(x-1)$

$x^{n+1} = [(x-1)+1]x^n$

$= (x-1)x^n + x^n$

$\geq 1(x-1) + x + (n-1)(x-1)$

$= x + n(x-1)$

Proof By simple induction we can show that $x^n \geq x + (n-1)(x-1)$

Let $M \in \mathbb{R}$, and $N := \lceil \frac{M}{x-1} \rceil$ so for $n \geq N+1$,

$x^n \geq x^{N+1} \geq x + N(x-1)$

$\geq x + \left(\frac{M}{x-1}\right)(x-1)$

$\geq x + M$

$> M$

Hence, $\lim_{n\to\infty} a_n = \infty$. $\square$

Theorem 2.44 Limit laws involving ∞

1. Let $\varepsilon > 0$. Let $u > 0$ be an upper bound of $\{b_n\}_{n=1}^{\infty}$. Since $\lim a_n = \infty$, there exists $N \in \mathbb{N}$ with $a_n \geq \frac{2u}{\varepsilon}$ given $n \geq N$. So, $\left|\frac{b_n}{a_n} - 0\right| \leq \frac{u}{a_n} \leq \frac{\varepsilon}{2} < \varepsilon$. Hence, $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$.

2. Let $M \in \mathbb{R}$. Then there exists $N \in \mathbb{R}$ so given $n \geq N$, $a_n \geq |\lceil M \rceil| + 1$ and $b_n \geq |\lceil M \rceil| + 1$. Thus, $a_n + b_n \geq 2|\lceil M \rceil| + 2 \geq 2M + 2 \geq M$, and similarly, $a_n b_n \geq (|\lceil M \rceil| + 1)^2 \geq M$. Accordingly, $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n b_n = \infty$.

3. & 4. Let $M \in \mathbb{R}$, $c > 0$. There exists $N \in \mathbb{R}$ so given $n \geq N$, $a_n \geq \frac{M}{c}$. As such, $ca_n \geq M$ therefore $\lim_{n\to\infty} ca_n = \infty$. Part 4 follows similarly. $\square$

Example 2.45

Ideas

Case 1

$0 \leq x \leq 1$

$x^n \leq x$

$x \leq x^{\frac{1}{n}} \leq 1$ lest $(x^{\frac{1}{n}})^n > x$

$x+1$ is an upper bound in both cases + nd / ni

Since $x^{\frac{1}{n}} \geq 0$, $\lim_{n\to\infty} x^{\frac{1}{n}} \geq 0$.

In case 1, 1 is an ub (lest ...).

If $u < 1$ is an ub,

Case 2

$x > 1$

$x^n > x$

$x > x^{\frac{1}{n}}$

$\Downarrow$

$L_x := \lim_{n\to\infty} x^{\frac{1}{n}}$ exists

$L_x^k = \lim_{n\to\infty} (x^{\frac{k}{n}}) = L_{x^k}$

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N$

$|L_x - x^{\frac{1}{n}}| < \varepsilon$

For $x > 1$,

$x^{\frac{k}{n}} > x^{\frac{1}{n}} > L_x$

$y = x^{\frac{1}{n}}$

$y^n = x$

$\lim_{n\to\infty} y^n = \infty$ for $y > 1$

$\lim_{n\to\infty} y^n = 0$ for $0 \leq y \leq 1$

$\lim_{k\to\infty} \lim_{n\to\infty} (x^{\frac{k}{n}}) = \infty$ if $L_x^k > 1$

$L_x > 1$

2-42 Ideas

$a_{0+1} = 0 + \frac{1}{2}(x - 0^2) = \frac{1}{2}x > 0 := a_0$

Assume $\sqrt{x} \geq a_{n+1} \geq a_n \geq 0$

$a_{(n+1)+1} = -\frac{1}{2}(a_{n+1} - 1)^2 + \frac{1}{2}(x+1)$

$a_n + \frac{1}{2}(x - a_n^2) \geq a_n$

$x - a_n^2 \geq 0$

$x \geq a_n^2 \geq 0$

$\sqrt{x} \geq a_n$

$0 \geq \sqrt{x} - 1 \geq a_{n+1} - 1$

$(a_{n+1} - 1)^2 \geq (\sqrt{x} - 1)^2$

$-\frac{1}{2}(\sqrt{x} - 1)^2 \geq -\frac{1}{2}(a_{n+1} - 1)^2$

$-\frac{1}{2}(\sqrt{x} - 1)^2 + \frac{1}{2}(x+1) = -\frac{1}{2}(x - 2\sqrt{x} + 1) + \frac{1}{2}x + \frac{1}{2}$

$= \sqrt{x} - \frac{1}{2}x + \frac{1}{2}x - \frac{1}{2} + \frac{1}{2}$

$= \sqrt{x}$

$\geq -\frac{1}{2}(a_{n+1} - 1)^2 + \frac{1}{2}(x+1)$

$= a_{(n+1)+1}$

For any $n \in \mathbb{N}$:

$0 \leq a_n \leq 1$

$-a_n^2 \leq 0$ $\qquad -1 \leq -a_n^2$

$\frac{1}{2}(x - a_n^2) \leq \frac{1}{2}x \leq \frac{1}{2}$ $\qquad -\frac{1}{2} \leq \frac{x-1}{2} \leq \frac{1}{2}(x - a_n^2)$

$\frac{1}{2}(x - a_n^2) \leq \frac{1}{2}$ $\qquad -\frac{1}{2} \leq \frac{1}{2}(x - a_n^2)$

$a_n + \frac{1}{2}(x - a_n^2) \leq a_n + \frac{1}{2}$ $\qquad a_n - \frac{1}{2} \leq a_n + \frac{1}{2}(x - a_n^2)$

$-\frac{1}{2}(a_n - 1)^2 + \frac{1}{2}(x+1)$

$-1 \leq a_n - 1$

$(a_n - 1)^2 \leq 1$

$-\frac{1}{2}(a_n - 1)^2 \leq -\frac{1}{2}$

$a_n - 1 \leq 0$

$(a_n - 1)^2 \geq 0$

$-\frac{1}{2}(a_n - 1)^2 \geq 0$

$a_{n+1} \geq \frac{1}{2}(x+1)$

$\frac{1}{2}(x - L^2) = 0$

$x = L^2$

$L = \sqrt{x}$

$(1-a_{n+1})^2 \leq (1-x)^2 = 1-2x+x$

$x-2\sqrt{x}+1 \leq 1-2a_{n+1}+a_{n+1}^2.$

$x-2\sqrt{x} \leq a_{n+1}^2 - 2a_{n+1}$

$x-2\sqrt{x}+1 \leq 1$

$x \leq 2\sqrt{x}$

$x^2 \leq 2x$

$x(x-2) \leq 0$

$0 \leq x \leq 2$

$a_{n+1} = a_n + \frac{1}{2}(x-a_n)(x+a_n)$

$x \geq a_n \geq 0$

$x \geq x-a_n \geq x-\sqrt{x} \quad 2\sqrt{x} \geq x+\sqrt{x} \geq x+a_n$

$\frac{1}{2}(x-a_n)(x+a_n) \leq \frac{1}{2}(x)(2\sqrt{x})$

$\leq \frac{1}{2}(2\sqrt{x})$

$\leq \sqrt{x}$

$a_n \leq \sqrt{x}$

$a_n + \frac{1}{2}(x-a_n^2) \leq \sqrt{x}$

$\frac{1}{4}a_n^4 - a_n^3 - \frac{x}{2}a_n^2 + (a_n-1)x + \frac{1}{4}x^2 \leq 0$

$a_{n+1} = -\frac{1}{2}(a_n^2 - 2a_n) + \frac{1}{2}x = -\frac{1}{2}(a_n-1)^2 + \underbrace{\frac{1}{2}x+\frac{1}{2}}^{\frac{1}{2}(x+1)}$

$0 \geq x-1 \geq a_n - 1 \geq -1$

$1 \geq (a_n-1)^2$

$-\frac{1}{2}(a_n-1)^2 \geq -\frac{1}{2}$

$a_{n+1} \geq \frac{1}{2}(x+1) - \frac{1}{2}$

$= \frac{1}{2}x$

$a_n + \frac{1}{2}(x-a_n^2) \leq$

$0 + \frac{1}{2}(x-0^2) = \frac{1}{2}x$

$\frac{1}{2}x + \frac{1}{2}(x - \frac{1}{4}x^2) = x - \frac{1}{8}x^2$

$x - \frac{1}{8}x^2 + \frac{1}{2}(x-(x-\frac{1}{8}x^2)^2) = -\frac{1}{128}x(x^3-16x^2+80x-192)$

$-\frac{1}{2}(a_n-1)^2 + \frac{1}{2}(x+1) \geq \frac{1}{2}(x+1)$

2 - 42 Proof

We see that since $\sqrt{x} \geq x \geq \frac{1}{2}x = a_1 \geq a_0 := 0$. Hence assume $\sqrt{x} \geq a_{n+1} \geq a_n$. So, $-\frac{1}{2}(\sqrt{x}-1)^2 \geq -\frac{1}{2}(a_n-1)^2$, and accordingly, $a_{(n+1)+1} := a_{n+1} + \frac{1}{2}(x - a_{n+1}^2)$ $= -\frac{1}{2}(a_{n+1}-1)^2 + \frac{1}{2}(x+1) \leq -\frac{1}{2}(\sqrt{x}-1)^2 + \frac{1}{2}(x+1) = \sqrt{x}$. Furthermore, this means $x \geq a_{n+1}^2$ thus $a_{n+1} + \frac{1}{2}(x - a_n^2) := a_{(n+1)+1} \geq a_{n+1}$.

Therefore, we have shown $\sqrt{x} \geq a_{(n+1)+1} \geq a_{n+1}$. As such, $\sqrt{x} \geq a_{n+1} \geq a_n$ for any $n \in \mathbb{N}$ by induction. Since $\sqrt{x}$ is an upper bound of $\{a_n\}_{n=1}^{\infty}$ which is nondecreasing, the Monotone Sequence Theorem informs us that $\{a_n\}_{n=1}^{\infty}$ converges to some limit L. It is clear that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$

By the Limit Laws of Theorem 2.14, and Theorem 2.17 $\left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} \frac{1}{2}\right)\left[\left(\lim_{n\to\infty} x\right) - \left(\lim_{n\to\infty} a_n\right)^2\right] = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} \frac{1}{2}\right)\left[\left(\lim_{n\to\infty} x\right) - \left(\lim_{n\to\infty} a_n^2\right)\right]$

$= \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} \frac{1}{2}\right)\left[\lim_{n\to\infty}(x - a_n^2)\right] = \left(\lim_{n\to\infty} a_n\right) + \lim_{n\to\infty}\left[\frac{1}{2}(x - a_n^2)\right] = \lim_{n\to\infty}\left[a_n + \frac{1}{2}(x - a_n^2)\right] = \lim_{n\to\infty} a_{n+1}$. Which is just L as stated above.

Consequently, $L + \frac{1}{2}(x - L^2) = L$ so $L = \pm\sqrt{x}$. Trivially, $L = -\sqrt{x}$ is impossible since $a_n \geq 0$ always (if $x > 0$). Wherefore, $\lim_{n\to\infty} a_n := L = \sqrt{x}$ $\square$

Example 2.45

Ideas $\lim_{n\to\infty} x^{\frac{k}{n}} = \left(\lim_{n\to\infty} x^{\frac{1}{n}}\right)^k$?

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N$

$|x^{\frac{k}{n}} - L| < \varepsilon$

$x > 1$:

$0 \leq x \leq 1$:

$x > 1$:
$0 < x^{\frac{1}{n}} < x^{\frac{k}{n}}$

$0 < x \leq 1$:
$0 < x^{\frac{k}{n}} \leq x^{\frac{1}{n}} \leq 1$

$|x^{\frac{1}{n}} - L| = x^{\frac{1}{n}} - L$
$\leq x^{\frac{k}{n}} - L$
$= |x^{\frac{k}{n}} - L|$
$< \varepsilon$

$|x^{\frac{1}{n}} - L| = L - x^{\frac{1}{n}}$
$\leq L - x^{\frac{k}{n}}$
$= |x^{\frac{k}{n}} - L|$
$< \varepsilon$

$\Rightarrow \lim_{n\to\infty} x^{\frac{1}{n}} = \left(\lim_{n\to\infty} x^{\frac{1}{n}}\right)^k$ indeed

$x^{\frac{1}{n}} > 1$
lest $x^{\frac{1}{n}} \leq 1$
so $x \leq 1$
$\Rightarrow x^{1+\frac{1}{n}} > x$
$x^{\frac{n+1}{n}} > x$
$x^{\frac{1}{n}} > x^{\frac{1}{n+1}}$

Decreasing / Non-increasing
$L < x^{\frac{1}{n}}$

$1 \geq x^{\frac{1}{n}} > 0$
lest $1 < x^{\frac{1}{n}}$
so $1 < x$
$\Rightarrow x \geq x^{1+\frac{1}{n}}$
$x \geq x^{\frac{n+1}{n}}$
$x^{\frac{1}{n+1}} \geq x^{\frac{1}{n}}$

Non-decreasing
$\Rightarrow L \geq x^{\frac{1}{n}}$ for all $n \in \mathbb{N}$

$L = L^k$
E.g. $L = L^2$
$L(L-1) = 0$
$L = 0$ or $L = 1$

$0 \geq x^{\frac{1}{n}}$
$\Rightarrow x^{\frac{1}{n}} = 0$
N.A.

Example 2-45

Proof

We claim that $\{x^{\frac{1}{n}}\}_{n=1}^{\infty}$ converges to some limit L:

Case 1 $0 < x \le 1$: $1 \ge x^{\frac{1}{n}} > 0$ lest $1 < x^{\frac{1}{n}}$ meaning $x < 1$, or $x^{\frac{1}{n}} = 0$ thus $x = 0$, a contradiction either ways. It follows that $x \ge x^{1+\frac{1}{n}} = x^{\frac{n+1}{n}}$, such that $x^{\frac{1}{n+1}} \ge x^{\frac{1}{n}}$. That is, $\{x^{\frac{1}{n}}\}_{n=1}^{\infty}$ is bounded above by 1, and is nondecreasing hence $L \ge x^{\frac{1}{n}}$ (for any $n \in \mathbb{N}$).

Case 2 $x > 1$: $x^{\frac{1}{n}} > 1$ lest $x^{\frac{1}{n}} \le 1$ so $x \le 1$, a contradiction as before. As such, $x < x^{1+\frac{1}{n}} = x^{\frac{n+1}{n}}$, telling us that $x^{\frac{1}{n+1}} < x^{\frac{1}{n}}$. In other words, $\{x^{\frac{1}{n}}\}_{n=1}^{\infty}$ is bounded below by 1, and decreasing / nonincreasing implying $L < x^{\frac{1}{n}}$ (for each $n \in \mathbb{N}$).

In any case, $\{x^{\frac{1}{n}}\}_{n=1}^{\infty}$ converges as expected by the Monotone Sequence Theorem. Next, we shall see that $L = L^k$ for every $k \in \mathbb{N}$. Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $|x^{\frac{k}{n}} - L| < \varepsilon$ for $n \ge N$. Again, evaluate casewise,

Case 1 $0 < x \le 1$: Since $0 < x^{\frac{1}{n}} \le 1$, $0 < x^{\frac{k}{n}} \le x^{\frac{1}{n}}$ holds. Therefore, $|x^{\frac{1}{n}} - L| = L - x^{\frac{1}{n}}$ (as $L \ge x^{\frac{1}{n}}$) $\le L - x^{\frac{k}{n}} = |x^{\frac{k}{n}} - L| < \varepsilon$.

Case 2 $x > 1$: Because $1 < x^{\frac{1}{n}} < x^{\frac{k}{n}}$, $|x^{\frac{1}{n}} - L| = x^{\frac{1}{n}} - L < x^{\frac{k}{n}} - L = |x^{\frac{k}{n}} - L| < \varepsilon$.

Indeed, combined with Theorem 2.17, $L := \lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} x^{\frac{k}{n}} = \left(\lim_{n\to\infty} x^{\frac{1}{n}}\right)^k = L^k$. For instance, when $k = 2$, $L(L-1) = 0$. Consequently, $L = 0$ or $L = 1$. $L = 0$ is impossible since for $0 < x \le 1$, $L \ge x^{\frac{1}{n}}$ would mean $x^{\frac{1}{n}} = 0$ thence $x = 0$, while for $x > 1$, 1 is a lower bound already. But $x > 0$.

Wherefore, $L := \lim_{n\to\infty} x^{\frac{1}{n}} = 1$. $\square$

.. $x<1$, or x ...

. decreasing hence $L \geq x^{\frac{1}{n}}$ (for any $n\in\mathbb{N}$).

.. $x^{\frac{1}{n+1}} < x^{\frac{1}{n}}$. In other words,

Exercises

2-43 The limit of a sequence $\{a_n\}_{n=1}^{\infty}$ is negative infinity iff for all $M\in\mathbb{R}$ there exists $N\in\mathbb{N}$ so when $n\geq N$, $a_n \leq M$.

2-45 (a) Since $\{1\}_{n=1}^{\infty}$ is obviously bounded and $\frac{1}{x}>1$ implies $\lim_{n\to\infty}\left(\frac{1}{x}\right)^n=\infty$ by example 2.43, so by Theorem 2.44 (the limit laws involving ∞),

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{x}\right)^n} = 0.$$

(b) Clearly, for all $n\in\mathbb{N}$, the inequality $x^n > x^{n+1} > 0$ holds. Suppose for a contradiction that $\{x_n\}_{n=1}^{\infty}$ does not go to 0. Then there is an $\varepsilon>0$ so for each $N\in\mathbb{N}$, there exists $n\geq N$ with $|x^n-0| = x^n \geq \varepsilon$. For any $n\in\mathbb{N}$, there exists $m\geq n$ with $x^n\geq x^m\geq\varepsilon$. Hence, the sequence $\{x^n\}_{n=1}^{\infty}$ is bounded below by ε. Let $L := \inf\{x^n \mid n\in\mathbb{N}\}$. Now, by exercise 1-16, because $\frac{L}{x}>L$, there is an $n\in\mathbb{N}$ so that $x^n<\frac{L}{x}$. But then $x^{n+1}=x\cdot x^n < x\frac{L}{x}=L$, a contradiction. Thus, $\lim_{n\to\infty}x^n=0$.

$\square$

2-46 See the self-proof for example 2.45

2-47 Ideas

$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3$

$n\geq 3$: $\quad n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$

$n^{n+1} > (n+1)^n$

$n^{1+\frac{1}{n}} > n+1$

$n > \left(1+\frac{1}{n}\right)^n$

$n^{\frac{1}{n}} > \frac{n+1}{n}$

$n^{1+\frac{1}{n}} > n+1$

$n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$

$3 > \left(1+\frac{1}{3}\right)^3 = \frac{64}{27}$

Assume $n > \left(1+\frac{1}{n}\right)^n$

$n+1 > n > \left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n+1}\right)^n$

$\Rightarrow n+\frac{1}{n+2} > n > \left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n+1}\right)^n$

$\Rightarrow \frac{n+1}{1+\frac{1}{n+1}} > \left(1+\frac{1}{n+1}\right)^n$

$n+1 > \left(1+\frac{1}{n+1}\right)^{n+1}$

Decreasing (for $n\geq 3$) thus the limit $L \leq n^{\frac{1}{n}}$ for all $n\geq 3$

$\frac{n+2}{n+1}$

$\frac{n}{\frac{n+2}{n+1}}$ $\quad$ $\frac{n+1}{\frac{n+2}{n+1}}$

$\frac{n(n+1)}{n+2}$ $\quad$ $\frac{(n+1)^2}{n+2}$

$=\frac{n^2+1}{n+2}$ $\quad$ $= n+\frac{1}{n+2}$

$= n-2+\frac{5}{n+2}$

$\lim_{n\to\infty} n^{\frac{k}{n}} = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^k$

$n^k > n \Rightarrow n^{\frac{k}{n}} \geq n^{\frac{1}{n}} \geq L$

$\forall\varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geq N$

$|n^{\frac{k}{n}}-L|<\varepsilon$

$\Rightarrow |n^{\frac{1}{n}}-L| = n^{\frac{1}{n}}-L$

$\leq n^{\frac{k}{n}}-L$

$<\varepsilon$

First notice $3>\left(1+\frac{1}{3}\right)^3=\frac{64}{27}$, and assume $n>\left(1+\frac{1}{n}\right)^n$. Then since $n+\frac{1}{n+2}>n>\left(1+\frac{1}{n}\right)^n>\left(1+\frac{1}{n+1}\right)^n$, $\frac{n+1}{1+\frac{1}{n+1}}>\left(1+\frac{1}{n+1}\right)^n$ so $n+1>\left(1+\frac{1}{n+1}\right)^{n+1}$. Thus, $n>\left(1+\frac{1}{n}\right)^n$ holds for any natural $n\geq 3$ by induction. Accordingly, $n^{\frac{1}{n}}>\frac{n+1}{n}$ and $n^{1+\frac{1}{n}}>n+1$, therefore $n^{\frac{1}{n}}>(n+1)^{\frac{1}{n+1}}$. Indeed, $\{\sqrt[n]{n}\}_{n=1}^{\infty}$ is decreasing for $n\geq 3$ and is bounded below by 0. By the Monotone Sequence Theorem, it converges to some limit L. Resultantly, $\lim_{n\to\infty}n^{\frac{k}{n}}=L^k$ by Theorem 2.17. Let $\varepsilon>0$. There exists $N\in\mathbb{N}$ so for every $n\geq N$, $|n^{\frac{k}{n}}-L|<\varepsilon$. As $n^{\frac{k}{n}}\geq n^{\frac{1}{n}}\geq L$, $|n^{\frac{1}{n}}-L|=n^{\frac{1}{n}}-L\leq n^{\frac{k}{n}}-L<\varepsilon$. In other words, $L=L^k$. For instance, in the case of $k=2$, $L(L-1)=0$. If $\sqrt[n]{n}<1$, $n<1$, a contradiction. As such, 1 is a lower bound implying $L\neq 0$. Consequently, $L=1$ as expected.

$\square$

2-48 (a)

Let $M \in \mathbb{R}$. For any $n \geq |\lceil M \rceil|$, $n \geq M$ and $-n^2 \leq -|\lceil M \rceil| \leq M$. Thus, it holds that $\lim_{n\to\infty} a_n = \infty$ while $\lim_{n\to\infty} b_n = -\infty$. Similarly, if $n \geq \lceil \sqrt{|M|+\frac{1}{4}} \rceil$, $a_n + b_n = -(n+\frac{1}{2})^2 + \frac{1}{4} \leq -(\lceil \sqrt{|M|+\frac{1}{4}} \rceil + \frac{1}{2})^2 + \frac{1}{4} < -\lceil \sqrt{|M|+\frac{1}{4}} \rceil^2 + \frac{1}{4} \leq -|M| \leq M$. Hence, $\lim_{n\to\infty} a_n + b_n = -\infty$. As such, $a_n + b_n$ must diverge.

(b) It follows from (a) and the limit laws that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$. Since $a_n + b_n = n - n = 0$, it is clear that $\{a_n + b_n\}_{n=1}^{\infty}$ converges to 0.

(c) Let $\{a_n\}_{n=1}^{\infty} = \{n+c\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{-n\}_{n=1}^{\infty}$. Then clearly, $\lim_{n\to\infty} a_n = \infty$ while $\lim_{n\to\infty} b_n = -\infty$ by (b) and limit laws. 13.

Furthermore, $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} c = c$.

(d) Let $\{a_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$. Part (a) with limit laws tells us $\lim_{n\to\infty} a_n = \infty$, and by Theorem 2.13, $\lim_{n\to\infty} b_n = 0$.

Thus, $\{a_n b_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ is clearly divergent.

(e) Let $\{a_n\}_{n=1}^{\infty} = \{cn\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$, so everything follows similarily as before.

(f) Simply let $\{a_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$.

(g) Fix $\{a_n\}_{n=1}^{\infty} = \{cn\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$.

2-49 Let $M \in \mathbb{R}$, then there exists $K \in \mathbb{N}$ so that if $k \geq K$, $a_k \geq \frac{M}{\varepsilon}$. Therefore, $a_k b_k \geq \frac{M}{\varepsilon} \cdot \varepsilon = M$. As such, $\lim_{n\to\infty} a_n b_n = \infty$.

2-50 (a) Suppose, without loss of generality, that $\{a_n\}_{n=1}^{\infty}$ is not bounded from above. Let $n_1 := 1$, $n_{k+1} :=$ the least $m \in \mathbb{N}$ with $a_m \geq a_{n_k} + 1$. Thus, $a_{n_k} \geq a_{n_1} + k - 1$. Now, given any $M \in \mathbb{R}$, and $k \geq \lceil M - a_{n_1} \rceil + 1$, $a_{n_k} \geq a_{n_1} + \lceil M - a_{n_1} \rceil \geq a_{n_1} + M - a_{n_1} = M$. So, $\lim_{k\to\infty} a_{n_k} = \infty$.

(b) Ideas

$\{a_n\}_{n=1}^{\infty}$ $\xrightarrow{BWT}$ convergent a_{ℓ_m} exists

L_ℓ

Should exist $\varepsilon > 0$ so $S_\varepsilon := \{n \in \mathbb{N} \mid |a_n - L_\ell| < \varepsilon\}$ and $T_\varepsilon := \{n \in \mathbb{N} \mid |a_n - L_\ell| \geq \varepsilon\}$ are both infinite.

If not, T_ε finite for all $\varepsilon > 0$. Given $n \geq \max(T_\varepsilon) + 1$, $|a_n - L_\ell| < \varepsilon$. Which would mean $\{a_n\}_{n=1}^{\infty}$ converges, a contradiction.

$b_1 := \min(T_\varepsilon)$, $b_{i+1} := \min\{m \in T_\varepsilon \mid m > n_k\}$. For that particular choice of ε, exists no $k \in T_\varepsilon$ with $|a_k - L| < \varepsilon$

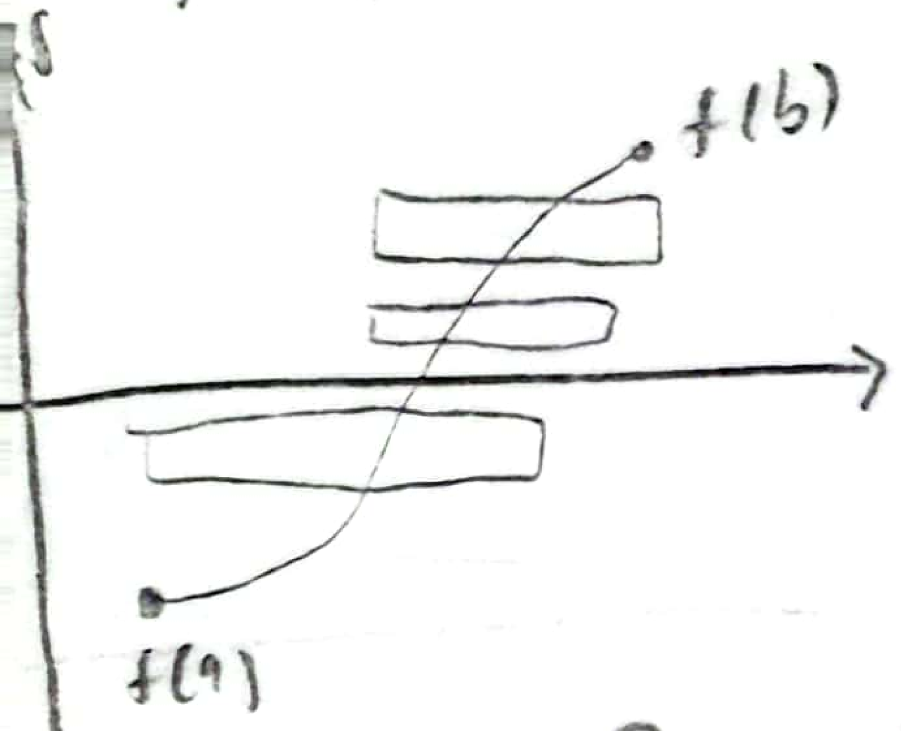
$\{a_{b_i}\}_{i=1}^{\infty}$ is bounded $\Rightarrow$ BWT $\to$ convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$.

Thus, $|a_{n_k} - L| < \varepsilon$ isn't possible for any $k \in \mathbb{N}$. So, $\lim_{m\to\infty} a_{\ell_m} \neq \lim_{k\to\infty} a_{n_k}$.

Proof

By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\{a_{\ell_m}\}_{m=1}^{\infty}$. Let $L_\ell := \lim_{m\to\infty} a_{\ell_m}$. There must exist some $\varepsilon > 0$ with $S_\varepsilon := \{n \in \mathbb{N} \mid |a_n - L_\ell| < \varepsilon\}$ and $T_\varepsilon := \{n \in \mathbb{N} \mid |a_n - L_\ell| \geq \varepsilon\}$ are both infinite. Otherwise, if T_ε was finite for all $\varepsilon > 0$, then given $n \geq \max(T_\varepsilon) + 1$, $|a_n - L_\ell| < \varepsilon$. Thus, $\{a_n\}_{n=1}^{\infty}$ converges, a contradiction. Indeed, T_ε is finite for some $\varepsilon > 0$, for which we define $b_1 := \min(T_\varepsilon)$ and $b_{i+1} := \min\{m \in T_\varepsilon \mid m > n_k\}$. Being a subsequence, $\{a_{b_i}\}_{i=1}^{\infty}$ is bounded so the Bolzano-Weierstrass Theorem again informs us that a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ must be present. Since $a_{n_k} \in T_\varepsilon$, for any $N \in \mathbb{N}$, $|a_N - L_\ell| \nless \varepsilon$. Consequently, $\lim_{m\to\infty} a_{\ell_m} \neq \lim_{k\to\infty} a_{n_k}$, even though they are both (convergant) subsequences.

- Proof of the Intermediate Value Theorem



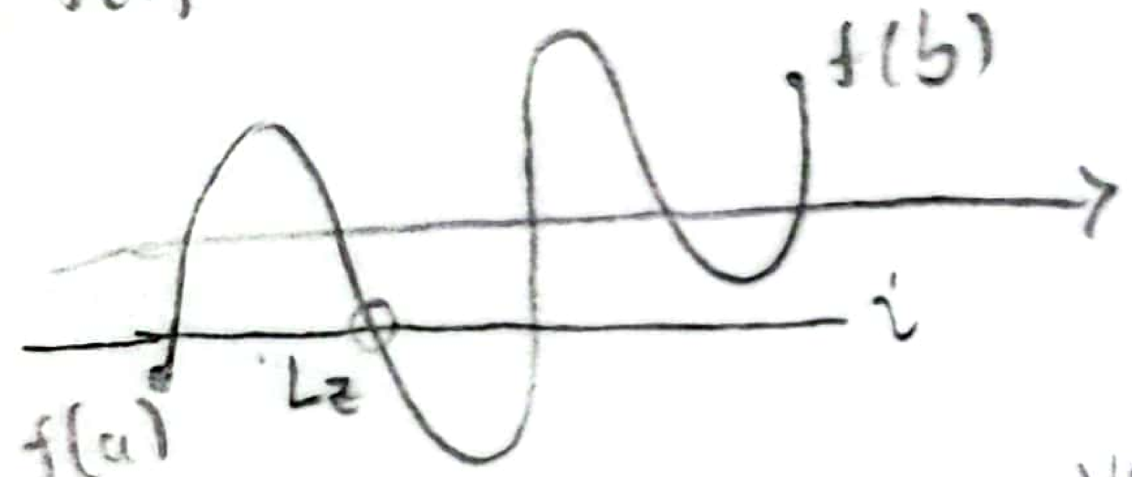
$S := \{f(a) < y < f(b) \mid y \notin f[a,b]\}$, $i := \inf S (> f(a))$

$i \in S$: For all sufficiently small $\varepsilon > 0$, there exists $m_\varepsilon \in (a,b)$ with $f(m_\varepsilon) = i - \varepsilon$

$\{z_n\}_{n=1}^{\infty} := \{c(\frac{1}{n})\}_{n=1}^{\infty}$ s.t. $\lim_{n\to\infty} z_n$ exists. & $\lim_{n\to\infty} f(z_n) = i$ $\quad L_z \in [a,b]$ as $z_n \in (a,b)$ for all $n \in \mathbb{N}$

(BWT)

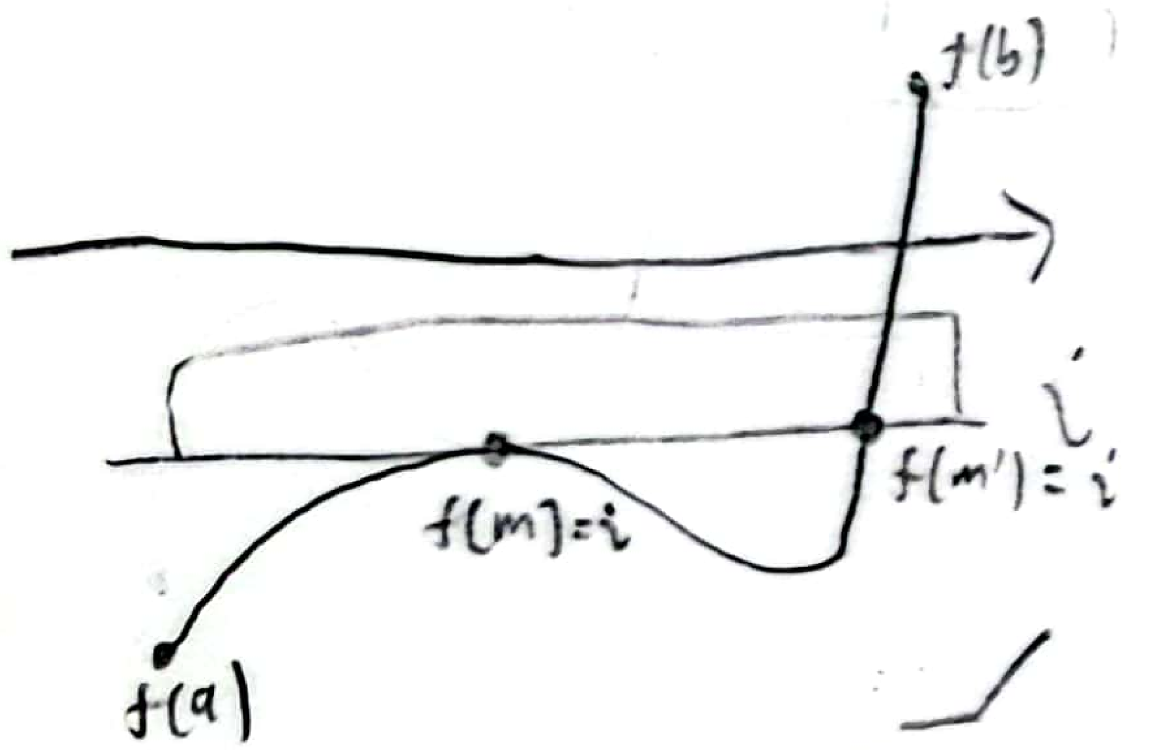
By continuity, $f(L_z) = i$, a contradiction. ☺

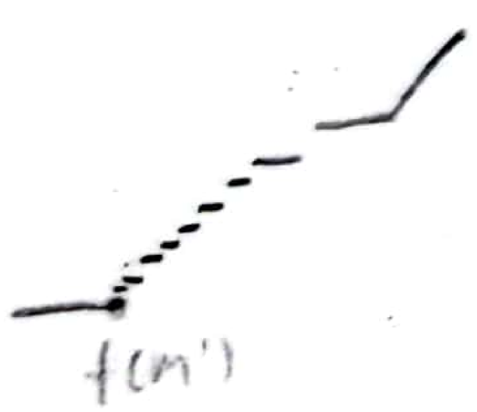


$i \notin S$: Let T be the set of $m \in [a,b]$ with $f(m) \doteq i$ $\quad$ If there is a sequence $\{z_n\}_{n=1}^{\infty} \to i$ in $(m, b]$

$\exists m \exists \delta \forall x$: if $x \in (m, m+\delta)$, then $f(x) > i$? $\quad$ will.

$\forall \varepsilon > 0$
$\exists x \in S$
$x - i < \varepsilon$ $\quad \forall m \forall \delta \exists x$: $x \in (m, m+\delta)$ but $f(x) \leq i$
$f(m) = i$





4-25

Ideas

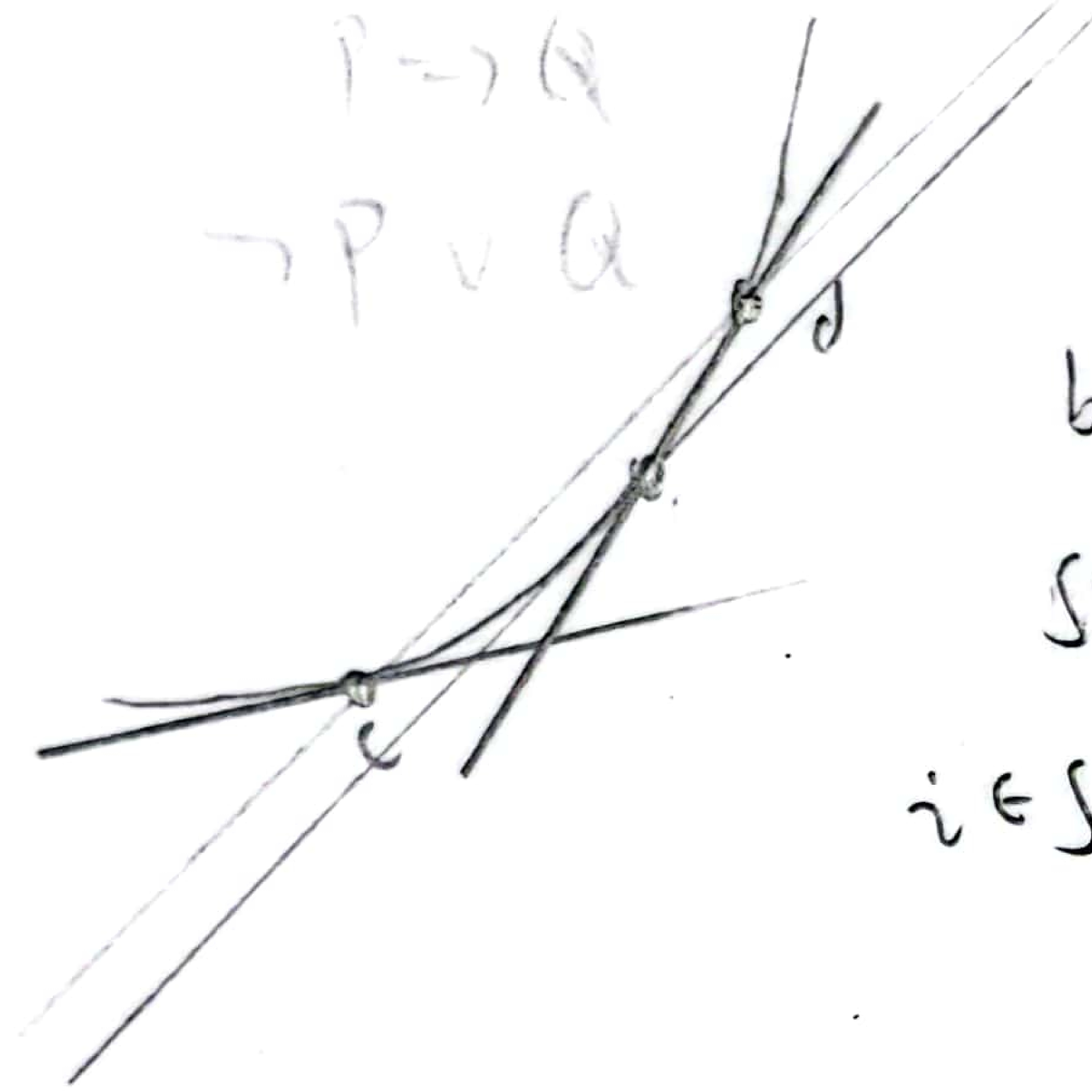
$\forall c \forall d \forall v$ $\quad$ $\forall v \, (\exists c \exists d \, (f(c) < v < f(d)) \Rightarrow \dots)$ $\quad$ $\forall c \forall d \forall v \, (f(c) < v < f(d) \Rightarrow \dots)$

$\forall c \forall d \forall v \, ($

$P \to Q$

$\neg P \lor Q$

When $f'(c) < v < f'(d)$



but for all $m \in (c,d)$, $f(m) \neq v$,

$S := \{ u \notin f[(a,b)] \mid f'(c) < u < f'(d) \}$, $i := \inf S$

$i \in S$: $\forall \varepsilon > 0$ small enough $\quad f(m_\varepsilon) = i - \varepsilon$ for some $m_\varepsilon \in (c,d)$

Self-Exercise 3.1(b)

Suppose $f:(a,b)\to\mathbb{R}$ is differentiable, but such that f' is discontinuous at some $x\in(a,b)$. What discontinuity can f be?

Ideas

If removable, $\lim_{z\to x} f'(z) = f'(x)$, a contradiction.

jump : $\lim_{z\to x^-} f'(z) \neq \lim_{z\to x^+} f'(z)$

Self-Exercise 3.1(a)

Let $f:(a,b) \to \mathbb{R}$ be differentiable. Give a counterexample to show that f' need not be continuous.

Ideas

We need $\lim_{z \to x} f'(z)$ to not exist but $f'(x)$ to exist

$f(x) := \begin{cases} \ldots & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} z \sin\left(\frac{1}{z}\right) = 0$ by IST since $-|z| \leq z \sin\left(\frac{1}{z}\right) \leq |z|$

Let $\varepsilon > 0$.

$\sin(x)$ is continuous in all $x \in \mathbb{R}$, so is $\frac{1}{x}$ for $x \neq 0$

$\Rightarrow$ Theorem 3.30: $\sin\left(\frac{1}{x}\right)$ is cont. for all $x \neq 0$

... then, $f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} z \sin\left(\frac{1}{z}\right) = 0$ ... The squeeze

Theorem since $-|z| \leq z \sin\left(\frac{1}{z}\right) \leq |z|$ for every $z \in \mathbb{R} \setminus \{0\}$. Furthermore, as $\sin(x)$ is continuous in all $x \in \mathbb{R}$ and $\frac{1}{x}$ for every $x \neq 0$, $\sin\left(\frac{1}{x}\right)$ is continuous as long as $x \neq 0$, which is ensured by Theorem 3.30. In fact, by yet-to-be-proven derivative rules,

4-24 Ideas

$\forall \varepsilon \exists \delta \forall y \in (a,b)\setminus\{x\}$ $|y-x|<\delta$

$$\lim_{y\to x}\lim_{z\to y}\frac{f(z)-f(y)}{z-y} \stackrel{?}{=} \lim_{z\to x}\lim_{y\to x} \dots \left(\lim_{y\to x}\right)\left|\lim_{z\to y}\frac{f(z)-f(y)}{z-y}-L\right|<\varepsilon \quad \text{vs} \quad \left(\lim_{z\to x}\right)\underbrace{\left|\lim_{y\to x}\frac{f(z)-f(y)}{z-y}-L\right|}_{\text{(cont)}}<\varepsilon$$

$$\left|\frac{f(z)-f(x)}{z-x}-L\right|$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{f(z_n)-f(y_m)}{z_n-y_m} \stackrel{?}{=} \lim_{n\to\infty}\lim_{m\to\infty}$$

y_m

$$= \lim_{n\to\infty}\frac{f(z_n)-f(x)}{z_n-x}$$

$|y-x|<\delta$

$$\left|\frac{f(z)-f(x)}{z-x}-L\right|<\varepsilon$$

$$\left|\lim_{z\to y}\frac{f(z)-f(y)}{z-y}\right|<\varepsilon+|L|$$

$\forall \varepsilon \exists \delta \forall z\, |z-y|<\delta_1$, $\exists\delta \forall y\, |y-x|<\delta_2$

$|z-x-(y-x)|=|z-y|$

$||z-x|-|y-x|| \le$

$\le |z-x| < \delta_1+|y-x|$

$$\text{Show } \left|\frac{f(z)-f(x)}{z-x}-L\right| < \left|\lim_{z\to y}\frac{f(z)-f(y)}{z-y}-L\right|<\varepsilon$$

$\overset{\delta?}{z_1 \quad x} \quad y \quad z_2$

$|y-x|<\delta$ $\quad \delta_1$

$\forall \varepsilon \exists \delta \forall y \forall \varepsilon \exists \delta \forall z$ $|z-y|<\delta$ $(\Rightarrow |z-x|<2\delta)$

$\forall \varepsilon \exists \delta \forall y$

$$\left|\frac{f(z)-f(y)}{z-y}-L_y\right|<\varepsilon, \quad |L_y-L|<\varepsilon, \quad |y-x|<\varepsilon$$

$|f(y)-f(x)|<\varepsilon$

$f(y)-f(x)<f(z)-f(y)$

$-f(z)<f(x)$

$$\left|\frac{f(z)-f(y)}{z-y}-L\right| = \left|\frac{f(z)-f(y)}{z-y}-L_y+(L_y-L)\right| \le \left|\frac{f(z)-f(y)}{z-y}-L_y\right|+|L_y-L|<\varepsilon$$

$|f(z)-f(y)-L(z-y)|<\varepsilon|z-y|$ $\quad$ z sufficiently close to y, y sufficiently close to x

$|f(z)-f(y)|<\varepsilon|z-y|+|L(z-y)|$

$\delta'' > |z-x| = |z-y-(x-y)|$ $\quad$ Find δ', δ'' so $|y-x|<\delta''$ & $|z-x|<\delta'$

$\delta \ge |z-y|-|y-x|$ $\quad \Rightarrow |y-x|<\delta_1 \quad |z-y|<\delta_2$

OR: $\forall \varepsilon \exists \delta \forall z \in (a,b)\setminus\{x\}$

$\delta''+|y-x| \ge |z-y|$

$$\left|\frac{f(z)-f(x)}{z-x}-L+\frac{f(z)-f(y)}{z-y}-\frac{f(z)-f(x)}{z-x}\right| \ge \left|\frac{f(z)-f(x)}{z-x}-L\right|-\left|\frac{f(z)-f(y)}{z-y}\right|$$

$\delta'+\delta'' \ge |z-y|$

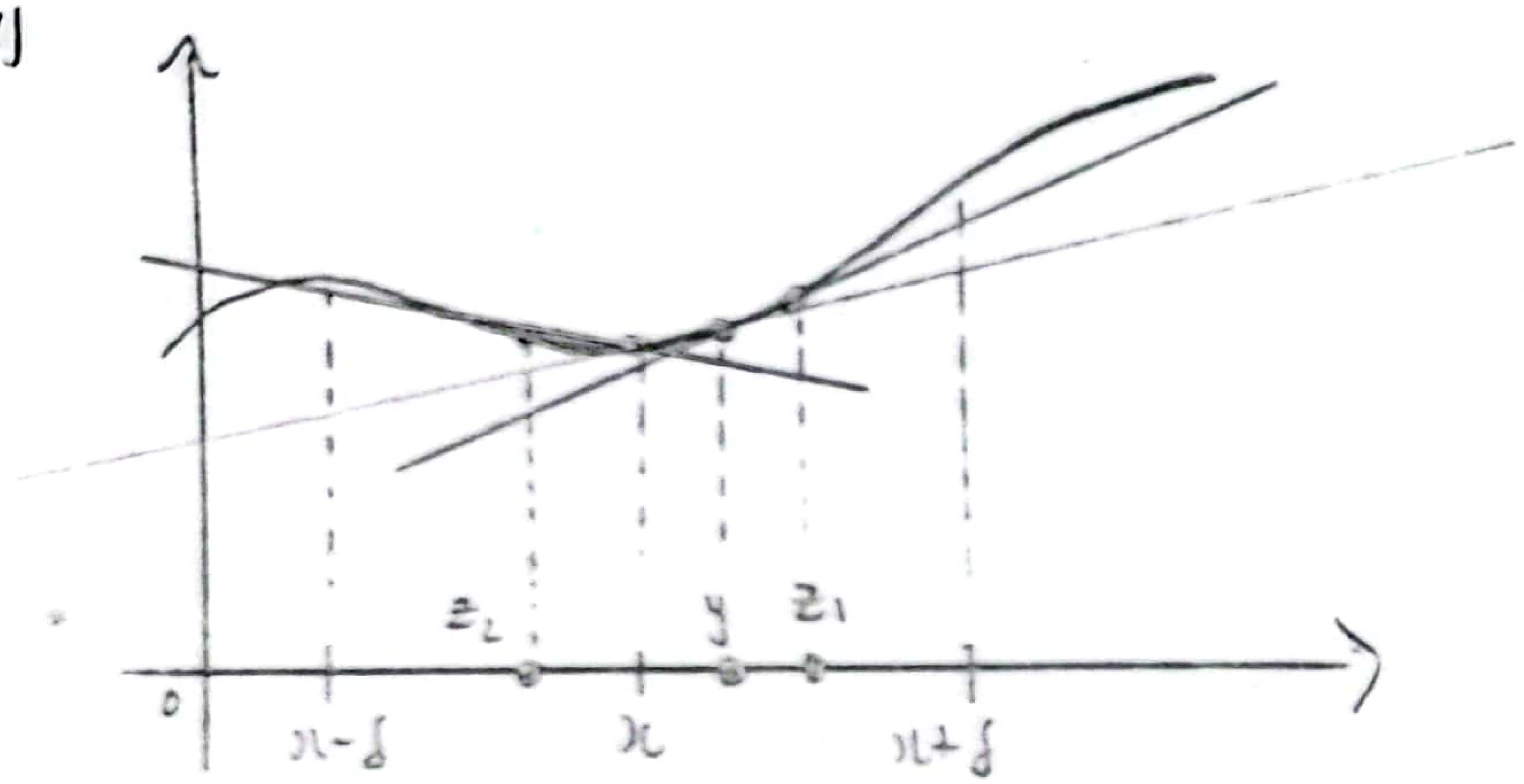
δ_2

$$\left|\frac{f(z)-f(x)}{z-x}-\frac{f(z)-f(y)}{z-y}+\frac{f(z)-f(y)}{z-y}-L\right| \le \left|\frac{f(z)-f(x)}{z-x}-\frac{f(z)-f(y)}{z-y}\right|+\left|\frac{f(z)-f(y)}{z-y}-L\right|$$

$<\varepsilon$

4-24 16:21 16:40

Ideas



$\lim_{z\to x} f'(z)$ $\quad \lim_{z\to x} f'(c_z)$ vs $\lim_{z\to x} f'(z)$?

$\lim_{z\to x}\lim_{y\to x} f'(c_{zy}) = \lim_{z\to x}\lim_{y\to x}\frac{f(z)-f(y)}{y-z}$

$\lim_{z\to x}\lim_{y\to x}\lim_{k\to y}$

$$\lim_{z\to x}\left(\lim_{y\to x}\frac{f(z)-f(y)}{z-y}\right) = \lim_{z\to x} f'(\lim_{y\to x} y)$$

$x-\delta < z < x+\delta$

$|c_z - z| < \delta_z$?

By MVT, for each $z \in (a,b)\setminus\{x\}$, $\exists c_z \in (a,b)\setminus\{x\}$ so $f'(c_z) = \frac{f(z)-f(y)}{z-y}$ and $y \in (a,b)$

$z < c_z < y$ or $y < c_z < z$ $\Rightarrow$ $\lim_{z\to x} c_z = x$ by The Squeeze Theorem $\forall z\in(a,b)\setminus\{x\}$ Fixing $y := x$

$\lim_{z\to x} f'(z)$ exists $\Rightarrow$ $\cancel{\lim_{z\to x}\lim_{y\to z}\frac{f(y)-f(z)}{y-z}}$

Consider $\{z_n\}_{n=1}^{\infty}$ in $(a,b)\setminus\{x\}$ & which converges to x. Let $\{c_n\}_{n=1}^{\infty}$ be the corresponding ... given by AC.

$$\lim_{z\to x} f'(c_z) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

Let $\varepsilon>0$. $\exists\delta>0$ $\forall y\neq z$ $|y-z|<\delta \Rightarrow \left|\frac{f(y)-f(z)}{y-z} - L\right| < \varepsilon$

$y\to z\to x$

$\lim_{z\to x}\lim_{y\to c_z}\frac{f(y)-f(c_z)}{y-c_z}$

$\lim_{z\to x} f'(z) = \lim_{z\to x}\lim_{y\to z}$

Proof

By the Mean Value Theorem, we have that for each $z \in (a,b)\setminus\{x\}$, there exists $c_z \in (a,b)\setminus\{x\}$ so $f'(c_z) = \frac{f(z)-f(x)}{z-x}$.

More specifically, $z < c_z < x$ if $z < x$ and $x < c_z < z$ if $x < z$. ~~Hence, the Squeeze Theorem says $\lim_{z\to x} c_z = x$.~~

Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $z \in (a,b)\setminus\{x\}$ and $|z-x|<\delta$, then $|f'(z) - \lim_{z\to x} f'(z)|$. By the above, $|c_z - x| < \delta$!

Therefore, $|f'(c_z) - \lim_{z\to x} f'(z)|$ holds. Consequently, we now have that $\lim_{z\to x} f'(z) = \lim_{z\to x} f'(c_z) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} := f'(x)$.

$\square$

4-24 Ideas $|y-x|<\delta_2$ $|z-x|<\delta_1 \Leftarrow \forall \varepsilon \exists \delta_1 \forall z \exists \delta_2 \forall y$

$\left|\frac{f(z)-f(y)}{z-y} - L_z\right| < \varepsilon$, $|L_z - L| < \varepsilon$

Checking (NOT Actual Proof)

By the existance of $\lim_{z\to x} f'(x)$, we have that for all $\varepsilon>0$, there exists $\delta_1>0$ so whenever $y \in (a,b)\setminus\{x\}$ has the property that $|y-x|<\delta_1$, $\left|\lim_{z\to y}\frac{f(z)-f(y)}{z-y} - L\right| < \frac{1}{4}\varepsilon$. And there exists δ_2 such that for each $z\in(a,b)\setminus\{y\}$ with $|z-y|<\delta_2$, $\left|\frac{f(z)-f(y)}{z-y} - L_y\right| < \frac{1}{4}\varepsilon$ where $L_y := \lim_{z\to y}\frac{f(z)-f(y)}{z-y}$.

Furthermore, for all $y \neq z$, there exists $\delta_3>0$ such that if $y\in(a,b)\setminus\{x\}$ and $|y-x|<\delta_3$, then $\left|\frac{f(z)-f(x)}{z-x} - L\right| < \frac{1}{2}\varepsilon$.

~~Let $\varepsilon>0$. There exists $\delta' = \frac{1}{2}\delta_1$ so $\left|\lim_{y\to x}\frac{f(z)-f(y)}{z-y} - L\right| < \varepsilon$, because for there exists $\delta'' := \frac{1}{2}\delta_1$ with $|y-x|<\delta''$~~

~~so $\left|\frac{f(z)-f(y)}{z-y} - L_z\right| < \varepsilon$ (where $L_z := \lim_{y\to x} \ldots$)~~

Let $\varepsilon>0$ and $\delta := \min\{\delta_1, \frac{\delta_2}{2}, \delta_3\}$. If $z\in(a,b)\setminus\{x\}$ and $y\in(a,b)\setminus\{x\}$ are so that $|z-x|<\delta$ and $|y-x|<\delta$, then $\delta > |z-x| = |z-y-(x-y)| \geq |z-y| - |y-x|$. Thus, $\delta_2 = \frac{\delta_2}{2} + \frac{\delta_2}{2} > \delta + |y-x| > |z-y|$. Therefore,

we know that $\left|\lim_{z\to y}\frac{f(z)-f(y)}{z-y} - L\right| < \frac{1}{4}\varepsilon$ and $\left|\frac{f(z)-f(y)}{z-y} - L_y\right| < \frac{1}{4}\varepsilon$. Which means

$$\left|\frac{f(z)-f(y)}{z-y} - L\right| = \left|\frac{f(z)-f(y)}{z-y} - L_y + (L_y - L)\right| \leq \left|\frac{f(z)-f(y)}{z-y} - L_y\right| + |L_y - L| < \frac{1}{2}\varepsilon.$$

As such,

$$\left|\frac{f(z)-f(x)}{z-x} - L\right| \leq \left|\frac{f(z)-f(x)}{z-x} - \frac{f(z)-f(y)}{z-y}\right| + \left|\frac{f(z)-f(y)}{z-y} - L\right| < \varepsilon.$$

2-50 (c) $\{a_n\}_{n=1}^{\infty}$ is either bounded or unbounded, so the result follows from (b) and (a) respectively; since it is divergent. The converse is straight forward from Proposition 2.40 (and noticing subsequences converging to $\pm\infty$ must diverge).

(d) Assume $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. So, $\{a_n\}_{n=1}^{\infty}$ cannot grow beyond / below all bounds. By (c), this leaves us with the sole possibility that there exists a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ (in the case where $\{a_n\}_{n=1}^{\infty}$ is divergent. If it is convergent, it itself immediately satisfies the Theorem).

$\square$

2-51 Ideas

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N$

$$\left| c - \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \right| < \varepsilon$$

$\forall M \in \mathbb{R} \ \exists N \in \mathbb{N} \ \forall n \geq N$

$b_n \geq M$.

(if $|c - \cdots| \geq 0$) $c - \varepsilon < \dfrac{a_n - a_{n-1}}{b_n - b_{n-1}}$ where $b_n - b_{n-1} > 0$

$(c-\varepsilon)(b_n - b_{n-1}) < a_n - a_{n-1}$

$\dfrac{b_n - b_{n-1}}{a_n - a_{n-1}}$ also converges by limit laws

$\dfrac{1}{b_n} \cdot \dfrac{b_n - b_{n-1}}{a_n - a_{n-1}}$ $\qquad \dfrac{a_n - a_{n-1}}{b_n(b_n - b_{n-1})}$

$\{a_n\}_{n=1}^{\infty}$ bounded $\Rightarrow c = 0$ statement true

$\{a_n\}_{n=1}^{\infty}$ unbounded (& div) $\rightarrow$ exists subsequences to ∞ and/or $-\infty$

If $c \neq 0$, cannot $\to \infty$ and $\to -\infty$.

If $\to \infty$, a_n should be "eventually inc" 'eventually', $a_n > a_{n-1}$

$b_n > b_n - b_{n-1}$ by positivity of terms

$\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} - c < \varepsilon$

$a_n - a_{n-1} < (c+\varepsilon)(b_n - b_{n-1}) < (c+\varepsilon) b_n$

Self - Proof of Theorem 3.2

Assume that f converges at x to some limit L and let $\{z_n\}_{n=1}^{\infty}$ be a sequence mapping to $I\setminus\{x\}$ so $\lim_{n\to\infty} z_n = x$. For $\varepsilon>0$, suppose wlog that $\varepsilon<\delta$. Thus, there exists $N\in\mathbb{N}$ with $|f(z_n)-L| = |g(z_n)-L| < \varepsilon$. Therefore, g converges at x to the same limit, L. The converse follows similarly. $\square$

Example 3.5

1. $|f(z_n)-x| = |z_n - x| < \varepsilon$.
2. For $\{z_n\}_{n=1}^{\infty}$ converging to x (w/o containing it), (for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ so if $n\geq N$,) $\big||z_n|-|x|\big| \leq |z_n - x| < \varepsilon$. Thence, $\lim_{z\to x}|z| = |x|$ as expected.
3. Let $\{z_n\}_{n=1}^{\infty}$ be defined by $z_{2m} = -\frac{1}{2m}$ and $z_{2m+1} = \frac{1}{2m}$, which must clearly converge to 0 and is always nonzero. For any $L\in\mathbb{R}$, select $\varepsilon = \frac{1}{2}$, so given $N\in\mathbb{N}$, even if $|f(z_{2N})-L| < \frac{1}{2}$ — meaning $L\in(-\frac{3}{2},-\frac{1}{2})$ — $f(z_{2N+1}) = 1$ thus $|f(z_{2N+1})-L| > \frac{3}{2} > \frac{1}{2}$.

Self - Proof of Theorem 3.7

Suppose $L\in\mathbb{R}$ is the limit of f at x and notice[1] the sequence $\{x+\frac{1}{n}\}_{n=1}^{\infty}$ converges to x. Now, letting $\varepsilon>0$, there exists an $n\in\mathbb{N}$ so the open interval $J_n := (x-\frac{1}{n}, x+\frac{1}{n})$ for which any $z\in J_n\setminus\{x\}$ is such that $|f(z)-L|<\varepsilon$; lest for each $n\in\mathbb{N}$ there is $y\in J_n\setminus\{x\}$ with $|f(y)-L|\geq\varepsilon$: But now we can define the sequence $\{y_n\}_{n=1}^{\infty}$ by $y_1 := C(J_1')$ and $y_{n+1} = C(J_M')$, where M is the least $m\in\mathbb{N}$ with $|y_n - x| \geq \frac{1}{m}$ so $y_n \notin J_M\setminus\{x\}$ (C the choice function on I, J_n' the set of $y\in J_n\setminus\{x\}$ with $|f(y)-L|\geq\varepsilon$). By Squeeze Theorem for sequences, $\lim_{n\to\infty} y_n = x$ even though for each $N\in\mathbb{N}$, $|f(y_N)-L|\geq\varepsilon$, a contradiction. As such, the existance of the required $n\in\mathbb{N}$ is certain, so by construction, for all $z\in I\setminus\{x\}$ with $|z-x|<\delta := \frac{1}{n}$, we have that $|f(z)-L|<\varepsilon$.

Conversely, instead presume that for all $\varepsilon>0$ there is a $\delta>0$ such that for all $z\in I\setminus\{x\}$ with $|z-x|<\delta$ we have that $|f(z)-L|<\varepsilon$. Thus, consider any $\{z_n\}_{n=1}^{\infty}$ mapping to $I\setminus\{x\}$ and which converges to x. As such, there must exist $N\in\mathbb{N}$ such that when $n\geq N$, $|z_n - x|<\delta$. Thus, by our above presumption, $|f(z_n)-L|<\varepsilon$. Consequently, $\lim_{n\to\infty} f(z_n) = L$, i.e. $\lim_{z\to x} f(z) = L$.

Wherefore the biconditional holds. $\square$

[1] If we really wanted to be more formal, we could use $\left\{x + \frac{\min\{x-a,\, b-x\}}{n}\right\}_{n=1}^{\infty}$ so the nth term of the sequence is in I and $J_n := \left(x - \frac{\min\{x-a,\, b-x\}}{n},\ x + \frac{\min\{x-a,\, b-x\}}{n}\right) \subseteq I$ with absolute certainty, but yeah this is kinda ugly so no.

Exercises

3-1 Because to assert otherwise would mean that for all sequences as specified in Definition 3·1, they each converge to L and L', even though $L \neq L'$. However, that would contradict the uniqueness of limits as shown in Proposition 2·4. (the only exception being $\operatorname{dom} f = \emptyset$).

3-2 See Self-Proof of Theorem 3·2

3-3 Suppose that $\{z_n\}_{n=1}^{\infty}$ converges to x. Let $\varepsilon > 0$, so there exists $N \in \mathbb{N}$ with $|z_n - x| < \frac{\varepsilon}{|m|}$ for each $n \geq N$. Then, $|(mz_n + b) - (mx + b)| = |m||z_n - x| < \varepsilon$, given $n \geq N$. Therefore, $\lim_{z \to x} mz + b = mx + b$ as expected. $\square$

$\square$

OR

Let $\varepsilon > 0$ and $\delta := \frac{\varepsilon}{|m|}$. So, for every $z \in I \setminus \{x\}$ with $|z - x| < \delta$, we have that $|(mz+b) - (mx+b)| = |m||z - x| < \varepsilon$. By Theorem 3·7, $\lim_{z \to x} mz+b = mx+b$. $\square$

3-4 Ideas

~~$\exists \varepsilon', \exists \{z_n\}_{n=1}^{\infty}, \forall N, \exists n \geq N, \quad |f(z_n) - L| \geq \varepsilon'$~~

~~$n_0 < n_1 < n_2 < \cdots$~~

$\exists \varepsilon \; \forall \delta \; \exists z \in (x - \delta, x + \delta) \;\&\; |f(z) - L| \geq \varepsilon$

$\exists z \; \delta \; A \; \exists \varepsilon$

$\frac{1}{n}$ $\qquad |z_n - x| < \frac{1}{n}$

$z_n = C(\frac{1}{n})$

Proof

By Theorem 3·7, there exists $\varepsilon > 0$ so for all $\delta > 0$, there exists $z \in I \setminus \{x\}$ with $|z - x| < \delta$ and $|f(z) - L| \geq \varepsilon$. By AC, there exists a function C of domain $\mathbb{R}$ with $C(\delta)$ being any of the x's specified above. Define the sequence $\{z_n\}_{n=1}^{\infty}$ by $z_n := C(\frac{1}{n})$, which must converge to x by Squeeze's Theorem, since $|z_n - x| < \frac{1}{n}$ for each $n \in \mathbb{N}$. But, as desired, $|f(z_n) - L| \geq \varepsilon$ now, for every $n \in \mathbb{N}$. $\square$

3-6 Ideas

$\frac{z-3}{z^2-9} = \frac{z-3}{(z-3)(z+3)} = \frac{1}{z+3}$ $\qquad (2, 4)$ $\qquad \lim_{n \to \infty} \frac{1}{z_n+3} = \frac{1}{(\lim_{n \to \infty} z_n) + 3} = \frac{1}{6}$

Proof

Let $\{z_n\}_{n=1}^{\infty}$ be any sequence converging to 3, with $z_n \in (2, 4) \setminus \{3\}$ for each $n \in \mathbb{N}$. Then, by limit laws it is clear that $\left\{\frac{z_n - 3}{z_n^2 - 9}\right\}_{n=1}^{\infty} = \left\{\frac{1}{z_n + 3}\right\}_{n=1}^{\infty}$ converges to $\frac{1}{6}$. In other words, $\lim_{z \to 3} \frac{z_n - 3}{z_n^2 - 9} = \frac{1}{6}$. $\square$

3-5 We notice that $\lfloor m + \frac{1}{n+1} \rfloor = m$ while $\lfloor m - \frac{1}{n+1} \rfloor = m - 1$ for each $n \in \mathbb{N}$, so $\lim_{n \to \infty} \lfloor m + \frac{1}{n+1} \rfloor = m$ but $\lim_{n \to \infty} \lfloor m - \frac{1}{n+1} \rfloor = m - 1$, even though $\lim_{n \to \infty} m + \frac{1}{n+1} = \lim_{n \to \infty} m - \frac{1}{n+1} = m$. $\square$

$x - \frac{\lfloor 10^n x \rfloor}{10^n} < 10^{-n}$ $\qquad 10^N > N > \frac{1}{\varepsilon}$ $\qquad \forall L\ \exists \varepsilon\ \forall \delta\ \exists z \quad |z-x|<\delta\ \&\ |f(z)-L| \geq \varepsilon$ $\qquad 0 < \delta - a \qquad \delta - a > \frac{\sqrt{2}}{m}$

$\varepsilon := \min(|L|, |L-1|) \qquad \frac{2z+\delta}{2} = z + \frac{\delta}{2} \qquad \varepsilon \begin{cases} -1 \\ -L \\ -0 \end{cases} \qquad \delta > \frac{\sqrt{2}}{m} + a$

$10^n x - \lfloor 10^n x \rfloor < 1$

Proof

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ with $\varepsilon > 10^{-N}$. For each $n \geq N$, since $10^n x - \lfloor 10^n x \rfloor < 1$, $|x - 10^{-n}\lfloor 10^n x \rfloor| < 10^{-n} \leq 10^{-N} < \varepsilon$. So, $\lim_{n\to\infty} 10^{-n}\lfloor 10^n x \rfloor = x$ and $\lim_{n\to\infty} 10^{-n}\lfloor 10^n x \rfloor + 10^{-n}\sqrt{2} = x$ by limit laws. Notice that $f(10^{-n}\lfloor 10^n x \rfloor) = 1$ and $f(10^{-n}\lfloor 10^n x \rfloor + 10^{-n}\sqrt{2}) = 0$ for all $n \in \mathbb{N}$. Consequently, $\lim_{n\to\infty} f(10^{-n}\lfloor 10^n x \rfloor) = 1$ while $\lim_{n\to\infty} f(10^{-n}\lfloor 10^n x \rfloor + 10^{-n}\sqrt{2}) = 0$. Wherefore, the Dirichlet function f does not converge at any $x \in \mathbb{R}$. $\square$

OR

Let $L \in \mathbb{R}$ and $\varepsilon := \frac{1}{2}$. For every $\delta > 0$, there exists $n \in \mathbb{N}$ with $\delta > 10^{-n}$. Again, as above, $|10^{-n}\lfloor 10^n x \rfloor - x| < \delta$. For some $m \in \mathbb{N}$, we also have $\delta - |10^{-n}\lfloor 10^n x \rfloor - x| > \frac{\sqrt{2}}{m}$, so $|10^{-n}\lfloor 10^n x \rfloor - \frac{\sqrt{2}}{m} - x| < \delta$. Therefore, if $|f(10^{-n}\lfloor 10^n x \rfloor) - L| = |L-1| < \varepsilon$ then $|f(10^{-n}\lfloor 10^n x \rfloor - \frac{\sqrt{2}}{m}) - L| = |L| \geq \varepsilon$, lest $1 > |L-1| + |L| \geq |L-1+L| = 1$. As such, we can again conclude the same thing. $\square$

3-8. There exists no open interval containing 0 that is simultaneously a subset of the domain of the square root function, namely, $\mathbb{R}^+_0$. To circumvent this, we could ~~either~~ introduce left/right handed limits (e.g. by using a half-open interval and stimpulating that $\{z_n\}_{n=1}^{\infty}$ must be such that z_n is always less than / greater than x) (or maybe create some funny definitions).

Self Proof of Theorem 3.10 Limit laws

Ideas

$z_n \to x$

$$(f(z_n)-L)(g(z_n)-M) = fg - Lg(z_n) - Mf(z_n) + LM$$

$$|f(z_n)\cdot g(z_n) - LM| \le |f(z_n)-L||g(z_n)-M| + |Mf(z_n)+Lg(z_n)|$$

$|M||f(z_n$

Proof

Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence with $z_n \in I\setminus\{x\}$ for all $n \in \mathbb{N}$, and which converges to x. Let $L := \lim_{z\to x} f(z)$ and $M := \lim_{z\to x} g(z)$.

1. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ for which if $n \ge N$, $|f(z_n)-L| < \frac{1}{2}\varepsilon$ and $|g(z_n)-M| < \frac{1}{2}\varepsilon$. Thus, $|(f+g)(z_n)-(L+M)| \le |f(z_n)-L| + |g(z_n)-M| < \varepsilon$

2. Clear from 1. and 3.

3.

Proof

A corollary of the Limit Laws for sequences.

□

Self-Proof of Theorem 3.12

$\forall \varepsilon \, \exists \delta \, \forall z$

Ideas

$|f(z_n)-L_f| < \frac{1}{2}\varepsilon \quad |g(z_n)-L_g| < \frac{1}{2}\varepsilon$

$|f(z_n)-L_f| + |g(z_n)-L_g| < \varepsilon$

$|f(z_n)+g(z_n)-(L_f+L_g)| < \varepsilon$

$L_f \le L_g$

$L_f - L_g \le 0$

$$L_f - L_g \le |L_f - f(z)| + |g(z)-L_g| + f(z)-g(z)$$
$$< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon + 0$$
$$= \varepsilon$$

Proof / Ex 3-11

Follows from Theorem 2.20.

OR

Let $\varepsilon > 0$. There exists $\delta > 0$ so for all $z \ne x$ with $|z-x| < \delta$, $|f(z)-L_f| < \varepsilon$ and $|g(z)-L_g|$. Thus, $L_f - L_g \le |L_f - f(z)| + |g(z)-L_g| + f(z)-g(z) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon + 0 = \varepsilon$. By exercise 25, $L_f \le L_g$.

□

Self-Proof of Theorem 3.13 The Squeeze Theorem / Ex 3-12

Idea:

$\forall \varepsilon \exists \delta \forall z \ |z-x|<\delta \Rightarrow |f(z)-L|<\varepsilon$ & $|h(z)-L|<\varepsilon$

$|g(z)-L|$ $\quad g(z) \leq L$

Proof

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence converging to x, with $z_n \in I \setminus \{x\}$ for all $n \in \mathbb{N}$. Then, since $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} h(z_n)$, thus by Theorem 2.21 (the Squeeze's Theorem for Sequences), we have that $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} h(z_n)$. $\square$

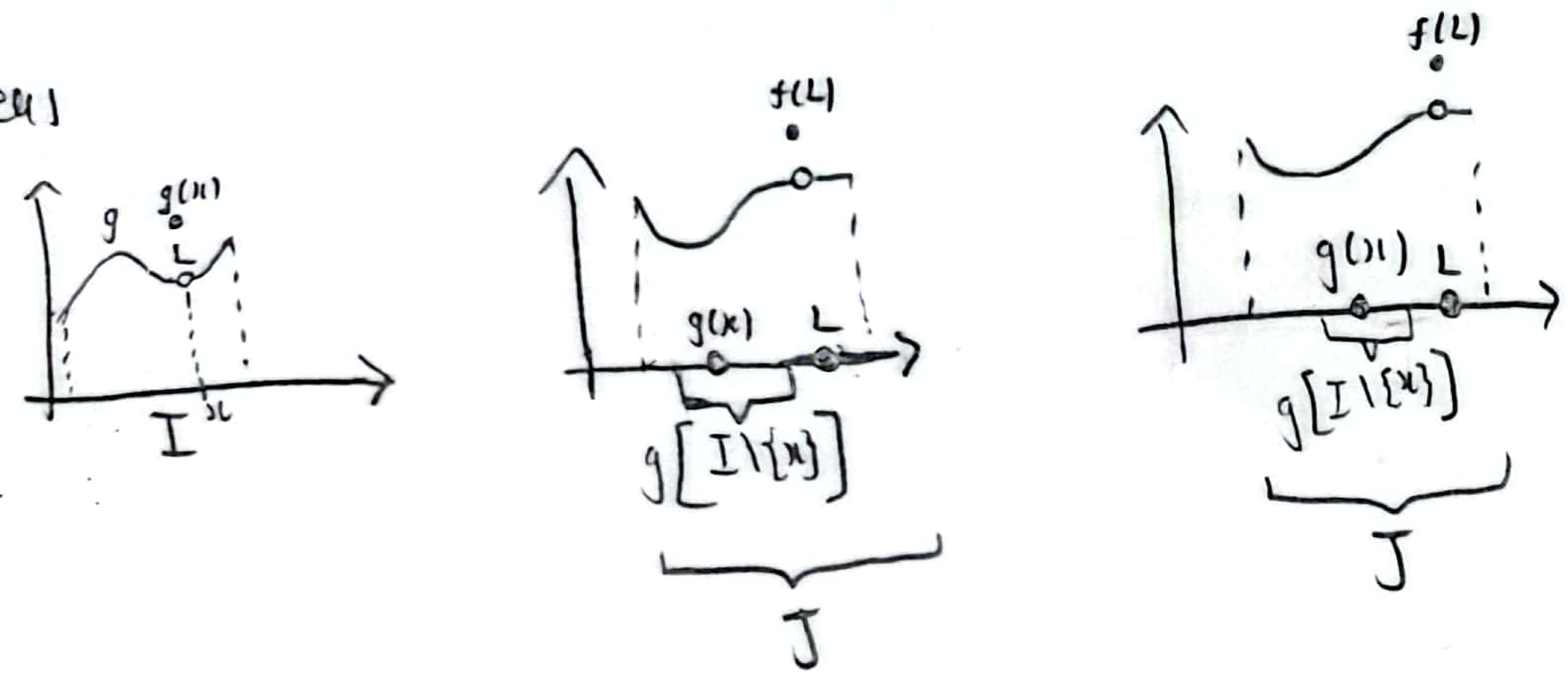
OR

Let $L := \lim_{z\to x} f(z) = \lim_{z\to x} h(z)$. For all $\varepsilon>0$, there exists $\delta>0$ so for each $z \in I\setminus\{x\}$ with $|z-x|<\delta$, both $|f(z)-L|<\varepsilon$ & $|h(z)-L|<\varepsilon$.

If $g(z) \leq L$, $|g(z)-L| \leq |f(z)-L| < \varepsilon$. Similarly, when $g(z) > L$, $|g(z)-L| \leq |h(z)-L| < \varepsilon$. Either ways, $|g(z)-L|<\varepsilon$ is guaranteed, such that $\lim_{z\to x} g(z) = L$ is certain. $\square$

Self-Proof of Theorem 3.14

Ideas



$|z-x|<\delta \Rightarrow |g(z)-L|<\varepsilon$

$|y-L|<\delta \Rightarrow |f(y)-M|<\varepsilon$ $\qquad |f(y)-f(L)|<\varepsilon$

Show $\forall \varepsilon\, \exists \delta\, \forall z \in I\setminus\{x\}$ $|z-x|<\delta \Rightarrow |f(g(z))-M|<\varepsilon$

for δ, $\exists \delta'$ $\forall z$ w/ $|z-x|<\delta' \Rightarrow |g(z)-L|<\delta \Rightarrow |f(g(z))-M|<\varepsilon$

$\exists N\, \forall n \geq N$ $|f(g(z_n))-M|<\varepsilon$ $\qquad$ $\exists N\, \forall n\geq N$ $|f(y_n)-M|<\varepsilon$ $\Rightarrow g(z)$ is a possible y

for all seq $y_n \to L$ e.g. $g(z_n)$ $\qquad |f(y_n)-f(L)|<\varepsilon$

Proof

Let $\varepsilon>0$. There exists $\delta>0$ so for all $y\in J\setminus\{L\}$ with $|y-L|<\delta$, $|f(y)-M|<\varepsilon$; for $M:=\lim_{y\to L} f(y)$ ~~which~~ exists by assumption. In turn, the existence of $L:=\lim_{z\to x} g(z)$ tells us there is some $\delta'>0$ for which given $z\in I\setminus\{x\}$ is such that $|z-x|<\delta'$, $|g(z)-L|<\delta$. Thus, if $g(z)\neq L$ $|f(g(z))-M|<\varepsilon$ follows. In the case that $g(z)=L$, $|f(g(z))-M|=|f(L)-f(L)|=0<\varepsilon$ still. Either ways, so long as $z\in I\setminus\{x\}$ satisfies $|z-x|<\delta'$, $|f(g(z))-M|<\varepsilon$. $\square$

Errata:
- $\lim_{z\to x} f\circ g(z)$ instead of $\lim_{z\to x} f\circ g(x)$
- L instead of $g(x)$, with the exception of the last sentence.

OR $M:=\lim_{y\to L} f(y)$

Let $\varepsilon>0$ and $\{z_n\}_{n=1}^{\infty}$ be a sequence converging to x, with $z_n\in I\setminus\{x\}$. If $\min\{m>k \mid g(z_m)\neq L\}$ does not exist for some $k\in\mathbb{N}$, then for each $n>k$, $|f(g(z_n))-M|=|f(L)-f(L)|=0<\varepsilon$. Even if it exists for all $k\in\mathbb{N}$, we can define the subsequence $\{\bar{z}_n\}_{n=1}^{\infty}$ by $\bar{z}_1:=$ and $\bar{z}_{n+1}:=z_{\min\{m>k \mid g(z_m)\neq L\}}$, where $\bar{z}_n=z_k$. So, since $M:=\lim_{y\to L} f(y)$ exists, there is some $N\in\mathbb{N}$ for which given $n\geq N$, $|f(g(\bar{z}_n))-M|<\varepsilon$. Accordingly, $|f(g(z_n))-M|<\varepsilon$. In any case, the result holds. $\square$

3-9 (b) Ideas

$\forall \varepsilon \exists \delta \forall z \; |z-x|<\delta \Rightarrow |f(z)-L_f|<\varepsilon$ & $|g(z)-L_g|<\varepsilon$

$\left|\frac{g(z)}{L_g}-1\right| < \frac{\varepsilon}{|L_g|}$

$|f(z)g(z)-L_fL_g| = |f(z)g(z) - L_g f(z) - L_f g(z) + 1 + L_g f(z) - L_f L_g + \cdots|$

$\leq |f(z)-L_f||g(z)-L_g| + |L_g||f(z)-L_f| + |L_f||g(z)-L_g|$

Show $|f(z)g(z)-L_fL_g|<\varepsilon$

$\left|f(z)\frac{g(z)}{L_g}-L_f\right| < \frac{\varepsilon}{|L_g|}$

$\left|\frac{f(z)}{L_f}\frac{g(z)}{L_g}-1\right| < \frac{\varepsilon}{|L_f||L_g|}$

$= \left|\frac{f(z)}{L_f}\frac{g(z)}{L_g} - \frac{f(z)}{L_f} - \frac{g(z)}{L_g} + 1 + \frac{f(z)}{L_f} - 1 + \frac{g(z)}{L_g} - 1\right|$

$\leq \left|\frac{f(z)}{L_f}-1\right|\left|\frac{g(z)}{L_g}-1\right| + \left|\frac{f(z)}{L_f}-1\right| + \left|\frac{g(z)}{L_f}-1\right|$

$< \frac{\varepsilon}{3(\varepsilon+1)}\cdot\frac{\varepsilon}{3(\varepsilon+1)} + \frac{\varepsilon}{3(\varepsilon+1)} + \frac{\varepsilon}{3(\varepsilon+1)}$

$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$

$= \varepsilon$

$\max\{1,\varepsilon\} > \frac{\varepsilon}{\varepsilon+1} > \left(\frac{\varepsilon}{\varepsilon+1}\right)^2$

$(a-1)(b-1) = ab-a-b+1$

$\left|\frac{f(z)}{L_f}-1\right|\left|\frac{g(z)}{L_g}-1\right| = \left|\frac{f(z)}{L_f}\frac{g(z)}{L_g} - 1 + 1 - \frac{f(z)}{L_f} + 1 - \frac{g(z)}{L_g}\right|$

$\leq \left|\frac{f(z)}{L_f}\frac{g(z)}{L_g}-1\right| + \left|\frac{f(z)}{L_f}-1\right| + \left|\frac{g(z)}{L_g}-1\right|$

$\cancel{\frac{ab}{|L_f||L_g|}+a+b=1}$ $\quad \cancel{a + 2|L_f||L_g|a = 1}$

$\cancel{\frac{a}{2|L_f||L_g|}+a = \frac{1}{2}}$ $\quad \cancel{a = \frac{1}{2|L_f||L_g|+1}}$

$ab+a+b=1$

$\frac{3}{2}a+\frac{1}{2}=1$

$a=\frac{1}{3}$

$|f(z)-L_f| < \frac{|L_f|}{3}\varepsilon \qquad |g(z)-L_g| < \frac{|L_g|}{2}\varepsilon$

$\left|\frac{f(z)}{L_f}-1\right| < \frac{1}{3}\varepsilon \quad \frac{\varepsilon}{3(\varepsilon+1)} \qquad \left|\frac{g(z)}{L_g}-1\right| < \frac{1}{2}\varepsilon$

$\cancel{ab+|L_g|a+|L_f|b = |L_f||L_g|}$

$\cancel{(|L_g|+\tfrac{1}{2})a = |L_f|(|L_g|-\tfrac{1}{2})}$

$\cancel{a = |L_f|}$

Proof

For all $\varepsilon>0$, there exists $\delta>0$, so for each $z\in I\setminus\{x\}$ with $|z-x|<\delta$, $|f(z)-L_f| < \frac{\varepsilon}{3(\varepsilon+1)(|L_g|+1)}$ and $|g(z)-L_g| < \frac{\varepsilon}{3(\varepsilon+1)(|L_f|+1)}$ where $L_f := \lim_{z\to x} f(z)$ and $L_g := \lim_{z\to x} g(z)$.

Consequently,

$$|f(z)g(z)-L_fL_g| = |f(z)g(z) - L_g f(z) - L_f g(z) + L_fL_g + L_g f(z) - L_g L_f + L_f g(z) - L_f L_g|$$

$$\leq |f(z)-L_f||g(z)-L_g| + |L_g||f(z)-L_f| + |L_f||g(z)-L_g|$$

$$< \frac{\varepsilon}{3(\varepsilon+1)(|L_g|+1)}\cdot\frac{\varepsilon}{3(\varepsilon+1)(|L_f|+1)} + \frac{|L_g|}{|L_g|+1}\cdot\frac{\varepsilon}{3(\varepsilon+1)} + \frac{|L_f|}{|L_f|+1}\cdot\frac{\varepsilon}{3(\varepsilon+1)}$$

$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$$

$$= \varepsilon.$$

Wherefore, $\lim_{z\to x}(f\cdot g)(z) = \lim_{z\to x} f(z)\cdot\lim_{z\to x} g(z)$. $\square$

the case that $g(z)=L$, $|f(g(z)) \ldots$

-14 (a) Errata states: The exercise would be (read) better as "Prove that $\lim_{z\to x} f(z)$ exists iff $\lim_{h\to 0} f(x+h)$ exists and in this case $\lim_{z\to x} f(z) = \lim_{h\to 0} f(x+h)$."

Ideas $|f(z) - L| < \varepsilon$. $\quad |f(x+h) - L| < \varepsilon \quad z = x+h$

$|z - x| < \delta \quad |h| < \delta \quad h = z - x$

Proof If $\lim_{z\to x} f(z)$ exists, then for each $\varepsilon > 0$ there exists $\delta > 0$ so for any $z \in I \setminus \{x\}$ with $|z - x| < \delta$, $|f(z) - L| < \varepsilon$. Since for every $h \in I \setminus \{0\}$ with $|h| < \delta$, $|(x+h) - x| < \delta$, therefore $|f(x+h) - L| < \varepsilon$. The converse is similar. $\square$

~~(b) : $\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 - x - 6} = \lim_{x\to 3}\left(1 - \frac{4}{x+2}\right) = 1 -$~~

Ideas $\left|\frac{1}{x+2} - \frac{1}{5}\right| < \varepsilon$

$(2, 4)$

$\frac{1}{x+2} - \frac{1}{5} < 0 \qquad |x - 3| < \delta$

$\frac{1}{x+2} < \frac{1}{5}$

$0 \le \frac{1}{x+2} - \frac{1}{5} < \varepsilon \qquad x > 3 \qquad \frac{1}{5} - \frac{1}{x+2} < \varepsilon$

$\frac{1}{5} \le \frac{1}{x+2} < \frac{5\varepsilon + 1}{5} \qquad \frac{1 - 5\varepsilon}{5} < \frac{1}{x+2}$

$3 \ge x > \frac{5}{5\varepsilon + 1} - 2 \qquad \frac{5}{1 - 5\varepsilon} - 2 > x > 3$

$0 \ge x - 3 > \frac{5}{5\varepsilon + 1} - 5 \qquad \frac{5}{1 - 5\varepsilon} - 5 > x - 3 > 0$

$0 \le 3 - x < 5 - \frac{5}{5\varepsilon + 1}$

(b) : We first notice that for any $\varepsilon > 0$, $\delta := \min\left\{-\frac{5}{5\varepsilon+1} + 5, \frac{5}{1-5\varepsilon} - 5\right\}$ is such that for each $x \in (2,4) \setminus \{3\}$ with $|x - 3| < \delta$, $\left|\frac{1}{x+2} - \frac{1}{5}\right| < \varepsilon$ via a routine calculation. In other words, $\lim_{x\to 3} \frac{1}{x+2} = \frac{1}{5}$. When combined with the limit laws, this tells us

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 - x - 6} = \lim_{x\to 3}\left(1 - \frac{4}{x+2}\right) \overset{1.4.3}{=} 1 - 4 \lim_{x\to 3} \frac{1}{x+2} = 1 - \frac{4}{5} = \frac{1}{5}. \quad \square$$

OR

By limit laws, $\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 - x - 6} = \lim_{x\to 3}\left(1 - \frac{4}{x+2}\right) = 1 - \frac{4}{\lim_{x\to 3}(x) + 2} = 1 - \frac{4}{3+2} = \frac{1}{5}$. $\square$

Self - Proof of Theorem 3.18 / Ex 3-17

Ideas

Assume the left limit of the function f exists at b, and is some $L \in \mathbb{R}$.

$z \in [a,b) \Rightarrow z < b$

$\delta > b - z > 0$

When $\exists \varepsilon \forall \delta \exists z_\delta \in I \setminus \{b\}$ w/ $|z_\delta - b| < \delta$ but $|f(z_\delta) - L| \geq \varepsilon$,

Show $\exists \{w_n\}_{n=1}^{\infty} \to b$ so $\forall N \exists n \geq N$ $|f(w_n) - L| \geq \varepsilon$

$w_{2n} := b - \frac{1}{2n}$ $w_{2n+1} := C(\frac{1}{2n}) > b - \frac{1}{2n}$

$b^{>}$

L

$\forall \varepsilon \exists N \forall n \geq N$ $|z_n - b| < \varepsilon$

Is be least N possible

Let $\varepsilon > 0$. $\forall \{z_n\}_{n=1}^{\infty}, \exists N\ \forall n \geq N$. $|f(z_n) - L| < \varepsilon$

exist $(b - \delta, b)$ with ... ?

If not, for all $(b - \delta, b)$, $\exists \{z_n\}_{n=1}^{\infty}, \exists k$ so $|f(z_k) - L| \geq \varepsilon$

If $\forall \varepsilon \exists \delta \forall z \in [a,b)$ w/ $|z - b| < \delta \Rightarrow |f(z) - L| < \varepsilon$, let $\{z_n\}_{n=1}^{\infty}$ converge to b

$\forall \varepsilon \exists N \forall n \geq N$ $|z_n - b| < \varepsilon$ (and show $|f(z_n) - L| < \varepsilon$)

Proof

When there exists $\varepsilon > 0$ so for all $\delta > 0$, there exists $z \in [a,b)$ with $|z - b| < \delta$ but $|f(z) - L| \geq \varepsilon$, let C be a function of domain $\mathbb{R}^+$ with $C(\delta)$ being one of the z's specified above. Thus, we can define the sequence $\{z_n\}_{n=1}^{\infty}$ by $z_{2n} := b - \frac{1}{2n}$ and $z_{2n+1} := C(\frac{1}{2n})$, and it is clear that it converges to b (simply notice $z_n \leq b - \frac{1}{n}$ for all $n \in \mathbb{N}$ and apply Squeeze's Theorem). However, for each $N \in \mathbb{N}$, $|f(z_{2N+1}) - L| \geq \varepsilon$ by construction. Taking the contrapositive, the $(\Rightarrow)$ direction holds.

Now assume that, instead, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $z \in [a,b)$ with $|z - b| < \delta$, we have $|f(z) - L| < \varepsilon$. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence that converges to b. Then, there exists $N \in \mathbb{N}$ so for every $n \geq N$, $|z_n - b| < \delta$, by definition. As such, $|f(z_n) - L| < \varepsilon$. In other words, the left limit of the function f at b exists, and is simply just L. The $(\Leftarrow)$ direction holds.

Wherefore, the biconditional is true. $\square$

Self - Proof of Theorem 3·19

Assume $L := \lim_{z\to x} f(z)$ exists. Let $\varepsilon>0$, $I := (a,b)$, $a' := \frac{a+x}{2}$ and $b' := \frac{x+b}{2}$. Then there exists $\delta>0$ so for each $z \in I\setminus\{x\}$ with $|z-x|<\delta$, $|f(z)-L|<\varepsilon$. Since $[a',x)$ and $(x,b']$ are subsets of I, this holds for any of their members z too. Hence, $L := \lim_{z\to x} f(z) = \lim_{z\to x^-} f(z) = \lim_{z\to x^+} f(z)$.

The converse follows a similarly straightforward procedure. □

Self - Proof of Theorem 3·22 / Ex 3-18

Ideas

$\exists M \forall \delta>0 \exists z\in[a,b)$ w/ $|z-b|<\delta$ but $f(z)\le M < M+1$ $\quad$ $\exists N \forall n\ge N \quad f(z_n)\ge M$

$\forall N \exists n\ge N \quad f(z_n) < M$

$\{z_n\}_{n=1}^\infty$: $z_{2n} := b-\frac{1}{2n}$ $\quad$ $z_{2n+1} := \underbrace{C\left(\tfrac{1}{2n}\right)}_{< M+1} > b-\frac{1}{2n}$.

$\exists\delta>0 \ \forall z\in[a,b)$ w/ $|z-b|<\delta \Rightarrow f(z) > M$. $\quad$ $\{z_n\}_{n=1}^\infty$: $z_n\in[a,b)$, $z_n\to b$ i.e. $\exists N \ \forall n\ge N \ |z_n-b|<\delta$

Proof

This is almost identical to the argument for Theorem 3·18 / Ex 3-17 (as roughly verified above). □

3-15 First assume the left limit L of f at b exists and let $\varepsilon>0$. There must exist $\bar\delta>0$ such that for any $z\in[a,b)$ with $|z-b|<\bar\delta$, $|f(z)-L|<\varepsilon$. So, for each $z\in[a,b)$, if $|z-b|<\min\{\delta,\bar\delta\}$, then $|g(z)-L|<\varepsilon$ by their equality on the specified subset. Thus, $\lim_{z\to x} g(z) = L$ as desired. By symmetry, the converse must also be true. Consequently, the biconditional holds. □

3-19 By limit laws, $\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x+3) = \lim_{x\to1^-}(x)+3 = 1+3=4$ and $\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x^2+2x+1) = \lim_{x\to1^+}(x^2)+2\lim_{x\to1^+}(x)+1 = 1+2+1=4$.

Hence, $\lim_{x\to1} f(x) = \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 4$ by Theorem 3·19. □

3-20 Ideas Let $\varepsilon>0$. Show $\exists\delta>0 \ \forall z\in(0,1]$ w/ $|z-0| = z < \delta \Rightarrow \underbrace{|\sqrt{z}-0| = \sqrt{z} < \varepsilon}_{\text{by ex 1-38}}$.

Proof

Let $\varepsilon>0$. If $z\in(0,1]$ is such that $|z-0| = z < \varepsilon^2$, then $\sqrt{z}<\varepsilon$ by 1-38. As such, $\lim_{z\to0^+}\sqrt{z} = 0$. The square root function is not defined over the negative reals anyways, so it wouldn't make sense to have, say $\lim_{z\to0}\sqrt{z} = 0$ via some funny definition. $\lim_{z\to0^+}\sqrt{z}$ already tells us all the information we have / need. □

3-21 (a) Ideas

$z < x \Rightarrow f(z) \le f(x)$

$f(z) \le \lim_{z \to x^-} f(z)$?

$s := \sup\{f(z) \mid z < x\}$

when $s > L$, $s - L > 0$

$\forall \varepsilon \exists \delta \forall z \in [a, x)$ w/ $|z - x| < \delta \Rightarrow |f(z) - L| < \varepsilon$

$a \le z < x$ $\quad 0 < x - z < \delta$

$0 < x - z$

let $\varepsilon > 0$

$|f(z) - s| < \varepsilon$

$|f(z) - \sup\{f(z) \mid z < x\}| < \varepsilon$

Find $|z - x| < \delta$

$f(y) < f(z)$ though $y > z$

$f(z)$ $\quad f(y)$

If $f(y) > L$, $0 < f(y) - L <$

$|f(z) - L| < |f(y) - L| = f(y) - L$ for some δ

$f(z) - L < f(y) - L$ or $L - f(z) > 0$

$f(z) < f(y)$ $\quad f(z) < L < f(y)$

Contradiction.

$\Rightarrow f(z) \le L$ for all $z \in [a, x)$

$\sup\{f(z) \mid z < x\} \le L$

Proof

~~Assume that $L := \lim_{z \to x^-} f(z)$ exists. Suppose for the sake of contradiction, that $f(z) > L$ for some $z \in [a, x)$. Then, by L's existence, there exists $\delta_0 > 0$ so we can define with $|f(x - \frac{\delta}{2}) - L| < f(z) - L$. So, either $f(x - \frac{\delta}{2}) - L < f(z) - L$ meaning $f(x - \frac{\delta}{2}) < f(z)$, or $f(x - \frac{\delta}{2}) - L < 0$, again implying $f(x - \frac{\delta}{2}) < L < f(z)$. A contradiction to f being nondecreasing. In other words, L is a upper bound of $\{f(z) \mid z < x\}$. Therefore, $L = \lim_{z \to x^-} f(z) \ge \sup\{f(z) \mid z < x\}$. It must also be that $s := \sup\{f(z) \mid z < x\} \ge L$, lest $L > s$ but then choosing $\varepsilon = L - s$ would contradict L being $\lim_{z \to x^-} f(z)$. Consequently, equality holds and $\lim_{z \to x^-} f(z) = \sup\{f(z) \mid z < x\}$.~~ □

Oh uhh ok yeah I misunderstood the qns lol.

Ideas $\forall \varepsilon \exists \bar{z} \in [a, x)$ with $s - f(\bar{z}) < \varepsilon$, lest $\exists \varepsilon \forall z$ $s - f(z) \ge \varepsilon$

$s - \varepsilon \ge f(z)$

cont. to the leastness of s

$\forall z \in [a, x)$ w/ $|z - x| < |\bar{z} - x|$

$x - z < x - \bar{z}$

$\bar{z} < z$

$f(\bar{z}) \le f(z)$

$s - f(z) \le s - f(\bar{z}) < \varepsilon$

Proof Let $s := \sup\{f(z) \mid z < x\}$ and $\varepsilon > 0$. There must exist $y \in [a, x)$ with $s - f(y) < \varepsilon$, lest we contradict the leastness of s. So, for each $z \in [a, x)$ with $|z - x| < |y - x|$, i.e. $x - z < x - y$, we have that $|f(z) - s| = s - f(z) \le s - f(y) < \varepsilon$ by nondecreasingness. Thence, $\lim_{z \to x^-} f(z) = \sup\{f(z) \mid z < x\}$. □

First assume the left ...

$[a, b)$ if $|z - b| < \min\{\delta, \delta\}$, then ...

Claim: $\lim_{z \to x^+}$ ...

$z < y \Rightarrow f(z) \le f(y)$

Proof

Let $i := \inf\{f(z) \mid z > x\}$ and $\varepsilon > 0$. There exists some $y \in (x, b]$ with $f(y) - i < \varepsilon$, lest we contradict the maximality of i. As such, for every $z \in (x, b]$ with $|z - x| < |y - x|$, by f being nondecreasing we have that $|f(z) - i| = f(z) - i \le f(y) - i < \varepsilon$. $\square$

3-22 Follows from limit laws for sequences involving ∞ (Theorem 2.44).

3-23 Similar to the self-proof of Theorem 3.19.

Self-Proof of Theorem 3.25

1. implies 2. follows from a similar procedure as done in the self-proof of Theorem 3.14 using sequences.

Similarly, 2. implying 3. is almost identical to what was done for Theorem 3.7, which itself directly tells us that, when 3. is true, so is 1. $\square$

Self-Proof of Theorem 3.27

1. to 4. follow from limit laws already established.

Ideas for 5. & 6.

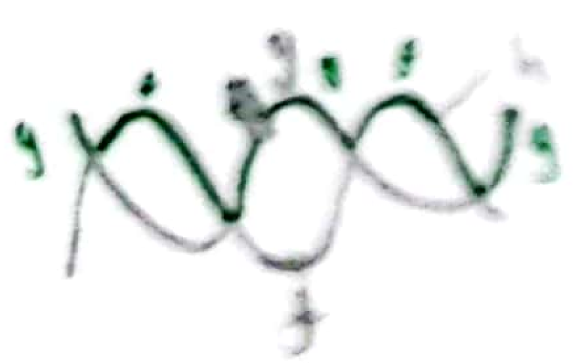
$h(x) := \max\{f(x), g(x)\}$ wlog, $h(x) = g(x)$,

$\nexists\, [a,b]$ where $\forall a \forall b,\ a = b$ or $x \notin [a,b]$ or $\exists z\,(z \in [a,b]\ \&\ h(z) \neq g(z))$

$\exists a \exists b\,(a \neq b\ \&\ x \in [a,b]\ \&\ \forall z\,(z \in [a,b] \Rightarrow h(z) = g(z)))$

$a \neq b$ & $x \in [a,b]$ & $\forall z \in [a,b]$; $h(z) = g(z)$, continuity is trivial

wlog, consider $a \neq b$ & $x \in [a,b]$



Let $\varepsilon > 0$, $\exists \delta > 0\ \forall z \in I$ w/ $|z - x| < \delta$, $|f(z) - f(x)| < \varepsilon$ & $|g(z) - g(x)| < \varepsilon$

wlog, $h(x) = g(x)$: i.e. $f(x) \le g(x)$

Show $\exists \delta > 0\ \forall z \in I$ w/ $|z - x| < \delta$, $|h(z) - h(x)| < \varepsilon$

$\overline{\quad |h(z) - g(x)| < \varepsilon \quad}$

OR

$|f(z) - g(x)| < \varepsilon$

If, instead, $\exists \varepsilon > 0\ \forall \delta > 0\ \exists z \in I$ w/ $|z - x| < \delta$ but $|h(z) - g(x)| \ge \varepsilon$

for $\delta := \bar{\delta}$, which $h(z) = f(z) > g(z)$ $\{z_n\}_{n=1}^{\infty} \to x$: $|f(z_n) - g(x)| \ge \varepsilon\ \forall n \in \mathbb{N}$

$\varepsilon \le |f(z_n) - g(z_n) + g(z_n) - g(x)|$

$z_n = f(\text{some } z)$ $|g(z_n) - g(x)| < \varepsilon\ \forall n \in \mathbb{N}$

$\varepsilon \le |f(z_n) - g(z_n)| + |g(z_n) - g(x)|$

$z_n \in I$

$\Downarrow$

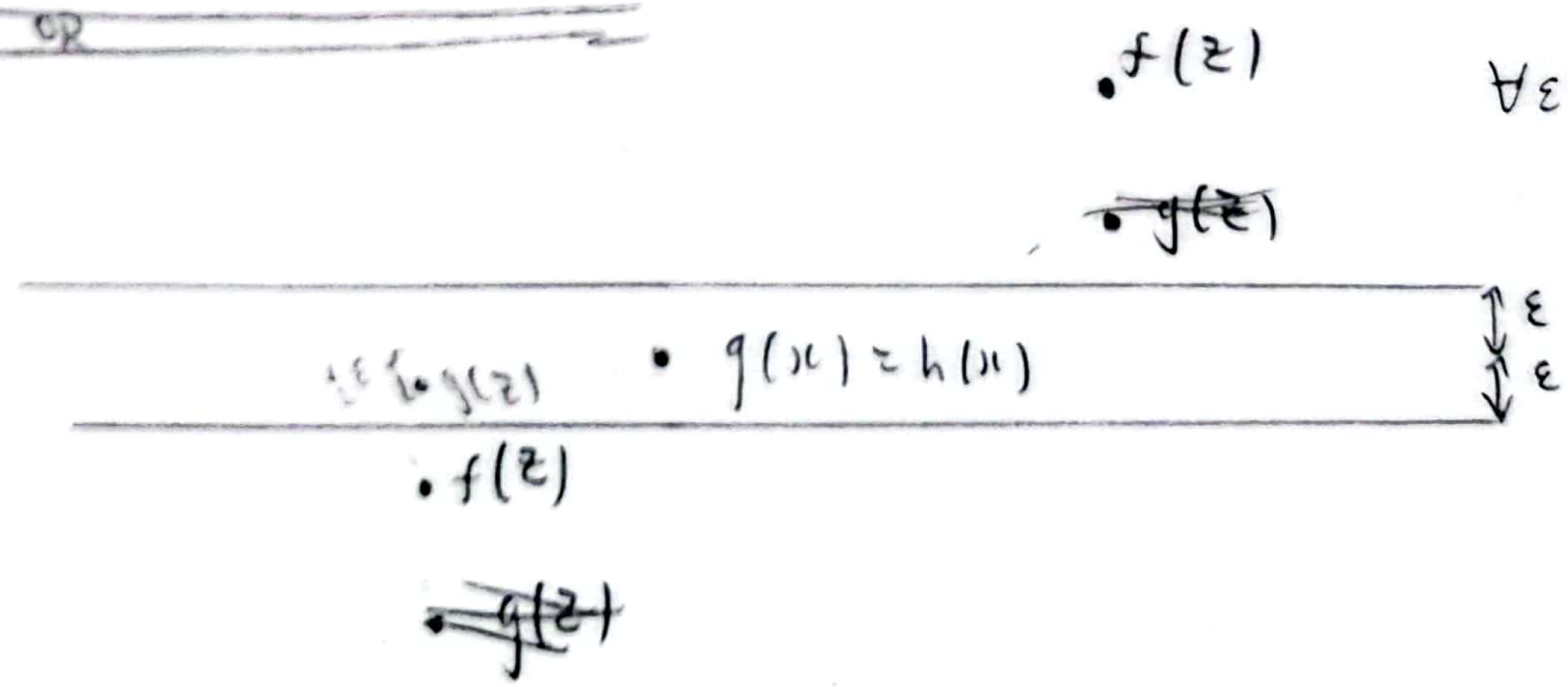
$\exists N\ \forall n \ge N$ $|f(z_n) - f(x)| < \varepsilon$ & $|g(z) - g(x)| < \varepsilon$

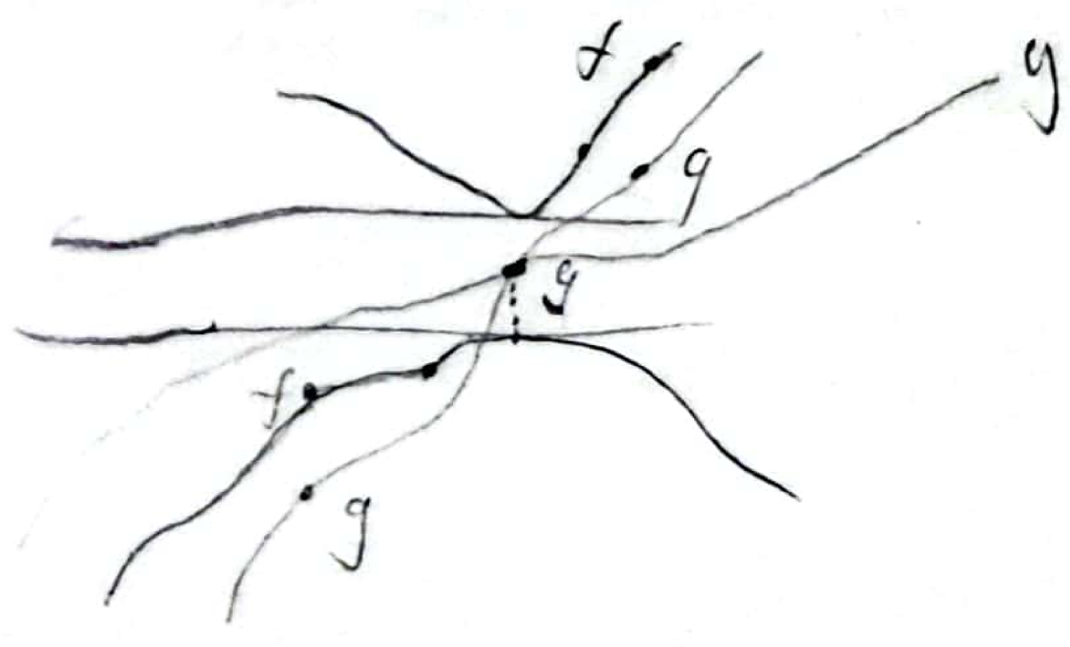
$\frac{1}{2}\varepsilon = \varepsilon' \le |f(z_n) - g(z_n)|$ $(\forall n \in \mathbb{N})$

Show $\exists M\ \forall m \ge M$ $|h(z_m) - g(x)| < \varepsilon$ Find large enough M so $|f(z_m) - g(x)| < \varepsilon$

OR

$\forall \varepsilon\ \exists N\ \forall n \ge N$ $|f(z_n) - f(x)| < \varepsilon$





Example 3.28 / Ex 3-28 (Accidentally saw the hint ¨)

Ideas

$n=0$ trivial

Assume true for $n \in \mathbb{N}$, $q(x) := \sum_{j=0}^{n}$

$$|p(z)-p(x)| < \varepsilon$$

$$\le |a_{n+1}z^{n+1} - a_{n+1}x^{n+1}| + \underbrace{|q(z)-q(x)|}_{\frac{1}{2}\varepsilon}$$

$$|a_{n+1}||z^{n+1} - x^{n+1}|$$

$\frac{1}{2|a_{n+1}|}\varepsilon$

$$|z^{n+1} - x^{n+1}| < \varepsilon$$

$$x^{n+1} - \varepsilon < z^{n+1} < x^{n+1} + \varepsilon$$

$$\delta := |x| - |x^{n+1} - \varepsilon|^{\frac{1}{n+1}} = |x^{n+1} + \varepsilon|^{\frac{1}{n+1}} - |x| > 0$$

$$2|x| = |x^{n+1} + \varepsilon|^{\frac{1}{n+1}} + |x^{n+1} - \varepsilon|^{\frac{1}{n+1}}$$

$\uparrow \square^{\frac{n+1}{n}}$

- z_1^n
- x^n
- z_0^n

$$|z-x| < \delta$$

$$x - \delta < z < x + \delta$$

$$\underbrace{x - |x|}_{0 \text{ or } -2x} + |x^{n+1} - \varepsilon|^{\frac{1}{n+1}}$$

Oh bruh just use Thm' 3.27 lmao (Example 3.24)

Proof When $n=0$, any polynomial of degree n is a constant function on $\mathbb{R}$, which is clearly continuous on $\mathbb{R}$. So, assume any polynomial of degree n is continuous on $\mathbb{R}$. Let p_{n+1} be a polynomial of degree $n+1$ with $p(x) = \sum_{j=0}^{n+1} a_j x^j$ for some $a_j \in \mathbb{R}$. By assumption, $\lim_{z\to x} \sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} a_j x^j$ and $\lim_{z\to x} a_{n+1} z^n = a_{n+1} x^n$ for any $x \in \mathbb{R}$. Combined with the fact that $\lim_{z\to x} z = x$ (Example 3.24), we have that $\lim_{z\to x} p(z) = x$ for each $x \in \mathbb{R}$, from Theorem 3.27. Hence, p is continuous on $\mathbb{R}$. By induction, any polynomial must be continuous on $\mathbb{R}$. $\square$

Self-Proof of Theorem 3.30

Since $\lim_{y\to g(x)} f(y) = f(g(x))$, it is certain that $\lim_{z\to x} (f\circ g)(z) = \lim_{y \to g(x)} f(y) = f(g(x)) = (f\circ g)(x)$ by Theorem 3.14. $\square$

Self - Proof of Theorem 3.32

Ideas

If both left and right limits exist at x, either they are equal (1.) or not (2.)

When at least one one-sided limit does not exist, either bounded or unbounded.
(4.) (3.)

$\neg L \; \exists \{z_n\}_{n=1}^{\infty} \; \exists \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \quad |f(z_n) - L| \geq \varepsilon$

$z_n \in [a, x)$
$\downarrow$
x

Proof

First consider $x \in D \cup \{x_1, \dots, x_n\}$ not being an endpoint. We evaluate casewise:

Case 1 Both $\lim_{z \to x^-} f(z)$ and $\lim_{z \to x^+} f(z)$ exist at x.

i. $\lim_{z \to x^-} f(z) = \lim_{z \to x^+} f(z)$ so there is a removable discontinuity at x.

ii. $\lim_{z \to x^-} f(z) \neq \lim_{z \to x^+} f(z)$, thus there is a jump discontinuity at x.

Case 2 One of $\lim_{z \to x^-} f(z)$ and $\lim_{z \to x^+} f(z)$ does not exist, say $\lim_{z \to x^-} f(z)$.

i. For all $\delta > 0$, f is unbounded in $\{z \in D \mid |z - x| < \delta\}$. Then, it is clear that there is an infinite discontinuity at x by constructing a suitable $\{z_n\}_{n=1}^{\infty}$

ii There exists $\delta > 0$ with f being bounded in $\{z \in D \mid |z - x| < \delta\}$. Suppose, for the sake of contradiction, that for any sequences $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$ we have $z_n, w_n < x$ (or $z_n, w_n > x$), and both $\lim_{n \to \infty} f(z_n)$ and $\lim_{n \to \infty} f(w_n)$ exist so that they're equal. Then, $\lim_{z \to x^-} f(z)$ exists and it is in fact just $\lim_{n \to \infty} f(x - \frac{1}{n})$. A contradiction. Therefore, there must be a discontinuity by oscillation at x.

The situation of x being an endpoint is similarly straightforward.

Consequently, every discontinuity at $x \in D \cup \{x_1, \dots, x_n\}$ is one of the 4 afforementioned. In fact, it is relatively clear that each of these types of discontinuities are unique (ie. no two can occur simultaneously at a common x).

Self-Proof of Theorem 3.27

Ideas for 5. & 6.

wlog, $f(x) \le g(x)$ so $h(x) = g(x)$

Suppose ftsoc, ... $|h(z) - g(x)| > \varepsilon$. $g(z) < g(x) + \frac{1}{2}\varepsilon$, $g(z) > g(x) - \frac{1}{2}\varepsilon$

For $\frac{1}{2}\varepsilon$, $\exists \delta > 0$ ... $|g(z) - g(x)| < \frac{1}{2}\varepsilon$

For $\frac{\delta}{n}$, $\exists z$, where $h(z) = f(x)$ $|\underbrace{h(z)}_{f(z)} - g(x)| \ge \varepsilon$

$g(x) - f(z_n) \ge \varepsilon$
$g(x) - \varepsilon \ge f(z_n)$
OR
$f(z_n) - g(x) \ge \varepsilon$
$f(z_n) \ge g(x) + \varepsilon$

$\frac{1}{2}\varepsilon$ $\forall \delta > 0$ (choosing $\frac{\delta}{n}$) $\exists z$ $|z - x| < \delta$ but $|f(z) -$

Define $\{z_n\}_{n=1}^{\infty}$ by $z_n := \cancel{f}(c(\frac{\delta}{n}))$ with $|f(z_n) - g(x)| \ge \varepsilon$ $\forall n \in \mathbb{N}$

[By $\Delta \neq$, $|f(z_n) - g(z_n)| + |g(z_n) - g(x)| \ge \varepsilon$ so $|f(z_n) - g(z_n)| \ge \frac{1}{2}\varepsilon \Rightarrow$
$\ge |f \cdots| + \frac{1}{2}\varepsilon$

$f(z_n) \ge g(z_n) + \frac{1}{2}\varepsilon$
OR
$g(z_n) - \frac{1}{2}\varepsilon \ge f(z_n)$

If ∞ number of n with $f(z_n) \ge g(x) + \varepsilon$, (can construct subseq so) $f(x) = \lim_{n\to\infty} f(z_n) \ge g(x) + \varepsilon > g(x)$, but. $f(x) \le g(x)$.

When ∞ number of $f(z_n) \le g(x) - \varepsilon$,

~~i. ∞ number of $g(z) \le f(z_n) - \frac{1}{2}\varepsilon \le g(x) - \varepsilon$~~
~~$g(z) + \varepsilon \le f(z_n) + \frac{1}{2}\varepsilon \le g(x) \Rightarrow g(z) \le g(x) - \varepsilon \Rightarrow g(x) = \lim_{n\to\infty} g(z_n) \le g(x) - \varepsilon < g(x)$~~

ii. ∞ no. of $\cancel{g(z_n) > f(z_n)}$

Contradict $h(z_n) = f(z_n)$.

Proof Let $h := \max\{f, g\}$ without loss of generality, we can assert $h(x) = g(x)$, i.e. $f(x) \le g(x)$. Suppose, for the sake of contradiction, that there exists $\varepsilon > 0$ so for all $\delta > 0$ there is some $x \in I$ with $|z - x| < \delta$ but $|h(z) - g(x)| \ge \varepsilon$. By AC, let c be a function with domain $\mathbb{R}^+$ so that $c(\delta)$ is one such z. Since g is continuous at x, for some $\delta > 0$ we have that for every $z \in I$ with $|z - x| < \delta$, $|g(z) - g(x)| < \varepsilon$.

Define the sequence $\{z_n\}_{n=1}^{\infty}$ by $z_n := c(\frac{\delta}{n})$, where $h(z_n) = f(z_n)$ is clear from the previous sentence. There are 2 cases to consider:

Case 1 There is a infinite number of $n \in \mathbb{N}$ with $f(z_n) \le g(x) - \varepsilon$. But $f(z_n) \le g(x) - \varepsilon < g(z_n)$ contradicts $h(z_n) = f(z_n)$.

Case 2 There is a infinite number of $n \in \mathbb{N}$ with $f(z_n) \ge g(x) + \varepsilon$. Then we can construct a subsequence $\{\bar{z}_n\}_{n=1}^{\infty}$ of such z_n so that

$f(x) = \lim_{n\to\infty} f(\bar{z}_n) \ge g(x) + \varepsilon > g(x)$, contradicting $h(x) = g(x)$.

Either ways, this is impossible. Therefore, 5. is established by contradiction. $\square$

Similarly, 6. is also certainly true.

3-26 Ideas

wlog, $f(x) \geq g(x)$

$$\lim_{z \to x} \underbrace{(h \circ i)(z)}_{\overset{?}{=} \max\{f, g\}(z)} = f(x)$$

3-27 The result that x^n is continuous is proven by 3-28 / induction + limit laws (Self-proof for Example 3.28). Then, by limit laws we also have that x^{-m} is continuous on $\mathbb{R}\setminus\{0\}$.

3-29 (a) See the self-proof of Theorem 3.30.

(b) Similar to the sequences self-proof of Theorem 3.19, but simpler.

3-30 If the interval D has zero length, it means $D=\emptyset$ and $f:\emptyset\to\mathbb{R}$. So, it doesn't really make sense to consider continuity at any $x\in\mathbb{R}$.

3-31 (a) Let $\varepsilon>0$. Then, for all $z\in\mathbb{R}\setminus\{0\}$ with $|z|<\varepsilon$, $\left|\frac{z^2}{z}-0\right| = |z| < \varepsilon$. Thus, $\lim_{z\to 0}\frac{x^2}{x}=0$. Since 0 is not an endpoint of $\mathbb{R}$, f has a removable discontinuity at 0.

(b) Let $\varepsilon>0$. For any $z\in\mathbb{R}^-$ with $|z-0|<1$, $|\lceil z\rceil - 0| = 0<\varepsilon$ so $\lim_{z\to 0^-}\lceil z\rceil = 0$. Similarly, for every $z\in\mathbb{R}^+$ with $|z-0|<1$, $|\lceil z\rceil - 1| = 0<\varepsilon$, thus $\lim_{x\to 0^+}\lceil z\rceil = 1$. By the inequality of left and right limits, there is a jump discontinuity at 0.

(c) Let $L\in\mathbb{R}$ and $\varepsilon>0$. For any $\delta>0$, $\left|\min\{\frac{1}{L+\varepsilon}, -\frac{\delta}{2}\}-0\right|<\delta$ and $\left|\frac{1}{\min\{\frac{1}{L+\varepsilon},-\frac{\delta}{2}\}}-L\right| \geq |L+\varepsilon-L| \geq \varepsilon$. So, $\lim_{z\to x^-}\frac{1}{x}$ does not exist. In fact, there is the sequence $\{-\frac{1}{n}\}_{n=1}^{\infty}$ such that given $M\in\mathbb{R}$, $\left|\frac{1}{-\frac{1}{\lceil |M|\rceil}}\right| = \lceil |M|\rceil \geq M$. And hence, $\lim_{n\to\infty}\left|\frac{1}{-\frac{1}{n}}\right| = \infty$. Which means there's an infinite discontinuity at 0.

(d) Ideas

$\frac{1}{2^n}\leq x\leq\frac{1}{2^{n-1}}$ $\qquad 0\leq 2x\leq 1$ $\qquad z_n := \frac{1}{2^n}$ $\quad g(2^n(\frac{1}{2^n}-\frac{1}{2^n})) = g(0) = 0$

$\qquad\qquad\qquad 0\leq -2x+2\leq 1$ $\qquad w_n := \frac{3}{2^{n+1}}$ $\quad g(2^n(\frac{3}{2^{n+1}}-\frac{1}{2^n})) = g(\frac{3}{2}-1) = g(\frac{1}{2}) = 1$

$1\leq 2^n x\leq 2$

$0\leq 2^n(x-\frac{1}{2^n})\leq 1$ $\qquad 0\leq f(x)\leq 1 \implies f$ is bounded in $\{z\in\mathbb{R} \mid |z-x|<1\}$

$0\leq 2^n(x-\frac{1}{2^n})\leq\frac{1}{2}$ $\qquad \frac{1}{2}\leq 2^n(x-\frac{1}{2^n})\leq 1$

$0\leq x-\frac{1}{2^n}\leq\frac{1}{2^{n+1}}$ $\qquad \frac{1}{2^{n+1}}\leq x-\frac{1}{2^n}\leq\frac{1}{2^n}$

$\frac{1}{2^n}\leq x\leq\frac{3}{2^{n+1}}$ $\qquad \frac{3}{2^{n+1}}\leq x\leq\frac{1}{2^{n-1}}$

Proof

It is clear that $0\leq g\leq 1$, and hence, $0\leq f\leq 1$. f is bounded in $\{z\in\mathbb{R} \mid |z-x|<1\}$. Consider $\{\frac{1}{2^n}\}_{n=1}^{\infty}$ and $\{\frac{3}{2^{n+1}}\}_{n=1}^{\infty}$ which converge to 0 but $f(\frac{1}{2^n}) = g(2^n(\frac{1}{2^n}-\frac{1}{2^n})) = 0$ and $g(2^n(\frac{3}{2^{n+1}}-\frac{1}{2^n})) = g(\frac{1}{2}) = 1$ tells us that $\lim_{n\to\infty} f(\frac{1}{2^n}) = 0 \neq 1 = \lim_{n\to\infty} g(\frac{1}{2})$. As such, it is clear that f has a discontinuity by oscillation at 0.

$\square$

3-32 (a) Ideas

$\frac{1}{n+1} \le \frac{9}{10n}$ [ ] - [ ] $[\frac{9}{10}, 1]$ $\otimes [\frac{9}{20}, \frac{1}{2}]$

$n \ge 9$ $\frac{10n}{9} \ge n+1$ $0 < x < \frac{1}{10}$ $[\ , \frac{1}{10}]\ [\frac{1}{10}, \ ]$ $(0, \frac{1}{10}] \subseteq U$

$\frac{1}{9}n > 1$ $\frac{1}{n+1} \ge \frac{9}{10n}$ $[\ , \frac{1}{11}]\ [\frac{9}{100}, \ ]$

$[\frac{9}{10(n+1)}, \frac{1}{n+1}] \cup [\frac{9}{10n}, \frac{1}{n}] = [\frac{9}{10(n+1)}, \frac{1}{n}]$

Let I be an interval

If $I \subseteq [-1, 0]$, $I \not\subseteq U \subseteq \mathbb{R}^+$

$\frac{10n}{9} \le n+1$

$\frac{1}{9}n \le 1$ $n \le 9$

least k with $x > \frac{9}{10k}$ $\Rightarrow \frac{9}{10k} < x \le \frac{9}{10(k-1)} \le \frac{1}{k}$

$(k \ge 10)$ $\Rightarrow x \in [\frac{9}{10k}, \frac{1}{k}]$

①: For $n \le 9$, $\frac{1}{n+1} \le \frac{9}{10n}$ so $\underbrace{[\frac{9}{10(n+1)}, \frac{1}{n+1}] \cup [\frac{9}{10n}, \frac{1}{n}]}_{\rule{0pt}{0pt}} \subset \underbrace{[\frac{9}{10(n+1)}, \frac{1}{n}]}_{\rule{0pt}{0pt}}$

②: For $n > 9$, $\frac{1}{n+1} > \frac{9}{10n}$ so ______ = ______

Check:

$n \le 9$

$\frac{1}{9}n \le 1$

$\frac{10n}{9} \le n+1$

$\frac{1}{n+1} \le \frac{9}{10n}$ ✓

$n > 9$

$\frac{1}{9}n > 1$

$\frac{10}{9}n > n+1$

$\frac{1}{n+1} > \frac{9}{10n}$ ✓

Let $x \in (0, \frac{1}{10}]$ and k be the least natural with $x > \frac{9}{10k}$.

From ②, $\frac{9}{10k} < x \le \frac{9}{10(k-1)} \le \frac{1}{k}$. Hence, $x \in [\frac{9}{10k}, \frac{1}{k}] \subseteq U$

Consequently, $(0, \frac{1}{10}] \subseteq U$.

$\frac{9}{10k} < x < \frac{1}{10}$

$\frac{9}{k} < 1$

$9 < k$

$k \ge 10$

$0 \le 2^n(x - \frac{1}{2^n}) = 1$

$0 \le 2^n(x - \frac{1}{2^n}) \le \frac{1}{2}$ $\frac{1}{2} \le 2^n(x - \frac{1}{2^n}) \le 1$

$\frac{1}{\ } \le \frac{1}{2^{n+1}}$ $\frac{1}{2^{n+1}} \le x - \frac{1}{2^n} \le \frac{1}{2^n}$

3-32 Replacing $\frac{1}{n}-\frac{1}{10n}$ with $\frac{1}{n}-\frac{1}{10n^2}$:

$\frac{1}{k+1} \geq \frac{1}{k}-\frac{1}{10k^2}$

$k^2 \geq k^2+k-\frac{1}{10}$

$\frac{1}{10} \geq k$

Ideas

If $0 \notin I$, $I \cap U = \emptyset$ lest there exists $x \in I \cap U$: $\exists k$. $x \in [\frac{1}{k}-\frac{1}{10k^2}, \frac{1}{k}]$

$\frac{1}{2}\left[\frac{1}{k+1}+\frac{1}{k}-\frac{1}{10k^2}\right] \in I$

$\Rightarrow I$ not $\subseteq D$, contradiction

$\Rightarrow I \subseteq [-1,0]$ or $I \subseteq U$ (distinct cases)

(a) Let I be an interval contained in the domain of f, D. If $0 \notin I$, then $I \cap U = \emptyset$, lest there exists $x \in I \cap U$, and correspondingly, some $k \in \mathbb{N}$ with $x \in [\frac{1}{k}-\frac{1}{10k^2}, \frac{1}{k}]$. Since $\frac{1}{n+1} < \frac{1}{n}-\frac{1}{10n^2}$ (otherwise $n \leq \frac{1}{10}$) for any $n \in \mathbb{N}$, $\frac{1}{2}\left(\frac{1}{k+1}+\frac{1}{k}-\frac{1}{10k^2}\right) \in I \setminus U$ is clear. But this contradicts $I \subseteq D$. Therefore, for $I \subseteq D$, there are two unique cases of $I \subseteq [-1,0]$ or $I \subseteq U$. Either ways, $f|_I$ is a constant function. As such, f is certainly continuous on I if we consider the continuity of endpoints with the corresponding one-sided limit.

(b) Let $x \in D$. Then either $x \in [-1,0]$, or $x \in U$ so there exists some $n \in \mathbb{N}$ with $x \in [\frac{1}{n}-\frac{1}{10n}, \frac{1}{n}]$, by definition.

(c) Notice that $\lim_{z \to 0^-} f(z) = 0 \neq 1 = \lim_{n\to\infty} f(z_n)$ for any sequence $\{z_n\}_{n=1}^{\infty}$ converging to 0 with $z_n \in U$, such as $\{\frac{1}{n}\}_{n=1}^{\infty}$. Thus this is similar to having a jump discontinuity. Hence, we shouldn't consider f to be continuous at 0 since the left and 'right' limits are unequal, in the above sense.

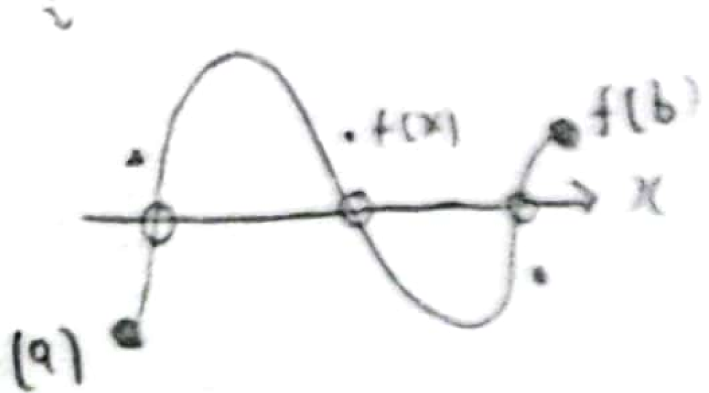
(d) Let the function $f: D \to \mathbb{R}$ be such that every $x \in D$ is contained in some interval $I \subseteq D$ of nonzero length. In other words, $f: \bigcup_{j \in J} I_j \to \mathbb{R}$. When $x \in D$ is such that for all sequences $\{z_n\}_{n=1}^{\infty}$ converging to x and $z_n \in D$, $\lim_{n\to\infty} f(z_n) = f(x)$, call f continuous at x.

Self-Proof of Theorem 3.34 Intermediate Value Theorem

Ideas

$S := \{y \in (f(a), f(b)) \mid \overbrace{(\forall c \in (a,b))[y \neq f(c)]}^{y \notin f[a,b]}\}$

$f(a) < \inf S < f(b)$ $\overset{!!}{?}$



If for all open intervals I with $0 \in I \subseteq (f(a), f(b))$, we have that $\exists y \in I$ with $y \in f[a,b]$,

For every $x \in (a,b)$,
when both one-sided lims are nonzero,

bounded on its domain, if $\{z_n\}_{n=1}^{\infty} \to x$ & $\{w_n\}_{n=1}^{\infty} \to x$ w/ $z_n, w_n < x$ (or $z_n, w_n > x$) then $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f(w_n)$

$\exists x$ s.t. one one-sided limit exists (and is just 0) $\Rightarrow \lim_{z \to x^-} f(z) = 0 \neq f(x)$, cont.

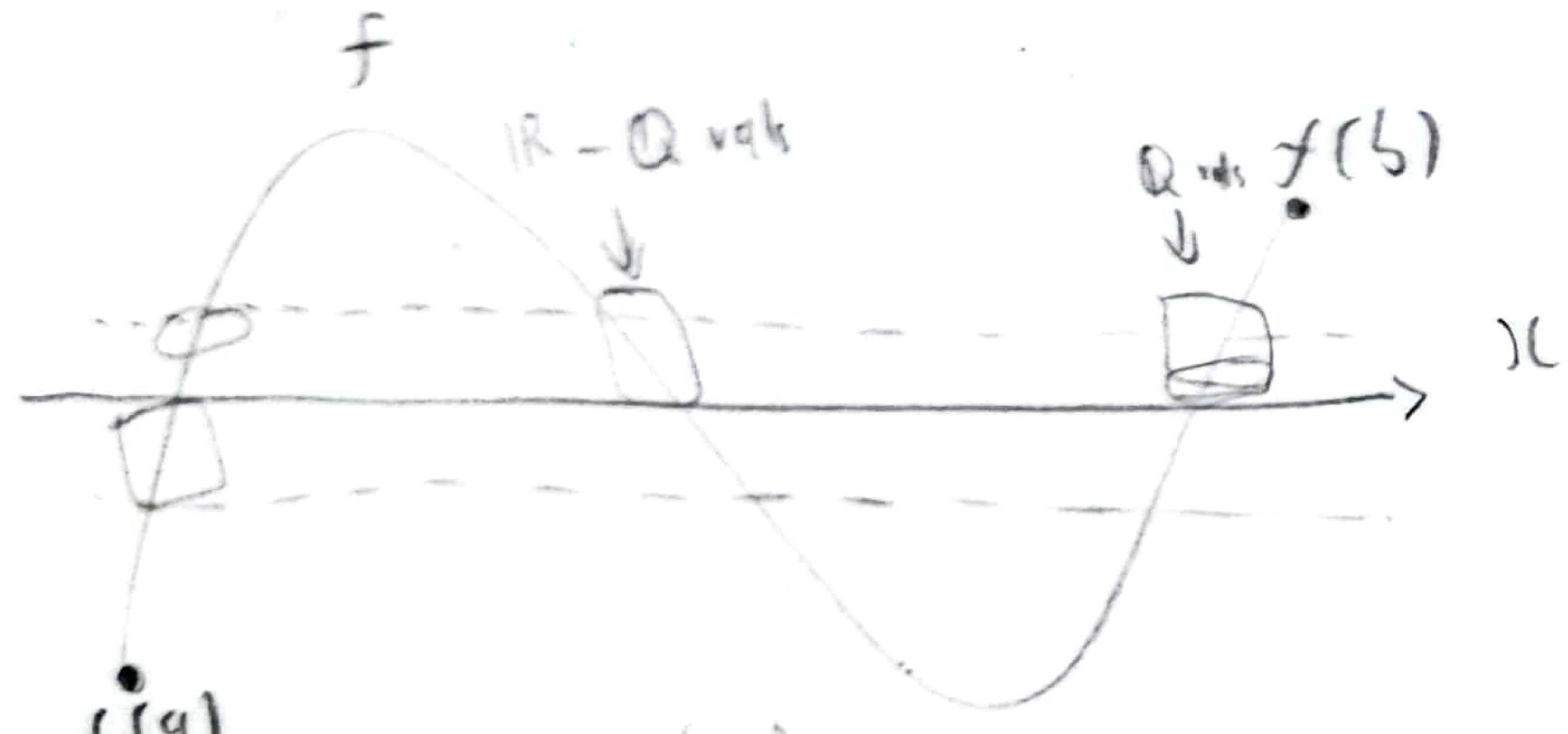
If continuous ..., $s := \sup S$,

$\exists N \in \mathbb{N}\ \forall n \geq N$

$\forall x \in [a,b]\ \forall \varepsilon > 0\ \forall \{z_n\} \to x$ & $z_n \in [a,b] \Rightarrow |f(z_n) - f(x)| < \varepsilon$

$\forall x \in [a,b]\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in [a,b]$ w/ $|z - x| < \delta \Rightarrow |f(z) - f(x)| < \varepsilon$

$\exists \varepsilon > 0\ \forall y : |y| < \varepsilon \Rightarrow \forall x : f(x) \neq y$



when $f(x) \neq 0$ for all x,

i. $\forall \varepsilon > 0\ \exists y : |y| < \varepsilon$ & $\exists x : f(x) = y$

exists defined y for all in interval

ε small
0
ε small

ii. $\exists \varepsilon'$ so finite no. of such y's

→ restrict ε to small enough value

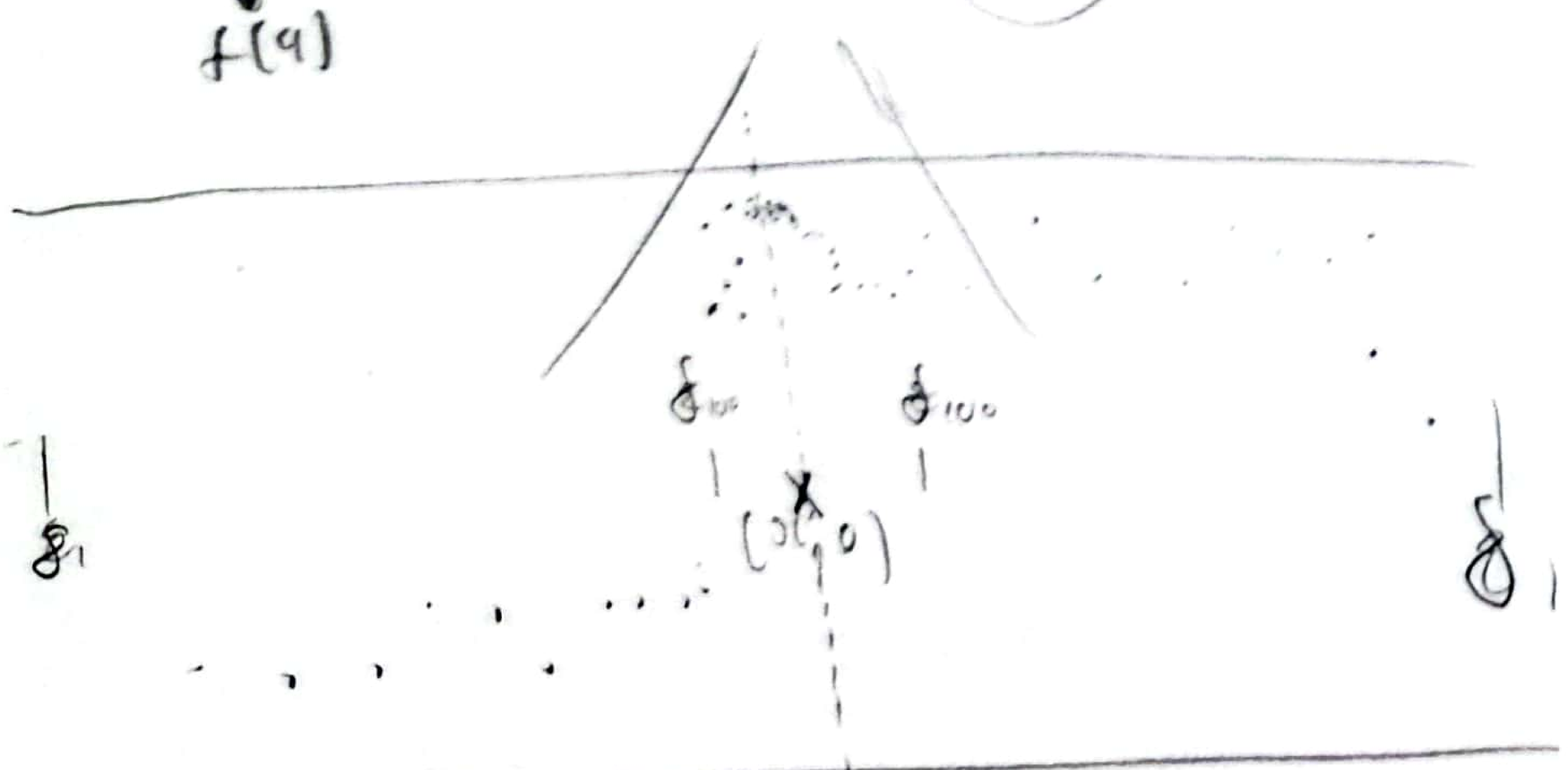
$f^{-1}(y) \neq \emptyset$

j. if $\exists \delta > 0$ ... $\exists x$, cont. (→ ¬ contr...)

jj. if $\forall \delta > 0\ \exists z\ |z - x| < \delta$ but $|f(z)| \geq \varepsilon'$ $\forall x$

$\{\frac{1}{n}\}_{n=1}^{\infty}$ (or $\varepsilon_{\text{some}}$ instead of 1)

$\{y_n\}_{n=1}^{\infty}$ $y_n := c(\frac{1}{n})$



$\forall x\ \forall \delta > 0\ \exists y\ |y| < \varepsilon$ but $\forall z\ f(z) \neq y$ (finite)

i.e. $\exists x\ \exists \delta > 0\ \forall y$'s $(|y| < \varepsilon_{\text{some}}) \Rightarrow \exists z\ f(z) = y$?

exists 'clustering pillar' ?

$i^- \in f[a,b]$

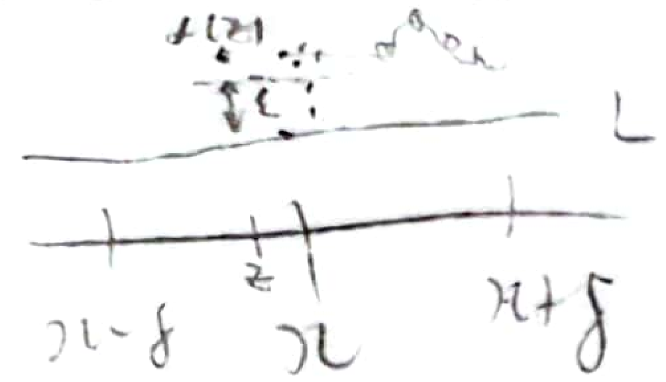
i least in $S \implies \forall \varepsilon > 0$, if $i - \varepsilon \in f[a,b]$, then $f^{-1}(i-\varepsilon) \neq \emptyset$

$\lim\limits_{z \to x^-} f(z) = i \neq f(x)$ or $\lim\limits_{z \to x^+} f(z) = i \neq f(x)$

Show $\exists x \; \exists \{z_n\}_{n=1}^{\infty} \to x \; \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; |f(z_n) - i| < \varepsilon$

What if $\forall x \; \forall \{z_n\}_{n=1}^{\infty} \to x \; \exists \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \; |f(z_n) - i| \geq \varepsilon$?

i.e. $\forall x \; \exists \varepsilon > 0 \; \forall \delta > 0 \; \exists z \in [a,b] \; |z - x| < \delta$ but $|f(z) - L| \geq \varepsilon$



$\exists x\ \forall\delta > 0\ \forall\varepsilon\ \exists$ infinite numbers of $|y| < \underbrace{\varepsilon}_{\text{e.g. } \frac{1}{n}}$ & $\underbrace{\exists z : |z - x| < \delta \ \&\ y = f(z)}_{y \in f[x-\delta,\, x+\delta]}$

$\{z_n\}_{n=1}^{\infty}$ $\quad z_n := C_2(f^{-1}(C_1(\frac{1}{n})))$

$y_n := C_1(\frac{1}{n})$

$f(z_n) = C_1(\frac{1}{n}) \qquad |f(z_n)| < \frac{1}{n}$

$\downarrow$

$0 \quad \Rightarrow \lim_{z \to x} f(z) = 0 \neq f(x) \Rightarrow \neg$ continuous. @ x

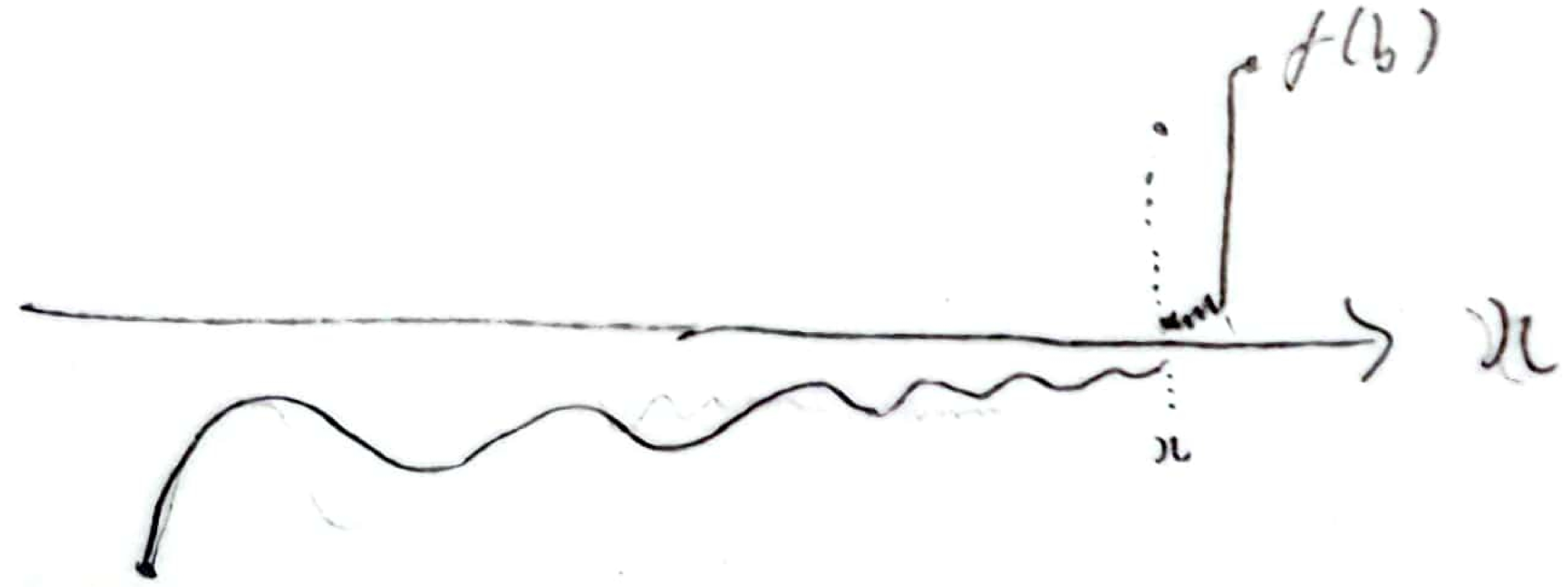
if it exists

$\forall x\ \exists\delta > 0\ \exists\varepsilon\ \exists$ only a finite no. of $|y| < \varepsilon$ & $y \in f[x-\delta,\, x+\delta]$

small enough δ and ε

if no. > 0, discont. at $C(f^{-1}(y))$

$\Rightarrow$ no. = 0



$\exists x\ \exists\delta : \forall z : x - \delta < z < x \Rightarrow f(z) < 0$, & $x < z < x + \delta \Rightarrow f(z) > 0$.

$\Longrightarrow$ Jump discontinuity

$\forall x\ \forall\delta\ \exists z : x - \delta < z < x$ but $f(z) > 0$, or $x < z < x + \delta$ but $f(z) < 0$.

$\delta = 1 \to x_1$

$\delta = \frac{|x - x_1|}{2} \to x_2$

Self-Proof of Theorem 3.36

Ideas

If $f(x_1) < y < f(x_2)$, the function $g: I \to \mathbb{R}$ def by $g(x) = f(x) - y$ is 0 at some $c \in I$ by IVT. Hence, $f(c) = y$. So, $y \in f[I]$.

Proof

For any $x_1, x_2 \in I$, if $f(x_1) < y < f(x_2)$, then we can define a function $g: I \to \mathbb{R}$ by $g(x) = f(x) - y$, which must be continuous according to limit laws. The Intermediate Value Theorem now informs us that there is some $c \in I$ for which $g(x) = 0$, that is, $f(c) = y$. Thus, $y \in f[I]$. Which tells us $f[I]$ is an interval. $\square$

Self-Proof of Theorem 3.38

Ideas

Let $\varepsilon > 0$, $\{y_n\}_{n=1}^{\infty}$ be a sequence converging to some $y \in f[I]$, $y_n \in f[I]$.

$\exists M \in \mathbb{N}\ \forall m \geq M$

$|y_m - y| < \varepsilon$

claim: $\{f^{-1}(y_n)\}_{n=1}^{\infty}$, $f^{-1}(y_n) \in I$, $\to f^{-1}(y)$?

$x_0 < x_1$, $f(x_0) \geq f(x_1)$

$x_2 < x_3$, $f(x_2) \leq f(x_3)$

injective $\Rightarrow$ increasing / decreasing

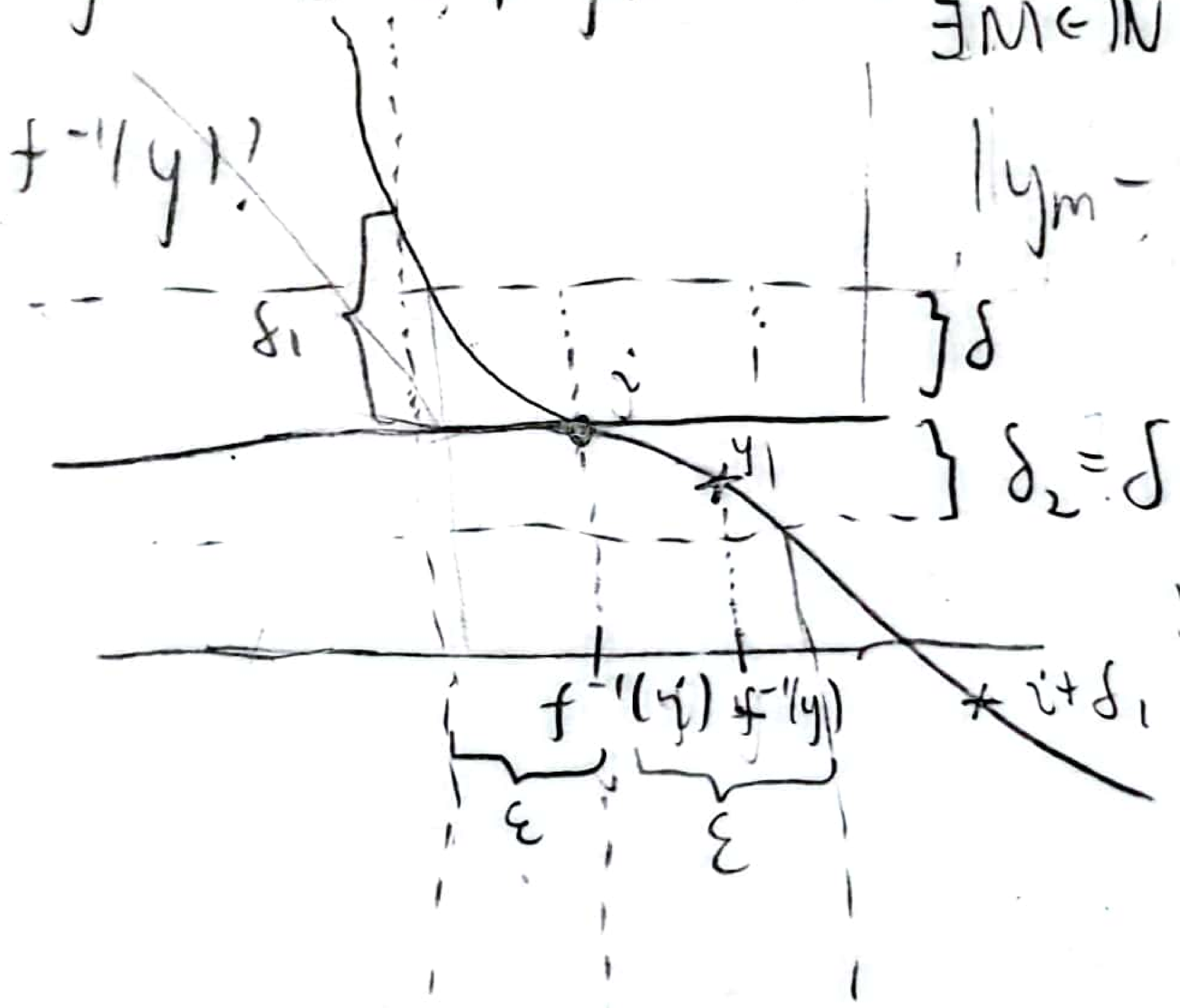
wlog $x \leq y \Leftrightarrow f(x) \leq f(y)$

Consider $i \leq f(z) < i + \delta \leq f(f^{-1}(i) + \varepsilon)$

$f^{-1}(i) \leq z < f^{-1}(i) + \varepsilon$

When $f(f^{-1}(i) - \varepsilon) \leq i - \delta < f(z) \leq i$

$f^{-1}(i) - \varepsilon < z < f^{-1}(i)$



$|z - f^{-1}(i)| < \varepsilon \Rightarrow |f(z) - i| < \delta$

$\delta_1 := \sup\{|f(z) - i| \mid z \in (f^{-1}(i) - \varepsilon, f^{-1}(i))\}$
$= i - f(f^{-1}(i) - \varepsilon) > 0$

$\delta_2 := \sup\{|f(z) - i| \mid z \in (f^{-1}(i), f^{-1}(i) + \varepsilon)\}$
$= f(f^{-1}(i) + \varepsilon) - i > 0$

$\delta := \min\{\delta_1, \delta_2\} \geq \delta_1, \delta_2$

$y \in f[I]$ &

$|y - i| < \delta \Rightarrow |f(z) - i| < \delta$

z

show $\forall i$ $\forall \varepsilon > 0\ \exists \delta > 0\ \forall y\ |y - i| < \delta \Rightarrow |f^{-1}(y) - f^{-1}(i)| < \varepsilon$

know: $\forall x$ $\forall \varepsilon > 0\ \exists \delta > 0\ \forall z\ |z - x| < \delta \Rightarrow |f(z) - f(x)| < \varepsilon$

$\forall z\ |z - f^{-1}(i)| < \delta \Rightarrow |f(z) - i| < \varepsilon$

Proof

We first show that f is increasing or decreasing on I. Suppose otherwise, for the sake of contradiction, that there exists $x_1 < x_2$ and $x_3 < x_4$ with $f(x_1) \geq f(x_2)$ and $f(x_3) \leq f(x_4)$. If $f(x_2) \leq f(x_3)$, then by IVT there exists $c \in (x_1, x_2)$ and $c' \in (x_3, x_4)$, such that $f(c) = f(c') = \min\{\frac{f(x_1)+f(x_2)}{2}, \frac{f(x_3)+f(x_4)}{2}\}$, contradicting injectivity. The case of $f(x_2) > f(x_3)$ is similar: again IVT says there exists distinct c's whose image under f is $\min\{\frac{f(x_2)+f(x_3)}{2}, \frac{f(x_3)+f(x_4)}{2}\}$, a contradiction. Thus, our claim holds; assume without loss of generality that f is increasing. Let $y \in f[I]$, $\varepsilon > 0$, and $\delta := \min\{y - f(f^{-1}(y) - \varepsilon), f(f^{-1}(y) + \varepsilon) - y\}$. Now, for any $z \in I$ so $|f(z) - y| < \delta$, we either have that $y \leq f(z) < y + \delta \leq f(f^{-1}(y) + \varepsilon)$ which tells us $f^{-1}(y) \leq z < f^{-1}(y) + \varepsilon$; or $f(f^{-1}(y) - \varepsilon) \leq y - \delta < f(z) \leq y$ saying that $f^{-1}(y) - \varepsilon < z < f^{-1}(y)$. In both cases, it is certain that $|z - f^{-1}(y)| < \varepsilon$. In other words, $\lim_{y' \to y} f^{-1}(y') = f^{-1}(y)$. Wherefore, continuity holds. $\square$

Self Proof of Corollary 3.40 (accidentally saw hint 😅)

Theorem 2.17 suffices to tell us x^d is continuous on $\mathbb{R}$ since x is. The rest follows from noticing $x^{\frac{1}{d}}$ is the inverse of x^d and applying Theorem 3.38.

Self-Proof of Theorem 3.44

Ideas

f must be bounded on $[a,b]$, lest $\forall M \exists x \; |f(x)| > M$: $\{z_n\}^\infty$ $z_1 := C(1)$ $z_{n+1} := C(f(z_n))$

$\forall x \exists \delta$ so f bounded in $\{z \in D \mid |z-x| < \delta\}$

Assume $\forall x \in [a,b] \; \exists z \in [a,b] . \; f(z) > f(x)$

If f bounded on $[a,b]$, say $\frac{-s}{-i}$ $\forall \varepsilon \exists y \; s-y < \varepsilon \stackrel{eg}{=} \frac{1}{n}$

By AC, can construct convergent sequence $\{z_n\}_{n=1}^\infty \to L$ so $\{f(z_n)\}_{n=1}^\infty \to s$
& BWT

suggests $\exists z \in [a,b]$ w/ $f(z) > f(L) = s$

Cont.

f is certainly unbounded from above ✓ $\to$ By AC can construct sequence $\{w_n\}_{n=1}^\infty \to K$ & $\{f(w_n)\}_{n=1}^\infty \to \infty$
& BWT
i.e. '$\lim_{z \to K} f(z) = \infty$'

Proof

Assume that $f : [a,b] \to \mathbb{R}$ is a function so for all $x \in [a,b]$, there exists $z \in [a,b]$ with $f(z) > f(x)$. If f is bounded on $[a,b]$, let $s := \sup f[a,b]$. By AC and the Bolzano-Weierstrass Theorem, it is clear that we can construct a convergent sequence $\{z_n\}_{n=1}^\infty$ in $[a,b]$ that converges to $L \in [a,b]$ whose image (under f) converges to s, that is, $\lim_{n\to\infty} f(z_n) = s$. Thus, it is certain that f is unbounded. Again, we now construct a sequence $\{w_n\}_{n=1}^\infty$ in $[a,b]$ converging to some $K \in [a,b]$, such that $\lim_{n\to\infty} |f(w_n)| = \infty$. This ensures f has an infinite discontinuity at K. Consequently, f cannot be continuous. Wherefore, taking the contrapositive gives the necessary result.

Self-Proof of Corollary 3.41

Ideas / checking

$x^{-\frac{p}{q}} = \left(\frac{1}{x}\right)^{\frac{p}{q}}$ $\quad x^{\frac{p}{q}} \cdot \left(\frac{1}{x}\right)^{\frac{p}{q}} = x$ $\quad$ or Thm 3.30 $\quad g(x) = x^{\frac{p}{q}} \quad g: \underset{\mathbb{R}\setminus\{0\}}{(0,\infty)} \to \mathbb{R}\setminus\{0\}$

$f(x) = \frac{1}{x} \quad f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$

Trivial from Theorem 2.17 and Corollary 3.40.

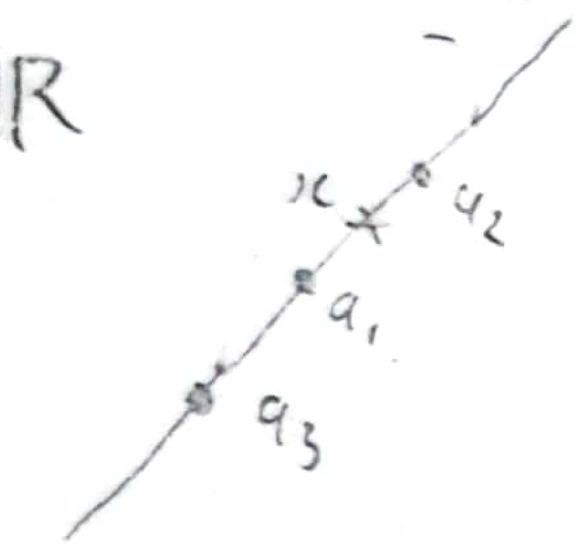
Self-Proof of Theorem 3.42

Ideas

$= (a_{n+1} - a_n)x + a_n$ $\qquad L := \lim_{n\to\infty} a_n$

$f : [a_1, \sup\{a_n \mid n \in \mathbb{N}\}] \longrightarrow \mathbb{R}$

oh lol i was overthinking



Proof

~~Let $x := \lim_{n\to\infty} a_n \geq 0$ and $\varepsilon > 0$~~ By Corollary 3.41 we have that $\lim_{n\to\infty} a_n^r = \left(\lim_{n\to\infty} a_n\right)^r$ since $\lim_{z\to x} x^r = x^r$ for $x := \lim_{n\to\infty} a_n \geq 0$.

Exercises

3-34 Simply apply the uniqueness criterion.

3-35 to 3-37 See self-proofs.

3-38 Either repeat Theorem 3.44's proof / self-proof, or apply Theorem 3.44 to $-f$ and notice that the absolute maximum of $-f$ is the absolute minimum of f.

3-39 Let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) := \frac{1}{x}$. Then $\lim_{z\to 0^+} f(z) = \infty$ is clear. Notice this divergence (to infinity) only occurs at the endpoint of our open interval domain, where f would be discontinuous if defined there. The mandate for a closed interval in Theorem 3.44 is the silver lining.

3-40 See the first part of the self-proof of Theorem 3.38.

3-41 Its contrapositive is just the Intermediate Value Theorem, which we know is true.

self proof of Theorem 3.47

Ideas lets try a direct proof for ($\Rightarrow$) this time!

($\Rightarrow$) Assume $f \to L$ at ∞, let $\varepsilon > 0$ $\quad \exists N \in \mathbb{N}\ \forall n \geq N \quad |f(z_n) - L| < \varepsilon$

$M := \inf\{ z_n \mid \{z_n\}_{n=1}^{\infty}$ is a sequence converging to ∞ & $|f(z_n) - L| < \varepsilon \}$

ok maybe a direct proof is ... hard or possibly impossible huh...

Proof

Similar to past self-proofs / proofs. $\square$

3-44 (a) Let $f : (t, \infty) \to \mathbb{R}$ be a function. We say f diverges to infinity at infinity and write $\lim\limits_{z \to \infty} f(z) = \infty$ iff for every sequence $\{z_n\}_{n=1}^{\infty}$ in (t, ∞) such that $\lim\limits_{n \to \infty}(z_n) = \infty$, we also have $\lim\limits_{n \to \infty} f(z_n) = \infty$.

(b) Same as what has been stated in 3-22 except that b^- is replaced with ∞.

(c) The function $f : (t, \infty) \to \mathbb{R}$ diverges to infinity at infinity iff for every $M \in \mathbb{R}$ there is a $N \in \mathbb{R}$ such that for all $z \geq N$ we have $f(z) \geq M$. Proof is similar to that of past statements. $\square$

3-45 There is no way to approach infinity from the right or negative infinity from the left.

Self Proof of Theorem 4.5

This essentially holds by definition.

Exercises

4-1 See my self-proofs for Example 4.2

4-2 (a)

By Theorem 4.5 f is $f'(x) = L$ iff there exists $\delta > 0$ so for each z with $|z-x| < \delta$, $|f(z) - [f(x) + L(z-x)]| \le \varepsilon |z-x|$, which is equivalent to

$$[f(x) + L(z-x)] - \varepsilon|z-x| \le f(z) \le [f(x) + L(z-x)] + \varepsilon|z-x|.$$

This shows that for each $\varepsilon > 0$, there exists a small enough region around x such that the curve $f(z)$ is within $\varepsilon|z-x|$ distance of the line $g(x) := f(x) + f'(x)(z-x)$. In other words, the inequality above supports the figure's illustration that $f(x)$ fully lies within the cone formed by the upper and lower bounding curves $u(x) := g(x) + \varepsilon|z-x|$ and $l(x) := g(x) - \varepsilon|z-x|$ in a small enough region around x.

(b) By limit laws, $\lim_{z\to 0} |z| - mz = \lim_{z\to 0} |z| - m \lim_{z\to 0} z = 0 - 0 = 0$.

(c) Again, limit laws (including exercise 2-12) inform us that $\lim_{z\to x} |f(z) - [f(x) + m(z-x)]| = \left|\lim_{z\to x} f(z) - f(x) + m \lim_{z\to x} z - x\right| = 0 + m\cdot 0 = 0$.

4-3 (a) This is easily seen to be true from Theorem 3.19.

(b) This was done in Example 4.2 (my self-proof of it).

Self-Proof of Theorem 4.6

Let $\varepsilon > 0$, $\delta > 0$ be such that if $|z-x| < \delta$ then $\left|\frac{f(z)-f(x)}{z-x} - L_f\right| < \frac{1}{2}\varepsilon$ and $\left|\frac{g(z)-g(x)}{z-x} - L_g\right| < \frac{1}{2}\varepsilon$. Now, $\left|\frac{f(z)+g(z)-f(x)-g(x)}{z-x} - L_f - L_g\right| \le \left|\frac{f(z)-f(x)}{z-x} - L_f\right| + \left|\frac{g(z)-g(x)}{z-x} - L_g\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. Similarly, letting $\delta' > 0$ such that $|z-x| < \delta$ implies $\left|\frac{f(z)-f(x)}{z-x} - L_f\right| < \frac{\varepsilon}{|c|}$, $\left|\frac{cf(z)-cf(x)}{z-x} - cL_f\right| = |c|\left|\frac{f(z)-f(x)}{z-x}\right| < |c|\cdot\frac{\varepsilon}{|c|} = \varepsilon$. (If $c=0$, f is just a constant function) So, $(f+g)'(x) = f'(x) + g'(x)$ and $(cf)'(x) = cf'(x)$ as expected.

Example 4.2 / Exercise 4-1

1. We notice that $\lim_{z \to x} \frac{z-x}{z-x} = \lim_{z \to x} 1 = 1$. Hence, $f'(x) = 1$.

2. Note that

$$\lim_{z \to 0^-} \frac{|z|-|0|}{z-0} = \lim_{z \to 0^-} -\frac{z}{z} = -1 \quad \text{but} \quad \lim_{z \to 0^+} \frac{|z|}{z} = \lim_{z \to 0^+} \frac{z}{z} = 1.$$

Hence, by the inequality of the left and right limits, $f'(0)$ must not exist. In other words, f is not differentiable at 0.

Self-Proof of Theorem 4.4

Ideas

$\exists G\ \forall \varepsilon \exists \delta\ \forall z \in (a,b)\ |z-x| < \delta \Rightarrow \left| \frac{f(z)-f(x)}{z-x} - G \right| < \varepsilon$

$$\left| \frac{f(z)-f(x)-G(z-x)}{z-x} \right| = \left| \frac{f(z) - Gz + Gx - f(x)}{z-x} \right|$$

$|z-x|G$

Show $\forall \varepsilon \exists \delta\ \forall z \in (a,b)\ |z-x| < \delta \Rightarrow |f(z) - f(x)| < \varepsilon$

$$|f(z)-f(x)| = \left| \frac{(z-x)[f(z)+f(x)]}{z-x} \right|$$

$f(z) \leq f(x) - \varepsilon$ / $f(x) + \varepsilon \leq f(z)$

Assume f is not continuous at x: $\exists \varepsilon\ \forall \delta\ \exists z \in (a,b)\ |z-x| < \delta\ \&\ |f(z)-f(x)| \geq \varepsilon$

$\forall L$
Show $\exists \varepsilon\ \forall \delta\ \exists z \in (a,b)\ |z-x| < \delta\ \&\ \left| \frac{f(z)-f(x)}{z-x} - L \right| \geq \varepsilon$

$$\lim_{z \to x} \left(\frac{f(z)-f(x)}{z-x} \cdot z - x \right) = \left(\lim_{z \to x} \frac{f(z)-f(x)}{z-x} \right) \cdot \lim_{z \to x} z - x$$

$$\lim_{z \to x} f(z) - f(x) = G \cdot 0 = 0$$

$$\lim_{z \to x} f(z) = f(x)$$

Proof

By limit laws, $\lim_{z \to x} f(z) - f(x) = \lim_{z \to x} \left[\frac{f(z)-f(x)}{z-x} \cdot (z-x) \right] = \left[\lim_{z \to x} \frac{f(z)-f(x)}{z-x} \right] \cdot \left[\lim_{z \to x} z - x \right] = 0$. In other words, $\lim_{z \to x} f(z) = f(x)$; f is continuous at x given it is differentiable there. As seen from Example 4.2 (c), continuity of a function at x doesn't guarantee its differentiability there. So, differentiability is indeed a strictly stronger condition than continuity.

Self-Proof of Theorem 4.7

Ideas

$$\left| \frac{\frac{f(z)}{g(z)} - \frac{f(x)}{g(x)}}{z-x} - \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} \right| = \left| \frac{\overbrace{\frac{f(z)(g(x))^2}{g(z)} - f(x)g(x)}^{\frac{g(x)(f(z)g(x)-f(x)g(z))}{g(z)}}}{(g(x))^2(z-x)} + \frac{f'(x)g(x)(x-z) - g'(x)f(x)(x-z)}{(g(x))^2(z-x)} \right|$$

$$\lim_{z\to x} \frac{\frac{f(z)}{g(z)} - \frac{f(x)}{g(x)}}{z-x} = \lim_{z\to x} \frac{f(z)g(x) - f(x)g(z)}{g(x)g(z)(z-x)} = \lim_{z\to x} \frac{g(x)\cdot\frac{f(z)-f(x)}{z-x} - f(x)\cdot\frac{g(z)-g(x)}{z-x} + \cancel{\frac{f(x)g(x)-f(x)g(x)}{z-x}}}{g(x)g(z)}$$

$$= \lim_{z\to x} \frac{g(x)\lim_{z\to x}\left(\frac{f(z)-f(x)}{z-x}\right) - f(x)\lim_{z\to x}\left(\frac{g(z)-g(x)}{z-x}\right)}{g(x)\lim_{z\to x} g(z)}$$

by continuity

$$= \lim_{z\to x} \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Reverse direction [i.e. starting from $\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$] gives a pedantic proof.

Self-Proof of Theorem 4.8

Ideas

$$f'(x)g(x) + g'(x)f(x) = g(x)\lim_{z\to x}\left(\frac{f(z)-f(x)}{z-x}\right) + f(x)\lim_{z\to x}\left(\frac{g(z)-g(x)}{z-x}\right)$$

$$= \lim_{z\to x} g(z) \lim_{z\to x}\left(\frac{f(z)-f(x)}{z-x}\right) + f(x)\lim_{z\to x}\left(\frac{g(z)-g(x)}{z-x}\right)$$

$$= \lim_{z\to x} \frac{f(z)g(z) - f(x)g(x)}{z-x}$$

by continuity

$g(z)f(z) - f(x)g(z) + f(z)g(z) - f(z)g$

$- \cancel{f(x)g(z)} + f(z)g(z) + \cancel{f(x)g(z)} - f(x)g(x)$

Self-Proof of Theorem 4.9

Ideas $\lim_{z\to x}\frac{z^n - x^n}{z-x} =$

$$
\begin{array}{r|l}
 & z^{n-1} + x z^{n-2} \\
z - x & z^n - x^n \\
 & -(z^n - x z^{n-1}) \\
 & x z^{n-1} - x^n \\
 & -(x z^{n-1} - x^2 z^{n-2}) \\
 & x^2 z^{n-2}
\end{array}
\qquad
\begin{array}{r|l}
 & z^n \\
z - x & z^{n+1} - x^{n+1} \\
 & -(z^{n+1} - x z^n) \\
 & x z^n - x^{n+1}
\end{array}
$$

Proof

When $n=1$, $\frac{d}{dx}(x^1) = 1$ is already known from Example 4.2. So, assume $\frac{d}{dx}(x^n) = n x^{n-1}$ for $n \in \mathbb{N}$. Now, from the Product Rule, $\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x)\cdot x^n + x \cdot \frac{d}{dx}(x^n) = x^n + n x^n = (n+1)x^n$ as expected. Therefore, $\frac{d}{dx}(x^n) = n x^{n-1}$ for all natural n. Furthermore, for such n, $\frac{d}{dx}(x^{-n}) = \frac{\frac{d}{dx}(1)\cdot x^n - 1\cdot \frac{d}{dx}(x^n)}{x^{2n}} = \frac{0 - n x^{n-1}}{x^{2n}} = -n x^{-n-1}$ too, according to the Quotient Rule. $\square$

Self-Proof of Theorem 4.10

Ideas $\lim_{z\to x}\frac{f(g(z)) - f(g(x))}{z-x} =$

Show $g(z) \neq g(x)$ for small enough δ

$\exists L\ \forall \varepsilon\ \exists \delta\ \forall y \in (c,d)\ \left|\frac{f(y) - f(g(x))}{y - g(x)} - L\right| < \varepsilon$

Show $= \lim_{y\to g(x)}\frac{f(y) - f(g(x))}{y - g(x)} \cdot \lim_{z\to x}\frac{g(z)-g(x)}{z-x}$

$\overset{?}{=} \lim_{z\to x}\frac{f(g(z)) - f(g(x))}{g(z) - g(x)} \cdot \lim_{z\to x}\frac{g(z)-g(x)}{z-x}$

$h : (c,d) \to \mathbb{R}$ with $h(y) := \frac{f(y) - f(g(x))}{y - g(x)}$ if $y \neq g(x)$,

$h(g(x)) := f'(g(x))$

h cont at $g(x)$ by defn

$\lim_{y\to g(x)} h(y) = \lim_{z\to x} h(g(z))$ since g cont. at x & $g[[a,b]] \subseteq (c,d)$

Proof

Exercises

4-4 This follows trivially from the other two parts of Thm 4.6 already proven (see self-proof).

4-5 See self-proof.

4-6 See self-proof

4-7 Ideas

$f(x) = |x^{n+1}|$ for odd n

1st $(n+1)x^n$ 2nd $n(n+1)x^{n-1}$ ith: $\frac{(n+1)!}{(n+1-i)!}x^{n+1-i}$ $n(n+1)x^{n-1}$ ✓ nth: $\frac{(n+1)!}{(n+1-n)!}x^{n+1-n} = (n+1)!\,x$ $(n+1)$th: $(n+1)!$

$-(n+1)!\,x$ $(n+1)!$

Proof

It is clear from the Power Rule that $\frac{d^i}{dx^i}x^n = \frac{(n+1)!}{(n+1-i)!}x^{n+1-i}$ for any $0 \le i \le n+1$; $f^{(n)}(x) = \begin{cases} (n+1)!\,x, & x \ge 0, \\ -(n+1)!\,x, & x < 0 \end{cases}$ so f is n times continuously differentiable, including at $x=0$ since the left and right limits agree. However, there is a jump discontinuity at $x=0$ for $f^{(n+1)}(x) = \begin{cases} (n+1)!, & x \ge 0 \\ -(n+1)!, & x < 0 \end{cases}$. Thus, f is not $(n+1)$-times differentiable.

4-8 Let $g(z) := z - x$; $g'(x) = 0$ is clear. Therefore, if $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists, it is identical to $\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$ by Theorem 3.14. Conversely, given the function $h(h) := x+h$, and that $\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$ exists, it's the same as $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ by the same theorem.

4-9 nx^{n-1}

Ideas $\frac{(x+h)^n - x^n}{h} = \frac{\left[\sum_{r=0}^{n}\binom{n}{r}x^{n-r}h^r\right] - x^n}{h} = \frac{\sum_{r=1}^{n}\binom{n}{r}x^{n-r}h^r}{h} = \frac{nx^{n-1}h}{h} + \sum_{r=2}^{n}\binom{n}{r}x^{n-r}h^{r-1}$

Proof

By Exercise 8 and the Binomial Theorem, $f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{nx^{n-1}h}{h} + \sum_{r=2}^{n}\binom{n}{r}x^{n-r}h^{r-1} = nx^{n-1} + \sum_{r=2}^{n}\binom{n}{r}x^{n-r}\left(\lim_{h\to 0}h\right)^{r-1} = nx^{n-1}$.

4-10 Ideas

$\frac{z-x}{(z-x)(\sqrt{z}+\sqrt{x})}$ $\frac{\sqrt{x}}{2x}$ $2x\sqrt{z} - (z+x)\sqrt{x}$

$\frac{\sqrt{z}-\sqrt{x}}{z-x} - \frac{1}{2\sqrt{x}} = \frac{2x(\sqrt{z}-\sqrt{x})}{2x(z-x)} - \frac{(z-x)\sqrt{x}}{2x(z-x)} = \frac{2x\sqrt{z}-(z+x)\sqrt{x}}{2x(z-x)}$

$= \frac{1}{\sqrt{z}+\sqrt{x}} - \frac{1}{2\sqrt{x}} < \varepsilon$

Proof We see that, by limit laws, $\lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x} = \lim_{z\to x}\frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{\left(\lim_{z\to x}\sqrt{z}\right)+\sqrt{x}} = \frac{1}{2\sqrt{x}}$.

4-12 A simple inductive proof using the Power Rule suffices. $\square$

4-13 When $n=1$, this is just the Product Rule. So assume this is true of $n \in \mathbb{N}$. Then by the Product Rule, $(fg)^{(n+1)} = \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx}\left(f^{(k)} g^{(n-k)}\right)$
$= \sum_{k=0}^{n} \binom{n}{k}\left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}\right) = \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} = \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} + \binom{n+1}{n+1} f^{(n+1)} g^{(0)} + \binom{n+1}{0} f^{(0)} g^{(n+1)}$
$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}$. Thus, the statement is true of $n+1$ too. By induction, it holds for all $n \in \mathbb{N}$, as required. $\square$

Self-Proof of Theorem 4.16

Ideas

$\frac{f(z)-f(m)}{z-m} \geq 0$ for $z \in (a,m)$ $\quad \leq 0$ for $z \in (m,b)$

Proof

For $z \in (a,m)$ we have $\frac{f(z)-f(m)}{z-m} \geq 0$ and for $z \in (m,b)$ we have $\frac{f(z)-f(m)}{z-m} \leq 0$, by the relative maximality of f at m. In other words, $\lim_{z \to m^-} \frac{f(z)-f(m)}{z-m} \geq 0$ and $\lim_{z \to m^+} \frac{f(z)-f(m)}{z-m} \leq 0$. Combined with the differentiability of f at m, which implies the equality of the above left and right limits, we see that $f'(m)=0$ is certain. $\square$

Self-Proof of Theorem 4.17 Rolle's Theorem

Since f is continuous on $[a,b]$, Theorem 3.44 informs us that it attains a (absolute) maximum and a (absolute) minimum at some M and m respectively. If $f(M)=f(m)$, f must be a constant function, in which case $f'(\frac{a+b}{2})=0$ is clear. Otherwise, when $f(M) \neq f(m)$, then $f(a)=f(b)$ implies at least one of $f(M)$ or $f(m)$ must not be $f(a)=f(b)$. Say, without loss of generality, that this refers to $f(M)$. Then, $M \in (a,b)$ is certain so that $f'(M)=0$ by Theorem 4.16. $\square$

Self - Proof of Theorem 4.18

Ideas

$g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$

$g(a) = f(a)$

$g(b) = \cancel{f(b)} - \cancel{f(b)} + f(a) = f(a)$ ✓ ·͜·

Proof

Let $g : [a,b] \to \mathbb{R}$ be the function differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$, which is defined by $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. We notice that $g(a) = f(a) = g(b)$ as expected, so Rolle's Theorem says there is a $m \in (a,b)$ with $g'(m) = 0$. In other words, $f'(m) = \frac{f(b)-f(a)}{b-a}$, just like we wanted. □

Oh yeah should have used 'c' here, but its alright too ·͜·. ✓

Self - Proof of Theorem 4.20

Assume that f is ~~nonincreasing~~ not increasing ≠ nonincreasing! Oops on $[a,b]$, thus there exists $a' < b'$ with $f(a') \geq f(b')$. By the Mean Value Theorem, there exists $c \in (a', b')$ so that $f'(c) = \frac{f(b')-f(a')}{b'-a'} \leq 0$. Therefore, $f'(x) > 0$ for all $x \in (a,b)$ is false. Taking the contrapositive gives us the desired result. □

✓ Rest is ok ·͜·!

Self - Proof of Theorem 4.21

Ideas

$f \circ f^{-1} = I$

$f'(f^{-1}(x))(f^{-1})'(x) = 1$

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$f[f^{-1}(x)] - f^{-1}(x) = x - f^{-1}(x)$

$g(x) := f(x) - x$

'reflection' : $-g(x) = x - f(x) \Rightarrow x - g(x) = 2x - f(x) \overset{?}{=} f^{-1}(x)$

$f^{-1} : f[(a,b)] \longrightarrow (a,b)$

$f : (a,b) \to f[(a,b)]$

ran $f^{-1} \subseteq$ dom f ✓

Show: f^{-1} differentiable at x

f differentiable at $f^{-1}(x)$ ✓

$\lim_{z \to x} \frac{f^{-1}(z) - f^{-1}(x)}{z - x}$

'$f(y) \to x$'? i.e. $y \to f^{-1}(x)$

$\frac{f^{-1}(f(y)) - f^{-1}(x)}{f(y) - x}$ $\quad \frac{y - f^{-1}(x)}{f(y) - x}$

Know: $f \circ f^{-1} = I$ is differentiable at x

$\lim_{z \to x} \frac{f(f^{-1}(z)) - f(f^{-1}(x))}{z - x} = (f \circ f^{-1})'(x) = \lim_{z \to x} \frac{z - x}{z - x} = 1$

$\lim_{y \to f^{-1}(x)} \frac{f(y) - f(f^{-1}(x))}{y - f^{-1}(x)} = f'(f^{-1}(x)) = \lim_{y \to f^{-1}(x)} \frac{f(y) - x}{y - f^{-1}(x)} \neq 0$

$\frac{1}{f'(f^{-1}(x))} = \frac{1}{\lim_{y \to f^{-1}(x)} \frac{f(y) - x}{y - f^{-1}(x)}} = \lim_{y \to f^{-1}(x)} \underbrace{\frac{y - f^{-1}(x)}{f(y) - x}}_{F}$

$\lim_{z \to x} f^{-1}(z) = f^{-1}(x)$ by continuity

$\underbrace{f^{-1}}_{G}[f[(a,b)] \setminus \{x\}] = (a,b) \setminus \{f^{-1}(x)\} \subseteq (a,b) \setminus \{f^{-1}(x)\}$ ✓

$\lim_{z \to x} (F \circ G)(z) = \lim_{y \to f^{-1}(x)} F$ 'L'

$\lim_{z \to x} \frac{f^{-1}(z) - f^{-1}(x)}{z - x} = \lim_{y \to f^{-1}(x)} \frac{y - f^{-1}(x)}{f(y) - x}$

Self-Proof of Theorem 4.21

Proof

Since $f'(f^{-1}(x))$ exists and is nonzero, $\underset{\frac{1}{f'(f^{-1}(x))} :=}{\wedge}$ $\lim_{y \to f^{-1}(x)} \frac{y - f^{-1}(x)}{f(y) - x}$ must also exist $\underset{\wedge}{\text{by limit laws.}}$ Furthermore, the continuity of f^{-1} fortold by Theorem 3.38 ensures $\lim_{z \to x} f^{-1}(z) = f^{-1}(x)$. Thence, $\underset{\vee}{}$ $f^{-1}[f[(a,b)] \setminus \{x\}] = (a,b) \setminus \{f^{-1}(x)\}$ ∴ (injectivity implies) $\frac{1}{f'(f^{-1}(x))} := \lim_{y \to f^{-1}(x)} \frac{y - f^{-1}(x)}{f(y) - x} = \lim_{z \to x} \frac{f^{-1}(z) - f^{-1}(x)}{z - x} := (f^{-1})'(x)$ holds from Theorem 3.14. $\square$

As such,

Actually, to be pedantic, we need Thms 3.36 & 4.16 so x is certain to not be an endpoint

---

<u>Check</u>

$f'(f^{-1}(x)) \neq 0 \Rightarrow \lim_{y \to f^{-1}(x)} \frac{f(y) - f(f^{-1}(x))}{y - x} = \lim_{y \to f^{-1}(x)} \frac{f(y) - x}{y - x} \Rightarrow \underbrace{\lim_{y \to f^{-1}(x)} \frac{y - x}{f(y) - x}}$ exists ✓ $\longrightarrow$ '$\lim_{y \to L} F(y)$'

f continuous on $(a,b) \subseteq I \Rightarrow f^{-1}$ is cont. on $(a,b) \Rightarrow \lim_{z \to x} f^{-1}(z) = f^{-1}(x)$ ✓

'$\lim_{z \to x} G(z) = L$'

$G[\underbrace{f[(a,b)]} \setminus \{x\}] = f^{-1}[f[(a,b)] \setminus \{x\}] = (a,b) \setminus \{f^{-1}(x)\}$ by injectivity as for $x' \neq x$, $f(x') \neq f^{-1}(x)$.

$\subseteq \underbrace{(a,b)} \setminus \underbrace{\{f^{-1}(x)\}}$

$G[I \setminus \{x\}] \subseteq J \setminus \underbrace{\{L\}}$ ✓

Theorem 3.14 $:\Rightarrow$ $\lim_{z \to x} (F \circ G)(z) = \lim_{y \to L} F(y)$

$$\lim_{z \to x} \frac{f^{-1}(z) - x}{\underbrace{z}_{f(f^{-1}(z)) = z} - x} = \lim_{y \to f^{-1}(x)} \frac{y - x}{f(y) - x} = \frac{1}{\lim_{y \to f^{-1}(x)} \frac{f(y) - x}{y - x}} = \frac{1}{f'(f^{-1}(x))}$$ ✓

Hmm why do we need Rolle's Thm?

Uh we don't? ... The author said it enables us to prove this thm but he didn't use it.. weird.

Self-Proof of Theorem 4.22

Ideas

Let $q \in \mathbb{Z}$, $(x^{\frac{1}{n}})^n = x$ $\quad \frac{d}{dx}(x^{\frac{1}{n}}) = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}$ $\quad \frac{p}{q} > \frac{p'}{q'}$, $pq' > p'q$ ① $\quad \lfloor \frac{p}{q} \rfloor \geq 1 \quad \lceil \frac{p}{q} \rceil \geq 2 \quad \frac{p}{q}-1>0$

$p \in \mathbb{N}$, $x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = px^{p-1} \cdot \frac{1}{q}(x^p)^{\frac{1}{q}-1}$ / $\frac{1}{q}x^{\frac{1}{q}-1} \cdot p(x^{\frac{1}{q}})^{p-1}$ $\quad \frac{z^{\frac{p}{q}} - 0^{\frac{p}{q}}}{z-0} = \frac{z^{\frac{p}{q}}}{z} = z^{\frac{p}{q}-1}$

$= \frac{p}{q}x^{p-1+\frac{p}{q}-p}$ $\quad = \frac{p}{q}x^{\frac{1}{q}-1+\frac{p}{q}-\frac{1}{q}}$ $\quad \lfloor \frac{p}{q}-1 \rfloor \geq 0 \quad \lceil \frac{p}{q}-1 \rceil \geq 1$

$= \frac{p}{q}x^{\frac{p}{q}-1}$ $\quad = \frac{p}{q}x^{\frac{p}{q}-1}$ $\quad x^{pq'} < x^{p'q}$ for $0<x<1$ $\quad |z|^{\frac{p}{q}-1} < |z|^{\frac{p-q}{q+1}}$

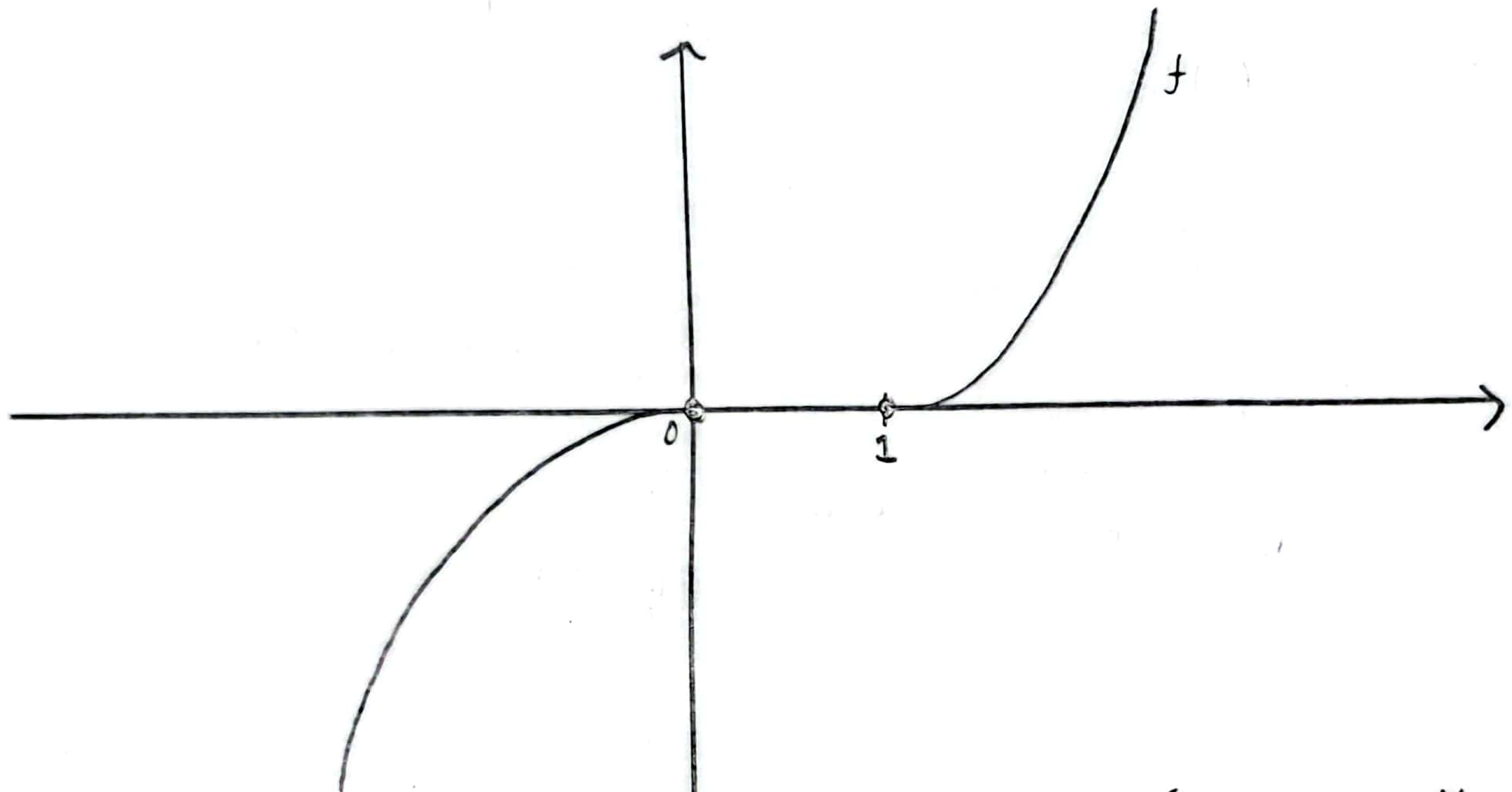
$x^{\frac{p}{q}} < x^{\frac{p'}{q'}}$ by 1-38 "q" $:= \frac{1}{qq'}$

Proof

Let $r \in \mathbb{Q} \setminus \{0\}$, so there exists $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ with $r = \frac{p}{q}$. First, notice that by Theorem 4.21, $\frac{d}{dx}(x^{\frac{1}{q}}) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q}x^{\frac{1}{q}-1}$ for each nonzero x ~~such that x^{q-1} is defined~~ ~~nonzero~~. Now, using the Chain Rule, we see that $\frac{d}{dx}(x^{\frac{p}{q}}) = \frac{d}{dx}(x^p)^{\frac{1}{q}} = px^{p-1} \cdot \frac{1}{q}(x^p)^{\frac{1}{q}-1}$ $= \frac{p}{q}x^{\frac{p}{q}-1}$, for any nonzero x. If x is zero and $\frac{p}{q}x^{\frac{p}{q}-1}$ is defined, $\frac{p}{q}-1>0$. Hence, $\left|\lim_{z\to 0} \frac{z^{\frac{p}{q}} - 0^{\frac{p}{q}}}{z-0}\right| = \lim_{z\to 0}|z^{\frac{p}{q}-1}| = 0$ by The Squeeze Theorem, since $0 \leq |z|^{\frac{p}{q}-1} \leq |z|^{\frac{p-q}{q+1}}$ for all $z \in [-1,1)$. $\square$

4-14 (a) For every point $x_m \in (0,1)$, let $\delta := \min\{x_m, 1-x_m\}$. Then, for any $x \in (x_m-\delta, x_m+\delta) \subseteq (0,1)$, $f(x_m) = f(x)$. $\square$

(b)



Depends on your intuition. It might make perfect sense to some, since no points around each $x \in (0,1)$ are larger than it. But for others, the notion of a maximum (or minimum) tends to bring across the idea of being strictly greater than; one may typically consider constant regions to not have any maximum points.

(c) We would have to exclude the case of f being a constant function in the statement of our theorem(s), which makes them less clean. Besides, the current definition does not hurt us in anyway; there's just a quirk about it for constant functions, and such functions are easy to notice and work with. $\square$

4-15 Similar to the self-proof of Theorem 4.16.

4-16 Similar to the self-proof of Theorem 4.20

4-17 Similar to the self proof of Theorem 4.20

4-18 Ideas

Assume $f'(x)>0$ $\forall x \in (a,b)$. Consider $x_1 < x_2$. $f(x_2)-f(x_1)$

Proof

Assume $f'(x)>0$ for all $x \in (a,b)$. Let us consider $x_1 < x_2$. By the Mean Value Theorem, there exists $c \in (a,b)$ so that
$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) > 0$. Therefore, $f(x_2) > f(x_1)$. Hence, f is increasing on $[a,b]$ as expected. $\square$

4-19 Notice $f'(x) = 2ax + b$ by the Power Rule. So, Theorem 4.20 suffices to give us the result desired.

4-20 Again, the Power Rule says $f'(x) = 3ax^2 + 2bx + c$. Therefore, $x = \frac{-2b \pm \sqrt{4b^2 - 4(3a)(c)}}{2(3a)} = -\frac{b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a}$ gives two points at which $f'(x) = 0$ if $4b^2 - 12ac > 0$. Call them x_l and x_r respectively. We see that $f'(x)$ is just $3a(x + \frac{b}{3a})^2 + c - \frac{b^2}{3a}$;

When $a > 0$, $f'(x) > 0$ for $x < x_l$ and $x > x_r$ while $f'(x) < 0$ for $x_l < x < x_r$. So, a simple application of Theorem 4.20 says f is increasing on $(-\infty, x_l) \cup (x_r, \infty)$ and decreasing on (x_l, x_r). Therefore, for any $x \in (-\infty, x_r)$, $f(x) \geq f(x_l)$; for all $x \in (x_l, \infty)$, $f(x) \leq f(x_r)$. In other words, x_l is a relative minimum and x_r a relative maximum.

The case where $a < 0$ can be similarly shown (or by taking $-f(x)$ and applying the above scenario) to be such that f is decreasing on $(-\infty, x_l) \cup (x_r, \infty)$ and increasing on (x_l, x_r). Correspondingly, x_l is a relative maximum and x_r a relative minimum.

Either way, the existence of 2 relative extrema is certain, provided $4b^2 - 12ac > 0$.

4-21 (a)-(b) The ($\Leftarrow$) direction can be proven with the contrapositive / using contradiction as in the self-proof / proof of Theorem 2.20.
To prove the ($\Rightarrow$) direction, take the contraposition again. In other words, assume $f'(x) < 0$ for all $x \in (a, b)$ (in the case of (a)). Then Theorem 2.20 (4-16 to be accurate) says f is decreasing on $[a, b]$, which means f is not nondecreasing. For (b), repeat a similar procedure. Hence, we establish the equivalences.

(c) The key outliers are stationary points of inflection. Consider x^3, as we know it is continuous and differentiable on $\mathbb{R}$, and also increasing on $\mathbb{R}$ — since its injective and $1^3 < 2^3$ — but its derivative at 0, $3x^2$, is $0 \ngtr 0$. So, Theorem 4.20 cannot be strengthened to a biconditional without weakening the condition of strictly increasing to nondecreasing and $f'(x) > 0$ to $f'(x) \geq 0$.

4-22 We notice that the inverse of $\sqrt{x}$ is x^2 since $\sqrt{x}^2 = x = \sqrt{x^2}$. Furthermore, x^2 is known to be differentiable for all $x \in \mathbb{R}$. Hence, Theorem 4.21 says the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

4-23 This "proof" implicitly assumes the differentiability of f^{-1} at x by using the chain rule on $f \circ f^{-1}$. However, that is a part of what we want to prove in Theorem 4.21; making the argument circular if used to claim the differentiability of f^{-1} at x.

Thm 6.7

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$, such that when $n \geq N$, we have

$$\left| \sum_{i=1}^{n+1} a_i - \sum_{i=1}^{n} a_i \right| < \varepsilon$$

$$|a_{n+1}| < \varepsilon.$$

$\square$

$$d(\omega, n) \leq d(\omega, m) + d(m, n)$$

$$\frac{1}{n} \leq (1 \leq) \frac{1}{m} + |m-n|$$

Thm 6.5

Since $b_j - a_j \geq 0$ for all j,
$$\sum_{j=1}^{\infty} b_j - a_j \geq 0$$
is clear. Hence Thm 6.4 tells us
$$\sum_{j=1}^{\infty} b_j \geq \sum_{j=1}^{\infty} a_j . \qquad \square$$

Self-Ideas

Definition Let $\{x_n\}_{n=1}^{\infty}$ be a sequence ~~converging to some limit L~~ in $\mathbb{R}$. Define m_n to be the least number such that $|x_{m_n} - L| < \frac{1}{n}$.

Version 1. We say the rate of convergence of $\{x_n\}_{n=1}^{\infty}$ is $r := \lim_{n\to\infty} \frac{m_{n+1}}{m_n}$,

Version 2 $r := \lim_{n\to\infty} \frac{m_{n+2} - m_{n+1}}{m_{n+1} - m_n}$

~~Version 2. We say the rate of convergence of $\{x_n\}_{n=1}^{\infty}$ is $r := \lim_{n\to\infty} (m_{n+1} - m_n)$.~~ $\to 0$ X

Question: Is there some range of values of r for which a sequence converges, and another for which $\sum x_n$ converges (provided $x_n \to 0$)?

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{n-1}$$

$$r_1 = \frac{n+2}{n+1} = 1 + \frac{1}{n+1} \qquad r_2 = \frac{n+3-(n+2)}{n+2-(n+1)} = \frac{1}{1} = 1$$

$\forall L \; \exists \varepsilon \; \forall N \; \exists n \geq N$
$$\left|\frac{m_{n+2} - m_{n+1}}{m_{n+1} - m_n} - L\right| \geq \varepsilon$$

Exercises

6-2 (d) $\sum_{j=4}^{\infty} \frac{2^{j-2}}{5^{j+4}} = \frac{1}{2500} \cdot \frac{2}{5} \left[\frac{1}{1-\frac{2}{5}} - \frac{1-\left(\frac{2}{5}\right)^3}{1-\frac{2}{5}} \right]$

$= \frac{4}{234375}$ ✓ Wolfram

6-3 (c) Let $a_j = 1$ and $b_j = -1$ for all j. Then,

$\sum_{j=1}^{\infty} a_j = \infty$ and $\sum_{j=1}^{\infty} b_j = -\infty$,

while

$\sum_{j=1}^{\infty} a_j + b_j = 0$.

(d) It is impossible. Assume $\sum_{j=1}^{\infty} a_j$ converges to some ~~limit~~ $S \in \mathbb{R}$, while $\sum_{j=1}^{\infty} b_j$ diverges. So, for any $L \in \mathbb{R}$, there exists $\varepsilon > 0$ and some $n \in \mathbb{N}$ for which

$\left| \sum_{j=1}^{n} a_j - S \right| < \varepsilon$ and $\left| \sum_{j=1}^{n} b_j - (L-S) \right| \geq 2\varepsilon$

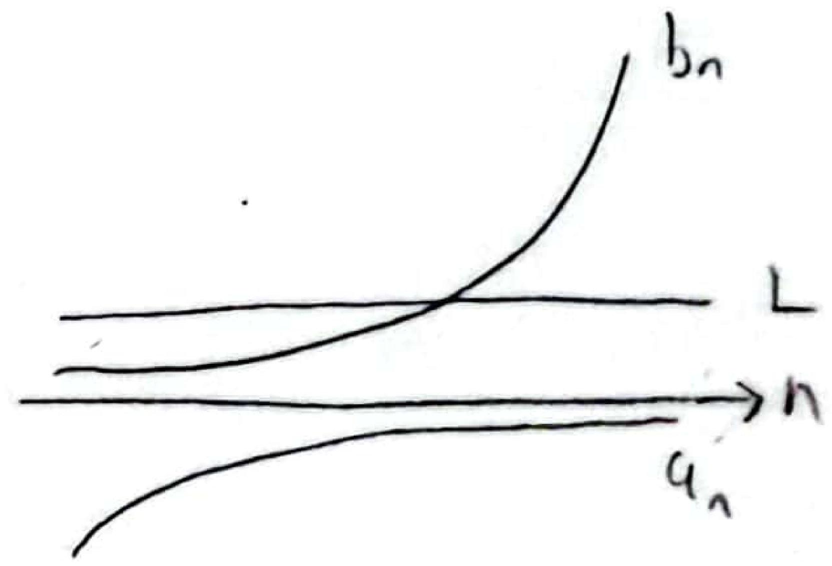
$$\left| \sum_{j=1}^{n} a_j + b_j - L \right| = \left| \left(\sum_{j=1}^{n} b_j - (L-S) \right) - \left(\sum_{j=1}^{n} a_j - S \right) \right|$$

$$\geq \left| \sum_{j=1}^{n} b_j - (L-S) \right| - \left| \sum_{j=1}^{n} a_j - S \right|$$

$$\geq 2\varepsilon - \varepsilon$$

$$= \varepsilon$$

$\square$



(e) For any divergent series $\sum_{j=1}^{\infty} a_j$, such as the harmonic series, $\sum_{j=1}^{\infty} c a_j$ trivially converges to 0 when $c = 0$.

But if $c \neq 0$, convergence of $\sum_{j=1}^{\infty} c a_j$ is clearly impossible.

6-5 (a) $0.25 = \frac{1}{4}$

(b)

$$\frac{1}{4} + \frac{1}{400} + \frac{1}{40000} + \cdots$$

$$\sum_{i=0}^{\infty} \frac{1}{4} \cdot 10^{-2i} = \frac{1}{4} \sum_{i=0}^{\infty} \left(\frac{1}{100}\right)^i = \frac{1}{4}$$

$$\sum_{i=1}^{\infty} 25 \cdot 10^{-2i} = 25 \sum_{i=1}^{\infty} \left(\frac{1}{100}\right)^i = \frac{25}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{25}{99}$$

(c)

$$\sum_{i=1}^{\infty} 9462 \cdot 10^{-4i} = \frac{9462}{10\,000} \cdot \frac{1}{1-\frac{1}{10\,000}} = \frac{9462}{9999} = \frac{3154}{3333}$$

(d)

$$\frac{1473}{9999}$$

(e)

$$12 + \frac{495}{999} \cdot 10^{-2} = \frac{1198800}{99900} + \frac{495}{99900} = \frac{1199295}{99900} = \frac{26651}{2220}$$

Self-Theorem

Let $\{d_j\}_{j=1}^{n}$ be a sequence of digits $(0, 1, \ldots, 9)$, Then,

$$0.\overline{d_1 d_2 \cdots d_n} = \frac{d_1 d_2 \cdots d_n}{\underbrace{99\cdots9}_{n \text{ times}}}.$$

$$\frac{3001}{250} \quad \frac{95}{99\,000} \quad \frac{297122750}{24750000} = \frac{1188\,491}{99\,000}$$

6-8 $\forall \varepsilon\ \exists N\ \forall n \ge N$

$$\left|\sum_{j=1}^{n} a_j - S\right| < \varepsilon \qquad |a_n| < \varepsilon$$

$$\left|\sum_{k=1}^{|m-n|} 2^k a_{2^k}\right| < \varepsilon \ ?$$

$\exists \varepsilon\ \forall N\ \exists m,n \ge N$

$$\varepsilon \le \left|\sum_{k=n+1}^{m} 2^k a_{2^k}\right| \le \left|\sum_{k=n+1}^{m-n}\right|$$

$$\left|\sum_{j=n+1}^{m-n} a_j\right| \ge \left|\sum_{k=n+1}^{m-n} a_{2^k}\right|$$

$\neg\forall L\ \exists \varepsilon\ \forall N\ \exists n \ge N$

$$\left|\sum_{k=1}^{n} 2^k a_{2^k} - L\right| \ge \varepsilon$$

$$\left|\sum_{j=1}^{m} a_j - L\right| =$$

$$\sum_{j=1}^{n} \left(\tfrac{1}{10}\right)^j = \frac{1}{10}\cdot\frac{1}{1-\frac{1}{10}} = \frac{1}{9}$$

$$\sum_{k=1}^{n} 2^k \left(\tfrac{1}{10}\right)^{2^k} =$$

Self-Proof the the harmonic series diverges
Let $N \in \mathbb{N}$.

$$\sum_{j=N+1}^{m} \frac{1}{j} = \qquad \frac{1}{j+1} - \frac{1}{j} = \frac{j-(j+1)}{j(j+1)} = \frac{-1}{j(j+1)}$$

$$\sum_{j=1}^{n} \frac{1}{j} - \frac{1}{j+1} = 1 - \frac{1}{n+1}$$

If $\sum_{j=1}^{\infty} \frac{1}{j} = S$, then $\sum_{j=1}^{\infty} \frac{1}{j+1} = S - 1$

$2^N / N^2 / 2N$

$$\sum_{j=N+1} \frac{1}{j} \ge \frac{1}{2}$$

↳ Huh. This thought was in the correct direction.

$2^0 + 1 = 1$ $\qquad 2^{0+1} = 2$

$2^1 + 1 = 3$ $\qquad 2^{1+1} = 4$

$2^2 + 1 = 5$ $\qquad 2^{2+1} = 8$

$2^3 + 1 = 9$ $\qquad 2^{3+1} = 16$

2^k

$2^k + 1 \qquad 2^{k+1}$

$$2^{k+1} - 2^k - 1$$
$$= 2^k - 1$$
$$\ge 0 \quad \text{for all } k$$

$2^k + 1 \le 2^{k+1}$ ✓

6-8

$(\Rightarrow)$ $\forall \varepsilon \exists N \forall n \geq N$

$|a_n| < \varepsilon$ $\quad \left| \sum_{j=1}^{n} a_j - S \right| < \varepsilon$

$\forall n: \ 0 \leq a_{n+1} \leq a_n$

$(\Leftarrow)$ $\exists \varepsilon \forall N \exists n,m \geq N:$

If $\sum_{j=1}^{\infty} a_j$ diverges: $\quad \sum_{j=n+1}^{m} a_j \geq \varepsilon$

$\sum_{j=1}^{\infty} a_j = \infty$ because $\forall n: 0 \leq a_{n+1} \leq a_n$

Let $M > 0$. If $\forall N \exists n \geq N$

$\sum_{j=1}^{n} a_j < M,$

Fix $N=1$, then $\forall m \geq n$

$\sum_{j=1}^{n} a_j < M$ by nonnegativity

A contradiction.

E.g.

$$\sum_{k=1}^{n} 2^k \cdot \frac{1}{2^k} = \sum_{k=1}^{n} 1 \geq \sum_{j=1}^{n} a_j$$

$$\sum_{k=1}^{n} 2^k a_{2^k} \geq \sum_{k=1}^{n} a_{2^k} \geq \cancel{\sum_{j=1}^{n} a_j}$$ ~~since a_j is nonincreasing.~~

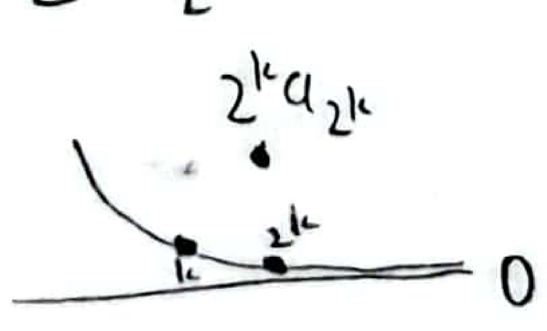
$$\sum_{k=1}^{n} 2^k a_{2^k} \geq \sum_{k=1}^{n} 2 a_{2^n} = 2n a_{2^n}$$

$\exists k:$

$2^k a_{2^k} \geq a_k$

If $2^k a_{2^k} < a_k \ \forall k$

then $\sum a_k$ must diverge, lest from the above we have that $\sum 2^k a_{2^k}$ converges.



$(\Leftarrow)$

$\exists E \forall \varepsilon \exists N \forall n \geq N$

$0 \leq E - \sum_{k=1}^{n} 2^k a_{2^k} < \varepsilon$ | by nonnegativity

$\forall \varepsilon \exists K \forall k \geq K$

$2^k a_{2^k} < \varepsilon$

$a_{2^k} < \varepsilon$

~~$\exists N \forall n \geq N$~~

$$\cancel{\sum_{k=1}^{n} 2^k a_{2^k} \geq \sum_{j=1}^{2^n} a_j \geq 0}\ ?$$

$(\Rightarrow)$ Assume $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges.

$$\sum_{j=1}^{2^n} a_j = a_1 + \sum_{j=2}^{2^n} a_j$$

$$= a_1 + \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} a_j$$

$$\geq a_1 + \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} a_{2^{k+1}}$$ since $\{a_j\}_{j=1}^{\infty}$ is nonincreasing

$$\geq a_1 + \sum_{k=0}^{n-1} 2^k a_{2^{k+1}}$$

$$\geq a_1 + \frac{1}{2} \sum_{k=0}^{n} 2^k a_{2^k} - \frac{1}{2} a_2$$

$\vdots$ $\quad 2^k$

$2^k + 1 \quad 2^{k+1}$

$2^{k+1} + 1 \quad \vdots$

$\forall j \exists n$

$\sum_{k=1}^{n} a_{2^k} \geq a_j$?

$\exists j \forall n$

$\sum_{k=1}^{n} a_{2^k} < a_j$

$$\sum_{j=1}^{2^n} a_j = a_1 + \sum_{k=0}^{n-1} \sum_{j=2^{k+1}}^{2^{k+1}} a_j$$

$$\leq a_1 + \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} a_{2^k}$$

$$\leq a_1 + \sum_{k=0}^{n-1} 2^k a_{2^k}$$

6-8 $\forall \varepsilon \, \exists N \, \forall n \geq N$

$$\left| \sum_{j=1}^{n} a_j - S \right| < \varepsilon \qquad |a_n| < \varepsilon$$

$$\left| \sum_{k=1}^{|m-n|} 2^k a_{2^k} \right| < \varepsilon \ ?$$

$\exists \varepsilon \, \forall N \, \exists m, n \geq N$

$$\varepsilon \leq \left| \sum_{k=n+1}^{m} 2^k a_{2^k} \right| \leq \left| \sum_{k=n+1}^{m-n} \right|$$

$$\left| \sum_{j=n+1}^{m-n} a_j \right| \geq \left| \sum_{k=n+1}^{m-n} a_{2^k} \right|$$

$\forall L \, \exists \varepsilon \, \forall N \, \exists n \geq N$

$$\left| \sum_{k=1}^{n} 2^k a_{2^k} - L \right| \geq \varepsilon$$

$$\left| \sum_{j=1}^{m} a_j - L \right|$$

$$\sum_{j=1}^{n} \left(\frac{1}{10}\right)^j = \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{9}$$

$$\sum_{k=1}^{n} 2^k \left(\frac{1}{10}\right)^{2^k} =$$

Self-Proof the the harmonic series diverges.
Let $N \in \mathbb{N}$.

$$\sum_{j=N+1}^{m} \frac{1}{j} = \frac{1}{j+1} - \frac{1}{j} = \frac{j-(j+1)}{j(j+1)} = \frac{-1}{j(j+1)}$$

$$\sum_{j=1}^{n} \frac{1}{j} - \frac{1}{j+1} = 1 - \frac{1}{n+1}$$

If $\sum_{j=1}^{\infty} \frac{1}{j} = S$, then $\sum_{j=1}^{\infty} \frac{1}{j+1} = S - 1$

$2^N / N^2 /$

$$\sum_{j=N+1}^{2N} \frac{1}{j} \geq \frac{1}{2}$$

↳ Huh. This thought was in the correct direction.

$2^0 + 1 = 1$	$2^{0+1} = 2$
$2^1 + 1 = 3$	$2^{1+1} = 4$
$2^2 + 1 = 5$	$2^{2+1} = 8$
$2^3 + 1 = 9$	$2^{3+1} = 16$
	2^k

$$\frac{2^k + 1 \qquad 2^{k+1}}{2^{k+1} - 2^k - 1}$$

$= 2^k - 1$

≥ 0 for all k

$2^k + 1 \leq 2^{k+1}$ ✓

$m_i' = \min\{|m_i|, |M_i|\}$

$M_i' = \max\{|m_i|, |M_i|\}$

$= |m_i| + |M_i| - m_i'$

$$U(|g|, P) - L(|g|, P) = \sum(\overbrace{M_i' - m_i'}^{|m_i| + |M_i| - 2m_i'})\Delta x_i$$

$$< \sum(|m_i| + |M_i| - 2m_i')\Delta x_i$$

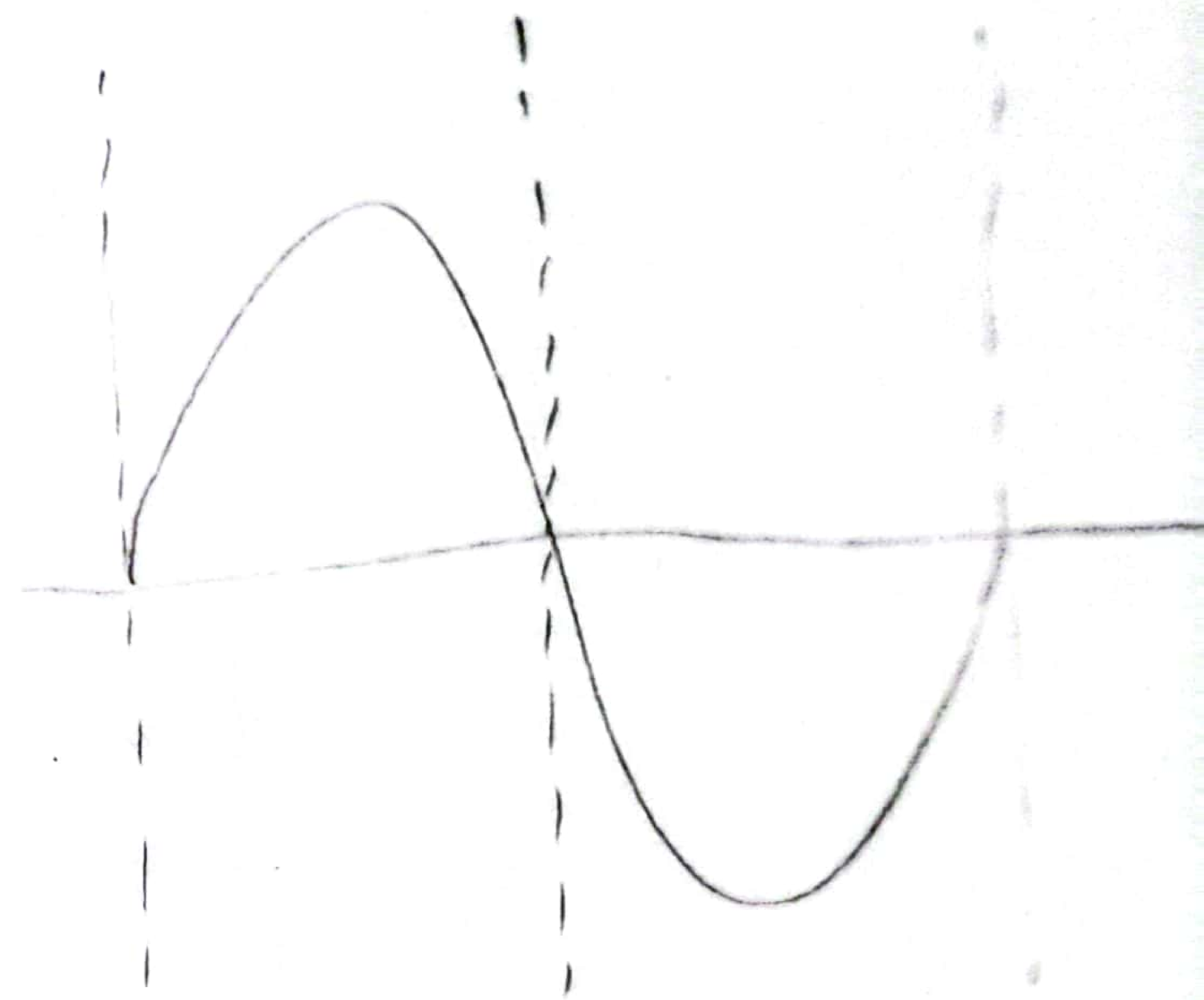
$$< \sum((|m_i| - m_i') + (|M_i| - m_i'))\Delta x_i$$

$$\leq \sum((|m_i| - m_i') + (|M_i| - m_i'))$$

wlog, $m_i' = |m_i|$

$$= \sum |M_i| - |m_i|$$

$\varepsilon > \frac{1}{n}$



$\lim_{k \to \infty} L(|f - s_n|, P_k) < \varepsilon$?

$\inf\{|f(x) - s_n(x)| \mid x \in [x_{i-1}^{(k)}, x_i^{(k)}]\}$

$I_n - L(|f - s_n|, P) < U(f, P) - L(f, P)$?

Exercises, Ideas

5-26 (a)

i. Let $n \in \mathbb{N}$ and, wlog, $x \in [x_{i-1}^{(k)}, x_i^{(k)})$. Then, $s_n(x) \doteq m_i^{(k)} \leq f(x)$.

ii. let f be continuous at $x \in [a,b]$. ~~Then, $\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} \inf\{f(y) \mid y \in [x_{i-1}^{(m)}, x_i^{(m)}]\}$~~

let $\varepsilon > 0$ and $\delta > \frac{1}{K}$. For $k \geq K$,

$$f(x) - s_n(x) = f(x) - m_i^{(k)} \leq |f(y) - f(x)| + |f(y) - m_i^{(k)}| < \varepsilon$$

$\exists \delta > 0 : \forall z : |z - x| < \delta \Rightarrow$
$|f(z) - f(x)| < \frac{1}{2}\varepsilon$

$\exists z \in [x_{i-1}^{(k)}, x_i^{(k)}]$:
$f(z) - s_n(x) < \frac{1}{2}\varepsilon$
$z \in [x - \frac{1}{k}, x + \frac{1}{k}]$

iii. $\forall \varepsilon \exists N \forall n \geq N$:

$\int_a^b |f - c_n| \, dx < \varepsilon$?

$$\underbrace{\int_a^b |f - c_n| \, dx}_{I_n} \leq \left| \sum_{i=1}^{n} |f(t_i) - c_n(t_i)| \Delta x_i - I_n \right| + \sum_{i=1}^{n} |f(t_i) - c_n(t_i)| \Delta x_i$$

$$I_n \leq \left| L(|f - c_n|, Q) - I_n \right| + \underbrace{L(|f - c_n|, Q)}_{0}$$

Integrability of $f - c_n$: Integral 'laws'

$|f - c_n|$: Show $\int f \Rightarrow \int |f|$

Let $\varepsilon > 0$. $\exists P \ldots$

$(0 \leq)\, U(f,P) - L(f,P) < \varepsilon$.

$(0 \leq)\, \sum (M_i - m_i) \Delta x_i < \varepsilon$

Find a partition Q of $[a,b]$ s.t. $L(|f - c_n|, Q) < \frac{1}{2}\varepsilon$

Let $Q = P_n$:

$L(|f - c_n|, P_n) = 0$?

$$l_i := \inf\{|f(x) - \overbrace{c_n(x)}^{m_i^{(n)}}| \mid x \in [x_{i-1}^{(n)}, x_i^{(n)}]\}$$
$= 0$?

Let $\varepsilon > 0$.

There exists $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$ s.t. $f(x) - m_i^{(n)} < \varepsilon$

$|m_i^{(n)} - f(x)| < \varepsilon$

$\Rightarrow l_i = 0$

Lemma

For any sequence of partitions $\{P_k\}_{k=1}^{\infty}$ with norm $\|P\|$ converging to 0,

$$\lim_{k\to\infty} L(g, P_k) = \int_a^b g(x) \, dx.$$

$$|L(g, P_k) - I| \leq |L(g, P_k) - R(g, P_k, T)| + |R(g, P_k, T) - I|$$

$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$

$= \varepsilon$

$L \leq R \leq U$ $\quad \frac{1}{2}\varepsilon \quad \frac{1}{2k}\varepsilon$

exists t_i s.t.

$g(t_i) - m_i < \frac{\varepsilon}{2k\Delta x_i}$

$$R(g, P_k, T) < \sum_{i=1}^{k} \left(m_i + \frac{\varepsilon}{2k\Delta x_i} \right) \Delta x_i$$
$$< \sum_{i=1}^{k} m_i \Delta x_i + \sum_{i=1}^{k} \frac{\varepsilon}{2k}$$
$$< L(g, P_k) + \frac{1}{2}\varepsilon$$

5-26 (b)

$y - b = \frac{b-d}{a-c}(x-a)$

Results similar to (a) can be proven for

$S_n := \left(\cdots M_i^{(n)} \cdots \right) + \cdots$,

where $M_i^{(n)} = \sup \cdots$

Let $x \in [x_{i-1}^{(n)}, x_i^{(n)}]$. There exists a_i and A_i in $[x_{i-1}^{(n)}, x_i^{(n)}]$, for which

$f(a_i) - m_i^{(n)} < \frac{1}{n}$ and $M_i^{(n)} - f(A_i) < \frac{1}{n}$

Define

$$c_n(x) := \begin{cases} f(A_i) + \frac{f(A_i) - f(a_{i-1})}{A_i - a_{i-1}}(x - A_i) & \text{if} \\ & \end{cases}$$

$$c_n(x) := f(a_i) + \frac{f(a_i) - f(a_{i-1})}{a_i - a_{i-1}}(x - a_i)$$

5-2 6 (a) iii. $\forall \varepsilon \exists N \forall n \geq N$

Show $\lim_{k \to \infty} L(|f - s_n|, Q_{n,k}) < \varepsilon$ OR $\forall \varepsilon \exists N \forall n \geq N$ $\int_a^b |f - s_n| \, dx < \varepsilon$

Let $Q_{n,k}$ be a refinement of P_k with $\|Q\| \to 0$. ~~converging~~

$\forall \varepsilon \exists N \forall n \geq N \, \forall \varepsilon', \exists \delta > 0 \, \forall P : \|P\| < \delta \, \forall T$

$l_i := \inf \{ |f(x) - s_n(x)| \mid x \in [x_{i-1}^{(n,k)}, x_i^{(n,k)}] \}$

E.g. $Q_{n,k} := P_n \cup P_k$

By integrability,

$\exists K \forall k \geq K$,

$L(f, P_k) = \int_a^b s_k(x) \, dx$

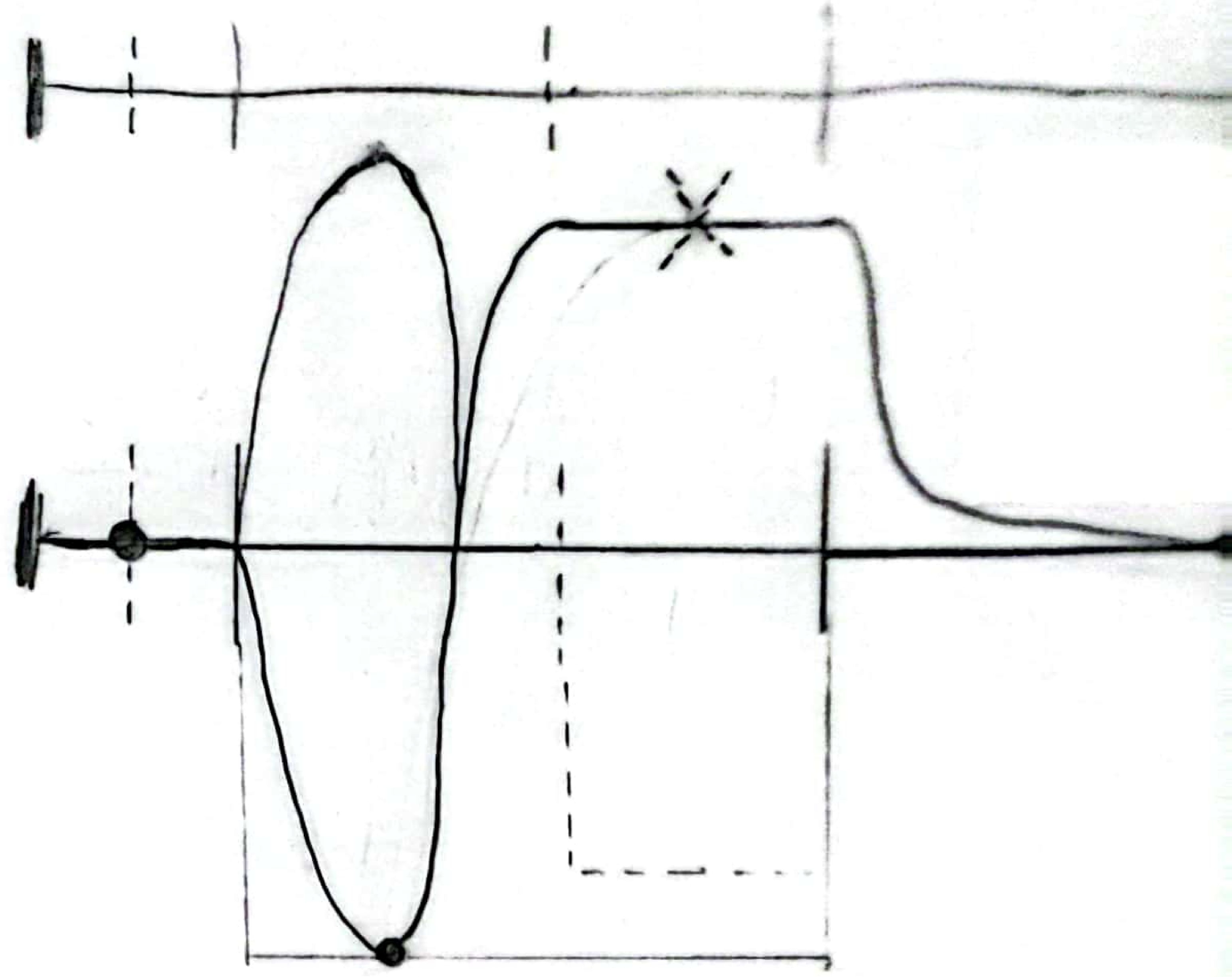
$\left| R(f, P_k, T_k) - \int_a^b f(x) \, dx \right| < \varepsilon$

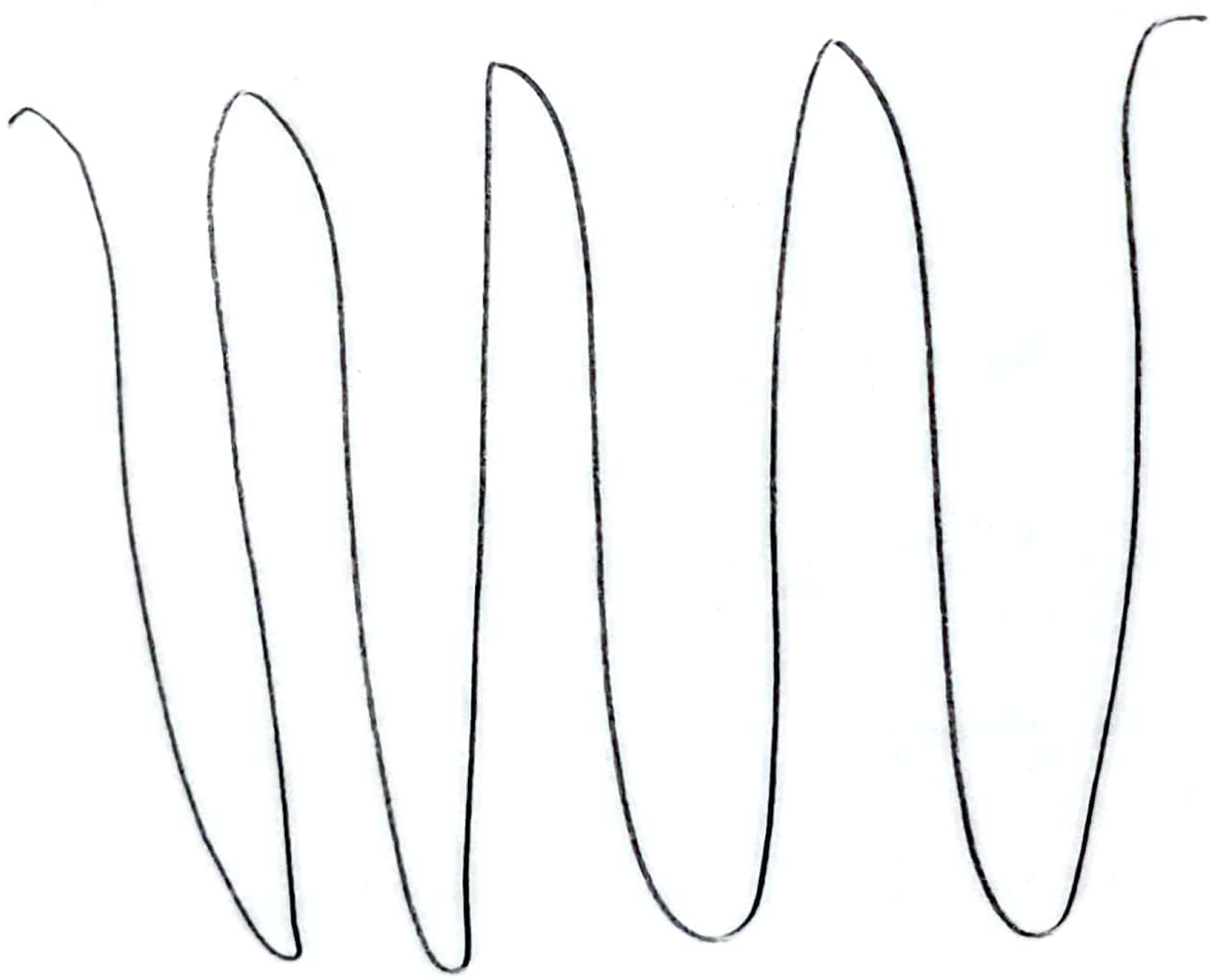
$\int_a^b f(x) \, dx - L(f, P_k) < \varepsilon$

$\left| \int_a^b f(x) - \int_a^b s_n(x) \, dx \right|$

$\leq \left| \int_a^b f(x) \, dx - L(f, P_k) \right|$

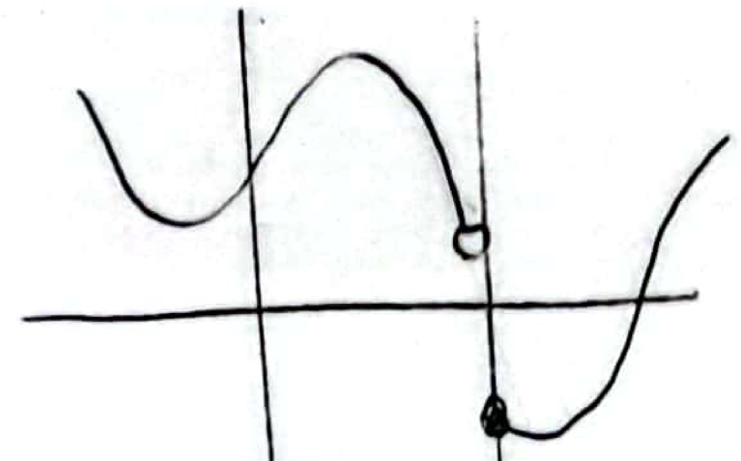
$+ \, |$





$\exists P : \forall i : \; f[x_{i-1}, x_i] \subseteq \mathbb{R}^+_0$

or $f[x_{i-1}, x_i] \supseteq \mathbb{R}^-_0$



$y - b = \frac{b-d}{a-c}(x - a)$

$\forall P \exists i$

partition Q of $[a, b]$ s.t.

2-51

$G = \mathcal{J}_x$ and $y = \frac{a_n - a_{n-1}}{b_n - b_{n-1}}(x - b_n) + a_n$

$\forall \varepsilon \exists N \forall n \geq N$

$$\left| \frac{a_n - a_{n-1}}{b_n - b_{n-1}} - c \right| < \varepsilon$$

$$\frac{\Delta a_n}{\Delta b_n} \sim c$$

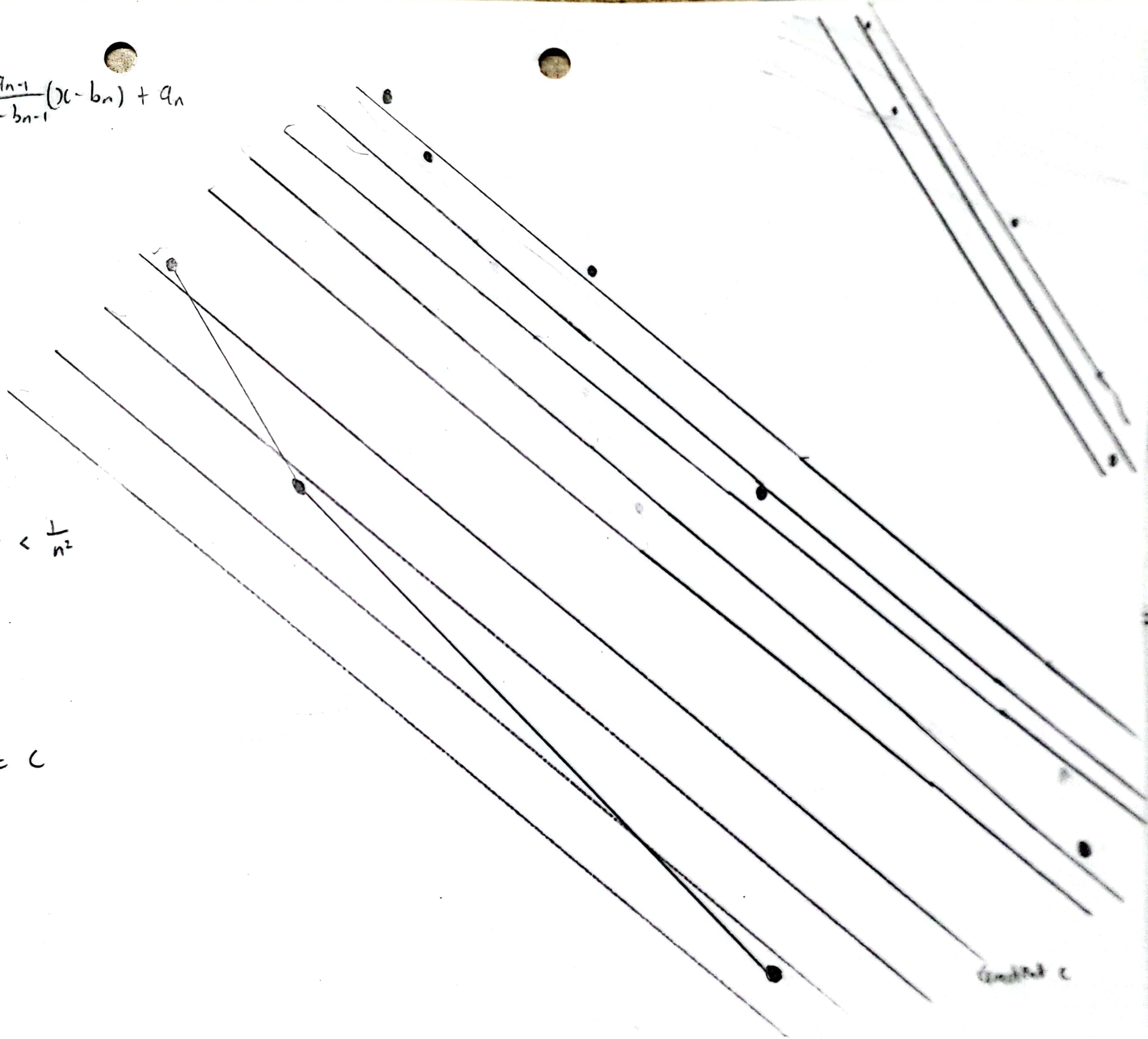
$$\Delta a_n \sim c \Delta b_n$$

$\exists n \forall M \exists m \geq M \forall x$

$$(a_m - x)^2 + (b_m - \beta + c(\alpha - x))^2 < \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} b_n} = 0 \checkmark$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} b_k \cdot \frac{a_k - a_{k-1}}{b_k - b_{k-1}}}{\sum_{k=1}^{n} b_k} = c$$



2-51

$$\frac{a_k}{b_k} - \frac{a_{k-1}}{b_{k-1}} = \frac{a_k b_{k-1} - a_{k-1} b_k}{b_k b_{k-1}}$$

$$\frac{a_k}{b_k} - \frac{a_k - a_{k-1}}{b_k - b_{k-1}} = \frac{\cancel{a_k b_k} - a_k b_{k-1} \; \cancel{- a_k b_k} + a_{k-1} b_k}{b_k (b_k - b_{k-1})}$$

$$= \frac{a_{k-1} b_k - a_k b_{k-1}}{b_k (b_k - b_{k-1})}$$

$b_{n+1} - b_1$

$$\lim_{n\to\infty} b_n = \infty$$

$$\overset{1}{\cancel{b_2} - b_1} + \overset{2}{\cancel{b_3} - \cancel{b_2}} + \overset{3}{b_4 - \cancel{b_3}}$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} b_{k+1} - b_k\right) = \lim_{n\to\infty}(b_{n+1} - b_1) = \infty$$

Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$

$$\left| \frac{a_{n+1}}{b_{n+1} - b_1} - c \right| < \varepsilon$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} (b_{k+1} - b_k)\left(\frac{a_{k+1} - a_k}{b_{k+1} - b_k}\right)}{\sum_{k=1}^{n} (b_{k+1} - b_k)} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_{k+1} - a_k}{\sum_{k=1}^{n} (b_{k+1} - b_k)} = c$$

$$\lim_{n\to\infty} \frac{a_{n+1} - a_1}{b_{n+1} - b_1} = c$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{b_{n+1} - b_1} = c$$

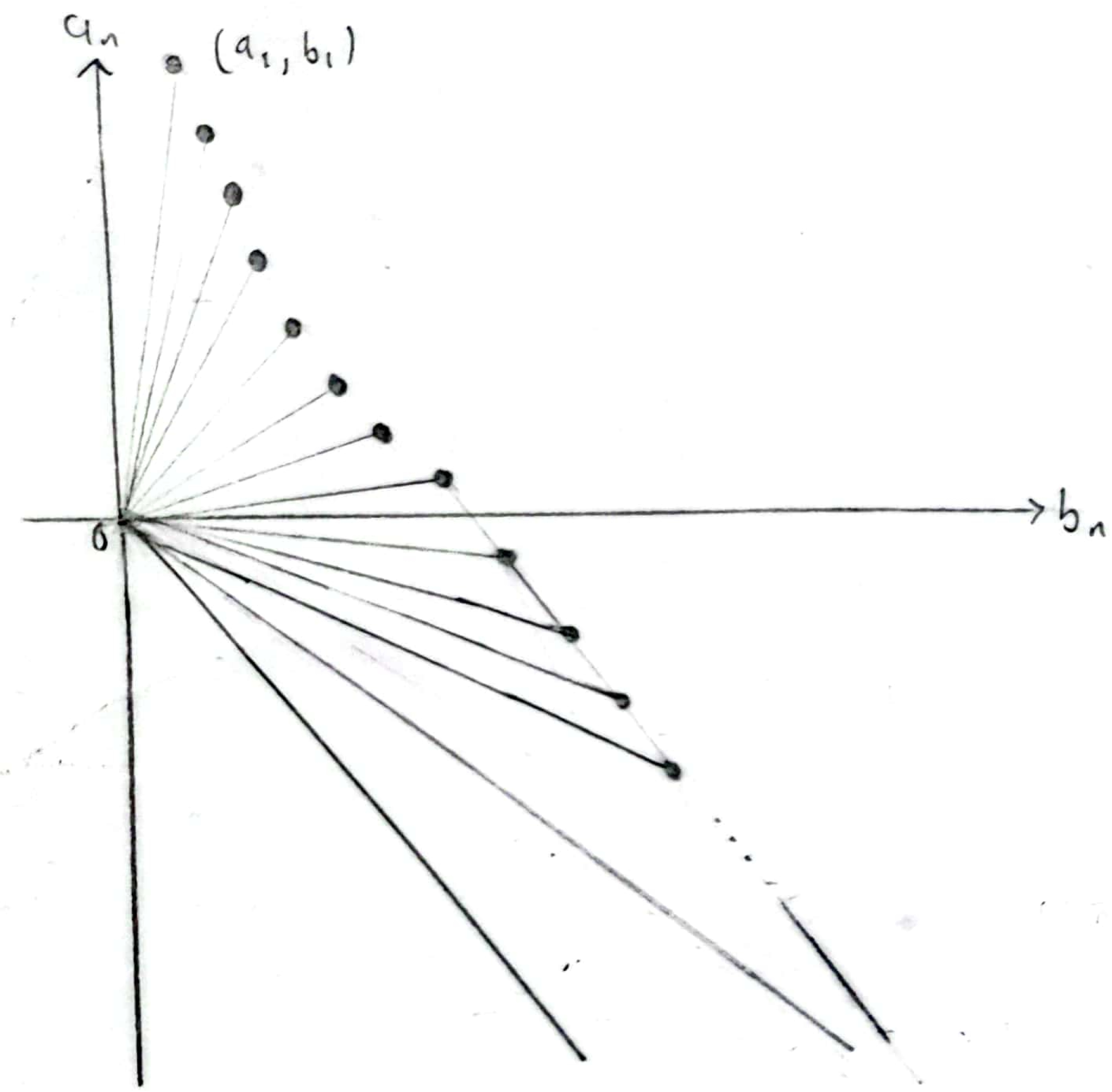
$$\left| \frac{a_{n+1}}{b_{n+1} - b_1} - \frac{a_{n+1}}{b_{n+1}} \right| < \varepsilon$$

$$\cancel{b_{n+1} - b_{n+1} + b_1}$$

$$\frac{a_{n+1} b_1}{b_{n+1}(b_{n+1} - b_1)}$$

$$= b_1 \cdot \frac{a_{n+1}}{b_{n+1} - b_1} \cdot \frac{1}{b_{n+1}}$$

2-51



E.g. $a_n = 2+n$, $b_n =$

$$\frac{2+n-(2+n-1)}{n-(n-1)} =$$

$a_n = n^2$, $b_n = e^n$

$$\frac{n^2-(n-1)^2}{e^n - e^{n-1}} = \frac{2n-1}{e^n(1-e^{-1})} \to 0$$

$a_n = 1$, $b_n = 1 - \frac{1}{n} \not\to \infty$

$$\frac{1-1}{1-\frac{1}{n}} = 0 \quad \frac{a_n}{b_n} = \frac{1}{1-\frac{1}{n}} = \frac{n}{n-1} = 1 + \frac{1}{n-1}$$

Assume that $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = c$

Let $\varepsilon > 0$.

$$\frac{a_n b_n - \underbrace{a_n b_{n-1}} - a_n b_n + \underbrace{a_{n-1} b_n}}{b_n(b_n - b_{n-1})}$$

$$\left|\frac{a_n}{b_n} - c\right| \le \left|\overbrace{\frac{a_n}{b_n} - \frac{a_n - a_{n-1}}{b_n - b_{n-1}}}\right| + \left|\frac{a_n - a_{n-1}}{b_n - b_{n-1}} - c\right|$$

$$y = \frac{a_n}{b_n} x \qquad y = \frac{a_n - a_{n-1}}{b_n - b_{n-1}}(x - b_n) + a_n$$

$b_n \to \infty$, $b_n > b_{n-1}$ $\quad \forall n \exists M \forall m \ge M, (b_m, a_m) \in S_n$?

$f(x) = c(x-\alpha) + \beta$ $\quad \exists n \forall M \exists m \ge M, (b_m, a_m) \notin S_n$?

$S_n = \{ Y \in N_{1/n}(X) \mid X \in f \}$.

Find some pt that is of sufficiently close gradient and which is far enough away from the origin...

$$y = Gx \qquad y = \frac{a_n - a_{n-1}}{b_n - b_{n-1}}(x - b_n) + a_n$$

$$Gx = \frac{a_n - a_{n-1}}{b_n - b_{n-1}}(x - b_n) + a_n$$

$$\left(G - \frac{a_n - a_{n-1}}{b_n - b_{n-1}}\right) x = a_n - b_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$$

$$x = \frac{a_n - b_n \frac{a_n - a_{n-1}}{b_n - b_{n-1}}}{G - \frac{a_n - a_{n-1}}{b_n - b_{n-1}}}$$

$$= \frac{\cancel{a_n b_n} - a_n b_{n-1} - \cancel{a_n b_n} + a_{n-1} b_n}{G(b_n - b_{n-1}) - a_n + a_{n-1}}$$

$$= \frac{a_{n-1} b_n - a_n b_{n-1}}{G(b_n - b_{n-1}) - a_n + a_{n-1}}$$

Self-Proof of Lemma 5.6

$\forall \varepsilon \exists N \forall k \geq N \quad \big| \|P_k\| \big| < \varepsilon$

Ideas

Show $\forall \varepsilon \exists N \forall k \geq N$ $|R(f, P_k, T_k) - I| < \varepsilon$

Know: $\forall \varepsilon \exists \delta \; \forall P \forall T \cdot \; \|P\| < \delta \Rightarrow |R(f, P, T) - I| < \varepsilon$

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - I \right| < \varepsilon$$

Let $\varepsilon > 0$, $\exists \delta \cdots \|P\| < \delta \Rightarrow \cdots$. $\exists N \forall k \geq N \; \|P_k\| < \delta$.

Proof

Let $\varepsilon > 0$. Since $f : [a, b] \to \mathbb{R}$ is Riemann integrable, there exists $I \in \mathbb{R}$ and $\delta > 0$ such that for all partitions P and all evaluation sets T, if $\|P\| < \delta$ then $|R(f, P, T) - I| < \varepsilon$. Hence, by virtue of $\lim_{k \to \infty} \|P_k\| = 0$, there must be some $K \in \mathbb{N}$ so, for each $k \geq K$, $\big| \|P_k\| \big| = \|P_k\| < \varepsilon$. As such, from the first sentence we now have $|R(f, P_k, T_k) - I| < \varepsilon$. In other words, $\lim_{k \to \infty} R(f, P_k, T_k) = I =: \int_a^b f(x)\,dx$. □

Example 5.7

Ideas

$P_k := \{0 < \frac{1}{k} < \frac{2}{k} < \cdots < \frac{k-1}{k} < 1\}$, $T_k := \{ f(\frac{i}{k}) \mid 1 \leq i \leq k \}$

$\|P_k\| = \frac{1}{k}$ so $\lim_{k \to \infty} \|P_k\| = 0$ is trivial.

$$\lim_{k \to \infty} \left(\sum_{i=1}^{k} \frac{i}{k} \cdot \frac{1}{k} \right) = \lim_{k \to \infty} \left(\frac{1}{k^2} \cdot \sum_{i=1}^{k} i \right) = \lim_{k \to \infty} \frac{1}{k^2} \cdot \frac{k(k+1)}{2} = \frac{1}{2} \lim_{k \to \infty} \left(1 + \frac{1}{k}\right) = \frac{1}{2}$$

Proof

If $f(x) = x$ is Riemann integrable on $[0, 1]$, then define the partition $P_k := \{0 < \frac{1}{k} < \frac{2}{k} < \cdots < \frac{k-1}{k} < 1\}$ and associated evaluation set $T_k := \{ f(\frac{i}{k}) \mid 1 \leq i \leq k \}$. So, by noticing $\lim_{k \to \infty} \|P_k\| = \lim_{k \to \infty} \frac{1}{k} = 0$, we apply Lemma 5.6 to see that $\lim_{k \to \infty} R(f, P_k, T_k) =$

$$\lim_{k \to \infty} \left(\sum_{i=1}^{k} \frac{i}{k} \cdot \frac{1}{k} \right) = \lim_{k \to \infty} \left(\frac{1}{k^2} \cdot \sum_{i=1}^{k} i \right) = \lim_{k \to \infty} \frac{1}{k^2} \cdot \frac{k(k+1)}{2} = \frac{1}{2} \lim_{k \to \infty} \left(1 + \frac{1}{k}\right) = \frac{1}{2} = \int_a^b x\,dx.$$ □

Self-Proof of Theorem 5.8

Ideas

$$\left|\sum_{i=1}^{n}(f+g)(t_i)\Delta x_i - I_f - I_g\right| = \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - I_f + \sum_{i=1}^{n} g(t_i)\Delta x_i - I_g\right|$$

$$\leq \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - I_f\right| + \left|\sum_{i=1}^{n} g(t_i)\Delta x_i - I_g\right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

wlog, $c \neq 0$:

$$\left|\sum_{i=1}^{n} cf(t_i)\Delta x_i - cI_f\right| \leq |c|\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - I_f\right|$$

$$\leq |c|\cdot\frac{\varepsilon}{|c|}$$

$$= \varepsilon$$

OR: Lemma 5.6

$$\lim_{k\to\infty}\sum_{i=1}^{k}(f+g)(t_i)\Delta x_i = \lim_{k\to\infty}\sum_{i=1}^{k} f(t_i)\Delta x_i + \lim_{k\to\infty}\sum_{i=1}^{k} g(t_i)\Delta x_i$$

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\lim_{k\to\infty}\sum_{i=1}^{k} cf(t_i)\Delta x_i = c\lim_{k\to\infty} f(t_i)\Delta x_i$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

(Sequence of partition can be chosen with ...

i.e. $x_{i_2+1} - x_{i_1-2} > x_{i_2} - x_{i_1-1}$

as $x_{i_2+1} > x_{i_2}$ & $x_{i_1-2} < x_{i_1-1}$

$x_{i_2-1} - x_{i_1} < x_{i_2} - x_{i_1-1}$

as $x_{i_2} - 1 \le x_{i_2}$ & $x_{i_1} > x_{i_1-1}$

Self-Proof of Proposition 5.10

$t_{i_1} \le t_{i_2}$

~~$t_{i_1} \in [c, x_{j+1}]$~~ ~~$t_{i_2} \in [x_{k-2}, d]$~~

Ideas $1 \le j \le k \le n$

For some ~~t~~, $c \in [x_{j-1}, x_j]$ $d \in [x_{k-1}, x_k]$

least i_1 greatest i_2 so $t_{i_1} \ge c$ & $t_{i_2} \le d$

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - (d-c)\right| < \varepsilon \ ?$$

$t_{i_1} \in [x_{i_1-1}, x_{i_1}]$ $t_{i_2} \in [x_{i_2-1}, x_{i_2}]$

$$\left|\left(\sum_{i=i_1}^{i_2} x_i - x_{i-1}\right) - (d-c)\right| = \left|(x_{i_2} - x_{i_1-1}) - (d-c)\right|$$

By least i_1, $x_{i_1-2} \le c$ & $x_{i_2+1} \ge d$ as $c \in [x_{i_1-2}, x_{i_1}]$

$d \in [x_{i_2-1}, x_{i_2+1}]$

$\le |x_{i_2} - d| + |x_{i_1-1} - c|$

$= |x_k - d| + |x_{j-1} - c|$

(x_{i_2-1}, x_{i_2+1})

$$\sum_{i=i_1}^{i_2} x_{i-1} = \sum_{i=i_1-1}^{i_2-1} x_i$$

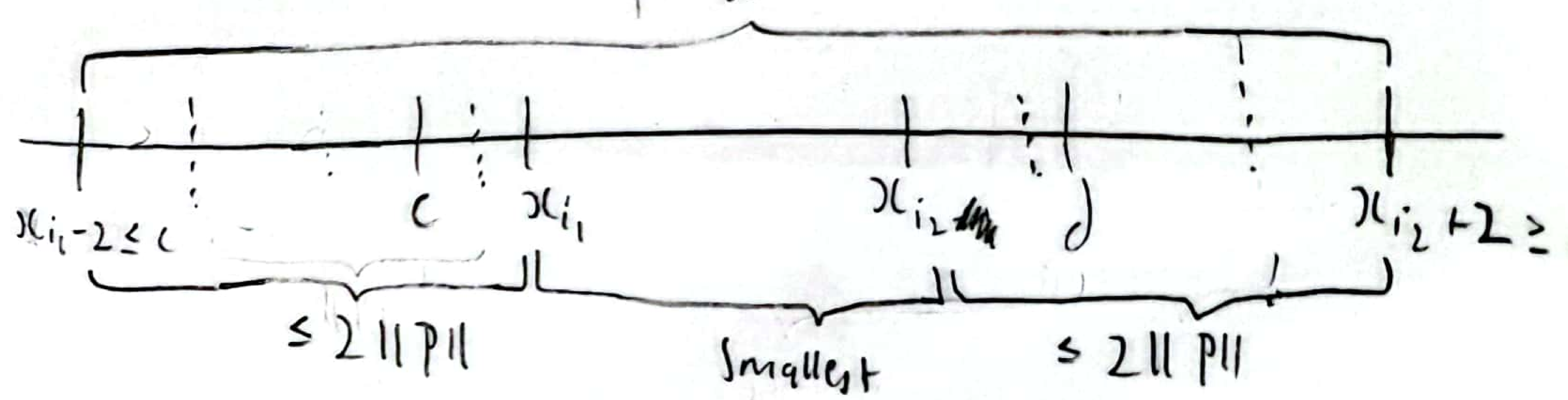
Find δ

$x_{i_2} - x_{i_1-1} \le (i_2 - i_1 + 1)\|P\| <$

$$\sum_{i=i_1}^{i_2} - \sum_{i=i_1-1}^{i_2-1} = x_{i_2} - x_{i_1-1} \le \|P\| < \delta$$

$= x_k - x_{j-1}$

$i_2 - 1 - i_1$ $i_2 + 1 - (i_1 - 2)$ $i_2 - i_1 + 3$



$$x_{i_2-1} - x_{i_1} < x_{i_2} - x_{i_1-1} < x_{i_2+1} - x_{i_1-2} \le (i_2 - i_1 + 3)\|P\| < d - c + \varepsilon$$

$$\|P\| < \frac{d-c+\varepsilon}{i_2 - i_1 + 3}$$

Proof

Let $\varepsilon > 0$, P be any partition with norm $\|P\| < \min\left\{\frac{\varepsilon}{2}, \frac{d-c}{3}\right\}$ and T be any corresponding evaluation set.

Further let $1 \le j \le k$ be the least j and greatest k, respectively, so $t_j \ge c$ and $t_k < d$. Hence, notice $x_{j-1}, c \in [x_{j-2}, x_j]$ and $x_k, d \in [x_k, x_{k+2}]$. In other words, $|x_{j-1} - c| \le |x_j - x_{j-2}| \le 2\|P\| < \frac{\varepsilon}{2}$ and $|x_k - d| \le |x_{k+2} - x_k| \le 2\|P\| < \frac{\varepsilon}{2}$. Thus,

(or if $j=1$, or $k=n$ ...)

$$\left|\sum_{i=1}^{n} \mathbf{1}_{[c,d)}(t_i)(x_i - x_{i-1}) - (d-c)\right| = \left|\left(\sum_{i=j}^{k} x_i - x_{i-1}\right) - (d-c)\right| = |x_k - x_{j-1} - d + c| \le |x_k - d| + |x_{j-1} - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as expected. Consequently, $\int_a^b \mathbf{1}_{[c,d)}\, dx = d - c$. $\square$

Self - Proof of Proposition 5.12 / Ex 5.5

By Theorem 5.8 and Proposition 5.10.

Exercises

5-1

~~Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If f is Riemann Integrable, then the necessary result follows from Lemma 5·6. Conversely, consider the case where there is an $I \in \mathbb{R}$ so that for every sequence $\{P_k\}_{k=1}^{\infty}$ of partitions of $[a,b]$ with $\lim_{k\to\infty} \|P_k\| = 0$ and any associated sequence of evaluation sets $\{T_k\}_{k=1}^{\infty}$ we have $\lim_{k\to\infty} R(f, P_k, T_k) = I$. Let $\varepsilon > 0$ and $P_k := \{\frac{b-a}{k} \cdot i \mid 1 \le i \le k\}$ with T_k being any associated evaluation set. Since $\lim_{k\to\infty} \|P_k\| = \lim_{k\to\infty} \frac{b-a}{k} = 0$, there must exist $K \in \mathbb{N}$ for which $|R(f, P_k, T_k) - I| < \varepsilon$, given $k \ge K$. Fix $\delta := \|P_K\|$. First notice that for any partition $P := \{a \le x_0 < \dots < x_n < b\}$ satisfying $\|P\| < \delta$ and each associated evaluation set $T := \{t_1, \dots, t_n\}$, $\|P\| \ge \|P_n\|$ since $n\|P\| \ge \sum_{i=1}^{n} x_i - x_{i-1} = b - a$. As such, $n \ge K$ so~~

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If f is Riemann integrable, then the necessary result follows from Lemma 5·6. Conversely, now consider the case where f is not Riemann Integrable. Then let $I \in \mathbb{R}$; there exists $\varepsilon > 0$ such that for each $\delta > 0$, there is a partition P with norm $\|P\| < \delta$ and an associated evaluation set T, so $|R(f, P, T) - I| \ge \varepsilon$. By setting $\delta := \frac{1}{k}$, and using AC, we can construct a sequence of partitions $\{P_k\}_{k=1}^{\infty}$, with $\|P_k\| < \frac{1}{k}$ so $\lim_{k\to\infty} \|P_k\| = 0$; and an associated sequence of evaluation sets $\{T_k\}_{k=1}^{\infty}$, having the property that $|R(f, P_k, T_k) - I| \ge \varepsilon$ for every $k \in \mathbb{N}$. Consequently, it is clear that $\lim_{k\to\infty} R(f, P_k, T_k) \ne I$. $\square$

5-2 (b)

Ideas: $P_k, T_k := \{ \frac{i}{k} \mid 1 \le i \le k \}$

$$\lim_{k\to\infty} \left(\sum_{i=1}^{k} \left(\frac{i}{k}\right)^3 \left(\frac{1}{k}\right) \right) = \lim_{k\to\infty} \frac{1}{k^4} \sum_{i=1}^{k} i^3 = \lim_{k\to\infty} \underbrace{\frac{1}{k^4}\left(\frac{1}{4}k^2\right)(k+1)^2}_{\frac{1}{4k^2}(k^2+2k+1) = \frac{1}{4} + \frac{1}{2k} + \frac{1}{4k^2}}$$

(Ex 1-33 (b))

Proof

Since f is Riemann integrable on $[0,1]$, Lemma 5.6 can be applied to the sequence of partitions $\{P_k\}_{k=1}^{\infty}$ and associated sequence of evaluation sets $\{T_k\}_{k=1}^{\infty}$ defined by $P_k := \{ \frac{i}{k} \mid 0 \le i \le k \}$ and $T_k := \{ \frac{i}{k} \mid 1 \le i \le k \}$; so with the help of exercise 1-33(b):

$$\int_0^1 x^3\,dx = \lim_{k\to\infty} R(f, P_k, T_k) = \lim_{k\to\infty} \sum_{i=1}^{k} \left(\frac{i}{k}\right)^3 \left(\frac{1}{k}\right) \overset{1-33(b)}{=} \lim_{k\to\infty} \frac{1}{k^4}\left(\frac{1}{4}k^2\right)(k+1)^2 = \lim_{k\to\infty} \frac{1}{4} + \frac{1}{2k} + \frac{1}{4k^2} = \frac{1}{4}$$

5-4 This is easily proven via an inductive proof.

5-6 By Theorem 5.8, $\int_a^b -g(x)\,dx = -\int_a^b g(x)\,dx$ so $\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b -g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$ as expected.

5-7 ~~Uh isn't this just Proposition 5.10?~~ An ~~almost~~ identical copy of Proposition 5.10,

5-8 By Theorem 5.8, since we know $\int_a^b \mathbf{1}_{[a,b]}\,dx = b-a$ from ~~Proposition 5.10~~ 5-7, $\int_a^b f_c(x)\,dx = \int_a^b c\mathbf{1}_{[a,b]}\,dx = c\int_a^b \mathbf{1}_{[a,b]}\,dx = c(b-a)$.

~~(Well, to be pedantic, if we wanted to show Riemann integrability of f_c over $[a,b]$, the above argument applies provided $\int_a^b \mathbf{1}_{[a,b]}\,dx = b-a$, but that comes rather trivially by making an almost identical copy of Proposition 5.10.)~~

5-9 Let the sequence of partitions $\{P_k\}_{k=1}^{\infty}$ and associated sequence of evaluation sets $\{T_k\}_{k=1}^{\infty}$ be defined by $P_k := \{ a + \frac{b-a}{k}\cdot i \mid 0 \le i \le k \}$ and $T_k := \{ a + \frac{b-a}{k}\cdot i \mid 1 \le i \le k \}$; it is clear that $\lim_{k\to\infty} \|P_k\| = \lim_{k\to\infty} \frac{b-a}{k} = 0$. Thus, ~~since~~ $m \le f(x) \le M$ holds for all $x \in [a,b]$, means $m\left(\frac{b-a}{k}\right) \le f\left(a + \frac{b-a}{k}\cdot i\right)\left(\frac{b-a}{k}\right) \le M\left(\frac{b-a}{k}\right)$ for any $1 \le i \le k$ — so $R(m\mathbf{1}_{[a,b]}, P_k, T_k) \le R(f, P_k, T_k) \le R(M\mathbf{1}_{[a,b]}, P_k, T_k)$ for each $k \in$
Moreover, exercise 5-7 in conjunction with Theorem 5.8 ensures the existence of the Riemann integrals $\int_a^b m\mathbf{1}_{[a,b]}\,dx = m(b-a)$ and $\int_a^b M\mathbf{1}_{[a,b]}\,dx =$
Consequently, Lemma 5.6 guarantees the existence of the limits $\lim_{k\to\infty} R(m\mathbf{1}_{[a,b]}, P_k, T_k) = m(b-a)$, $\lim_{k\to\infty} R(f, P_k, T_k) = \int_a^b f(x)\,dx$, and $\lim_{k\to\infty} R(M\mathbf{1}_{[a,b]}, P_k, T_k) = M(b-a)$. Finally, the above inequality implies these limits satisfy the identity $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

5-10 Proof by contrapositive

Assume that $f:[a,b]\to\mathbb{R}$ is an unbounded function. We can assume without loss of generality that f is unbounded from above. By AC and the Bolzano-Weierstrass Theorem, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in $[a,b]$ converging to $L\in[a,b]$ so $\lim_{n\to\infty} f(z_n)=\infty$.

Letting $I\in\mathbb{R}$, $\varepsilon:=1$, $\delta>0$, there exists a $n\in\mathbb{N}$ with $\frac{b-a}{n}<\delta$, allowing us to define the partition $P:=\{a+\frac{b-a}{n}\cdot i \mid 0\le i\le n\}$. Thus, $\|P\|=\frac{b-a}{n}<\delta$ and, for some $1\le j\le n$, $L\in[a+\frac{b-a}{n}\cdot(j-1),\ a+\frac{b-a}{n}\cdot j]$. By the construction of $\{z_n\}_{n=1}^{\infty}$, there is a $m\in\mathbb{N}$ with $|z_m-L|<\frac{b-a}{n}$ and $f(z_m)\ge\left(\frac{b-a}{n}\right)^{-1}\left[I+1-\max\left\{\sum_{\substack{i=1\\ i\neq k}}^{n} f(a+\frac{b-a}{n}\cdot i)\left(\frac{b-a}{n}\right) \,\middle|\, k=j-1,j,j+1\right\}\right]$.

Now let $r\in\{j-1,j,j+1\}$ such that $z_m\in[a+\frac{b-a}{n}\cdot(r-1),\ a+\frac{b-a}{n}\cdot r]$. As such, we define the evaluation set $T:=\{t_i \mid 1\le i\le n\}$ by $t_r=z_m$ and $t_i=a+\frac{b-a}{n}\cdot i$ if $i\neq r$. Consequently,

$$R(f,P,T)=f(z_m)\left(\frac{b-a}{n}\right)+\sum_{\substack{i=1\\ i\neq r}}^{n} f(a+\frac{b-a}{n}\cdot i)\left(\frac{b-a}{n}\right)\ge I+1=I+\varepsilon.$$

In other words, $|R(f,P,T)-L|\ge\varepsilon$. $\square$

5-10 Direct Proof

5-11 Proof

In the case that f is not Riemann integrable: for every $I \in \mathbb{R}$ there exists $\varepsilon > 0$ so for all $\delta > 0$ there is a partition Q and an associated evaluation set U with $|R(f,Q,U) - I| \geq \varepsilon$. Define the partition $P := \{a + \frac{b-a}{n} \cdot i \mid 0 \leq i \leq n\}$ and associated evaluation set $T := \{a + \frac{b-a}{n} \cdot i \mid 1 \leq i \leq n\}$.

Then for $I := R(f,P,T)$, there is some $\varepsilon > 0$ such that for each $\delta > 0$, there is a partition Q and associated evaluation set U, satisfying $|R(f,Q,U) - R(f,P,T)| \geq \varepsilon$. Check with mathcad.

5-12 (a) Let $\varepsilon > 0$. As f_1, f_2 are Riemann-Stieltjes integrable with respect to g, there exists $\delta > 0$ so for all evaluation sets $P := \{a = x_0 < \cdots < x_n = b\}$ having $\|P\| < \delta$ and associated evaluation sets $T := \{t_1, \ldots, t_n\}$, we have $\left|\left[\sum_{i=1}^{n} f_1(t_i)\Delta g_i\right] - \int_a^b f_1\,dg\right| < \frac{1}{2}\varepsilon$ and $\left|\left[\sum_{i=1}^{n} f_2(t_i)\Delta g_i\right] - \int_a^b f_2\,dg\right| < \frac{1}{2}\varepsilon$. Consequently,
$\left|\left[\sum_{i=1}^{n}(f_1+f_2)(t_i)\Delta g_i\right] - \left[\int_a^b f_1\,dg + \int_a^b f_2\,dg\right]\right| \leq \left|\left[\sum_{i=1}^{n} f_1(t_i)\Delta g_i\right] - \int_a^b f_1\,dg\right| + \left|\left[\sum_{i=1}^{n} f_2(t_i)\Delta g_i\right] - \int_a^b f_2\,dg\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. $\square$

(b) Consider $c \neq 0$ and Let $\varepsilon > 0$. As f is Riemann-Stieltjes integrable with respect to g, there is some $\delta > 0$ such that for any evaluation set $P := \{a = x_0 < \cdots < x_n = b\}$ and associated evaluation set $T := \{t_1, \ldots, t_n\}$, having the property $\|P\| < \delta$, we have $\left|\left[\sum_{i=1}^{n} f(t_i)\Delta g_i\right] - \int_a^b f\,dg\right| < \frac{\varepsilon}{|c|}$. As such, $\left|\left[\sum_{i=1}^{n} cf(t_i)\Delta g_i\right] - c\int_a^b f\,dg\right| = |c|\left|\left[\sum_{i=1}^{n} f(t_i)\Delta g_i\right] - \int_a^b f\,dg\right| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon$.
When $c = 0$, this is even simpler as we can choose any δ, such as $\delta = 1$, because $\left|\left[\sum_{i=1}^{n} 0 \cdot f(t_i)\Delta g_i\right] - 0 \cdot \int_a^b f\,dg\right| = 0 < \varepsilon$ by definition. $\square$

(c) Let $I \in \mathbb{R}$, $\varepsilon = 1$, $\delta > 0$. There must exist $n \in \mathbb{N}$ with $\frac{1}{n} < \delta$. Hence, define the partition $P := \{0 = x_0 < \cdots < x_{2n-1} = 1\}$ by $x_i := \frac{1}{2} - \frac{n-i}{2n}$ for $0 \leq i \leq n-1$ and $x_i := \frac{1}{2} + \frac{i+1-n}{2n}$ for $n \leq i \leq 2n-1$. Thus, $\|P\| = \max\{\frac{1}{2n}, \frac{1}{n}\} = \frac{1}{n} < \delta$. Also define the associated evaluation set $T := \{t_i = x_i \mid 1 \leq i \leq 2n-1\}$. Consequently, $|S_g(\mathbf{1}_{[0,\frac{1}{2})}, P, T) - I| = \left|\left[\sum_{i=1}^{2n-1} \mathbf{1}_{[0,\frac{1}{2})}(t_i)\Delta \mathbf{1}_{[\frac{1}{2},1]}\right] - I\right| \geq \left|\mathbf{1}_{[0,\frac{1}{2})}(\frac{1}{2} + \frac{1}{2n})\left(\mathbf{1}_{[\frac{1}{2},1]}(\frac{1}{2} + \frac{1}{2n}) - \mathbf{1}_{[\frac{1}{2},1]}(\frac{1}{2} - \frac{1}{2n})\right)\right|$
$= |1| = 1 \geq \varepsilon := 1$. Wherefore, $f = \mathbf{1}_{[0,\frac{1}{2})}$ is not Riemann-Stieltjes integrable over $[0,1]$ with respect to $g := \mathbf{1}_{[\frac{1}{2},1]}$. $\square$

5-11 Ideas

If f is not Rieman integrable, $\forall I\ \exists\varepsilon\ \forall\delta\ \exists P\ \exists T : \|P\| < \delta$ but $|R(f,P,T) - I| \geq \varepsilon$

show $\exists\varepsilon\ \forall\delta\ \exists P\ \exists Q\ \exists T\ \exists U : \|P\|, \|Q\| < \delta$ $|R(f,P,T) - R(f,Q,U)| \geq \varepsilon$

$|x-y| \leq ||x|-|y||$

$|x+y| \leq |x|+|y|$

$R(Q$

$R(P$

ε

I

5-12 Ideas

(a) Let $\varepsilon > 0$

$$\left| \sum_{i=1}^{n} (f_1+f_2)(t_i)(g(x_i)-g(x_{i-1})) - \int_a^b f_1\, dg - \int_a^b f_2\, dg \right| \leq \left| \Sigma \right.$$

$\Delta g_i = 1$

(b)

(c) $\exists\varepsilon\ \forall\delta\ \exists P := \{0 = x_0 < \cdots < x_n = 1\}, \exists T := \{t_1, \cdots, t_n\}$ $[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$

$$\sum_{i=1}^{n} f(t_i)(g(x_i) - g(x_{i-1}))$$

5-1 Ideas

Assume $\exists I \exists \varepsilon \forall \delta \exists P \exists T : \|P\| < \delta$ but $|R(f,P,T) - I| \geq \varepsilon$ ($\forall I$)

Show $\forall I \exists \varepsilon \exists \{P_k\}_{k=1}^{\infty} : \forall K \exists k \geq K : \lim_{k\to\infty} \|P_k\| = 0$ but $|R(f, P_k, T_k) - I| \geq \varepsilon$

By AC,

$\delta := \frac{1}{k} \implies \exists \{P_k\}_{k=1}^{\infty} : \|P_k\| < \frac{1}{k} \implies \lim_{k\to\infty} \|P_k\| = 0$ ✓

$\neg((a \wedge \neg b) \vee (\neg a \wedge b))$

$(\neg a \vee b) \wedge (a \vee \neg b)$

5-10 Ideas

Assume that $f : [a,b] \to \mathbb{R}$ is an unbounded function. $\forall M \exists x \; |f(x)| > M$

$f(x) < -M$ or $f(x) > M$

Let $L \in \mathbb{R}$, $\varepsilon := 1$, $\delta > 0$:

Intuitively, either $f(x)$ is unbounded from above or below (or both). If for all $M \in \mathbb{N}$ $\exists x \; f(x) > M$, but $\exists M \in \mathbb{N}$ so $\forall x \; f(x) \nless -M$

f is unbounded from above, vice versa. So consider the final case of ...

$\implies$ Yes, either uba or ubb. Wlog, unbounded from above. By AC & BWT, $\exists \{z_n\}_{n=1}^{\infty} \to L \in [a,b]$

So $\{f(z_n)\}_{n=1}^{\infty} \to \infty$

$L \in (x_{j-1}, x_j)$

Let $I \in \mathbb{R}$, $\varepsilon := 1$, $\delta > 0$: $\exists n \left(\frac{b-a}{n} < \delta\right)$

Define $P := \{a + \frac{b-a}{n} \cdot i \mid 0 \leq i \leq n\}$;

$R > I + 1$

$W + H > I + 1$

$W > I - H + 1$

$$\exists s \quad f(s) > \frac{-\sum_{\substack{i=1\\ i\neq j}}^{n} f\left(a + \frac{b-a}{n} \cdot i\right) \frac{b-a}{n} + I + 1}{\frac{b-a}{n}}$$

$T := \{a + \frac{b-a}{n}, a + \frac{b-a}{n} \cdot 2, \dots, a + \frac{b-a}{n} \cdot (j-1); s, a + \frac{b-a}{n} \cdot (j+1), \dots, b\}$

$T := \{s\} \cup \{a + \frac{b-a}{n} \cdot i \mid 1 \leq i \leq n \;\&\; i \neq j\}$

$R(f,P,T) = f(s)\left(\frac{b-a}{n}\right) + \Sigma > \frac{I - \Sigma + 1}{b-a} \cdot b-a + \Sigma = I + 1 \implies |R(f,P,T) - I| \geq 1 = \varepsilon$

5-10 Proof

Assume that $f:[a,b]\to\mathbb{R}$ is an unbounded function; we can assume without loss of generality that f is unbounded from above.

By AC ~~and the Bolzano-Weierstrass Theorem, there exists a sequence~~ $\{z_n\}_{n=1}^{\infty}$ ~~in~~ $[a,b]$ ~~converging to~~ $L\in[a,b]$ so $\lim\limits_{n\to\infty} f(z_n)=\infty$.

Letting $I\in\mathbb{R}$, $\varepsilon:=1$, $\delta>0$, there exists a $n\in\mathbb{N}$ with $\frac{b-a}{n}<\delta$, allowing us to define the partition $P:=\{p_i \mid 0\le i\le 2n\}$ by

$p_i := a+\frac{L-a}{n}\cdot i$ and $p_{i+n} := L+\frac{b-L}{n}\cdot i$ for $1\le i\le n$. Thus, $\|P\| = \max\{\frac{L-a}{n},\frac{b-L}{n}\}<\frac{b-a}{n}=\delta$. ~~As $\lim\limits_{n\to\infty} f(z_n)=\infty$, there is a $n\in\mathbb{N}$ with $|z_n-L|<p_{n+1}-p_{n-1}$~~

and $f(z_n) > \max\{(\frac{L-a}{n})^{-1}(I+1-\sum_{\substack{i=1\\ i\neq j}}^{2n} f(p_i)(p_i-p_{i-1})), ($ ^{b} . Let j be the natural number such that $z_n\in[p_{j-1},p_j]$ since $z_n\in[p_{n-1},p_{n+1}]$

As such, we now define the evaluation set $T:=\{t_i \mid 1\le i\le 2n\}$ where $t_j:=z_n$ and $t_i:=p_i$ if $i\neq j$. Then, we note that

$R(f,P,T) \geq f(z_n)($

L Didn't like phrasing, rewrote

Self-Proof of Lemma 5.14

Since $f(t_i) \in \{f(x) \mid x \in [x_{i-1}, x_i]\}$ by definition, $m_i \leq f(t_i) \leq M_i$ holds by construction for each $1 \leq i \leq n$. So, each summand satisfies the inequality $m_i \Delta x_i \leq f(t_i) \Delta x_i \leq M_i \Delta x_i$. As such, the sums also do: $L(f,P) \leq R(f,P,T) \leq U(f,P)$.

Self-Proof of Lemma 5.16

The case of $|Q| = 2$ is trivial as Q is the only subset of itself that is a partition. So assume that for all bounded functions f such that $f: [a,b] \to \mathbb{R}$ for some $a, b \in \mathbb{R}$, and any partitions $P \subseteq Q$ with $|Q| \leq n$, we have $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$. Now let f be a bounded function so $f: [a,b] \to \mathbb{R}$ for some $a, b \in \mathbb{R}$. Let the partition $Q' := \{a = x_0 < \cdots < x_{n+1} = b\}$ be a refinement of $P' := \{a = x_{w_0} < \cdots < x_{w_k} = b\}$. By further defining the following partitions on $[a, x_{w_{k-1}}]$, namely $P := \{a = x_{w_0} < \cdots < x_{w_{k-1}}\} \subseteq Q := \{a = x_0 < \cdots < x_{w_{k-1}}\}$, our assumption / induction hypothesis says

$$L(f|_{[a, x_{w_{k-1}}]}, P) \leq L(f|_{[a, x_{w_{k-1}}]}, Q) \leq U(f|_{[a, x_{w_{k-1}}]}, Q) \leq U(f|_{[a, x_{w_{k-1}}]}, P);$$ i.e.

$$\sum_{i=1}^{k-1} m_i (x_{w_i} - x_{w_{i-1}}) \leq \sum_{i=1}^{w_{k-1}} \bar{m}_i (x_i - x_{i-1}) \leq \sum_{i=1}^{w_{k-1}} \bar{M}_i (x_i - x_{i-1}) \leq \sum_{i=1}^{k-1} M (x_{w_i} - x_{w_{i-1}}).$$

We notice that $m_k \leq \bar{m}_i$ and $\bar{M}_i \leq M_k$ for $w_{k-1} < i \leq n+1$. Hence, $m_k (x_{w_i} - x_{w_{i-1}}) = \sum_{i=w_{k-1}+1}^{n} m_k (x_i - x_{i-1}) \leq \sum_{i=w_{k-1}+1}^{n} \bar{m}_i (x_i - x_{i-1})$ and, similarly $\sum_{i=w_{k-1}+1}^{n} \bar{M}_i (x_i - x_{i-1}) \leq M_k (x_{w_i} - x_{w_{i-1}})$. As such,

$\sum \cdots$

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Therefore... By induction...

Self-Proof of Lemma 5.17

This is a simple Corollary of Lemma 5.16.

Self-Proof of Lemma 5.19

Ideas

$x \; \forall \varepsilon \exists \delta_x \; |z-x| < \frac{\delta_x}{2} \Rightarrow |f(z)-f(x)| < \frac{\varepsilon}{2}$

$\forall \varepsilon \exists \delta \forall u \forall v$

$|u-v| \le |u-x| + |v-x| < \delta \qquad |f(u)-f(v)| \le |f(u)-f(x)| + |f(v)-f(x)|$

$|u-v| < \delta \qquad |u-v| \le |u-x| + |v-x|$

$|u-x| \le |u-v| + |v-x|$

$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists u_\delta \exists v_\delta \qquad x-\delta < z < x+\delta$

$|u_\delta - v_\delta| < \delta$ but $|f(u_\delta) - f(v_\delta)| \ge \varepsilon$

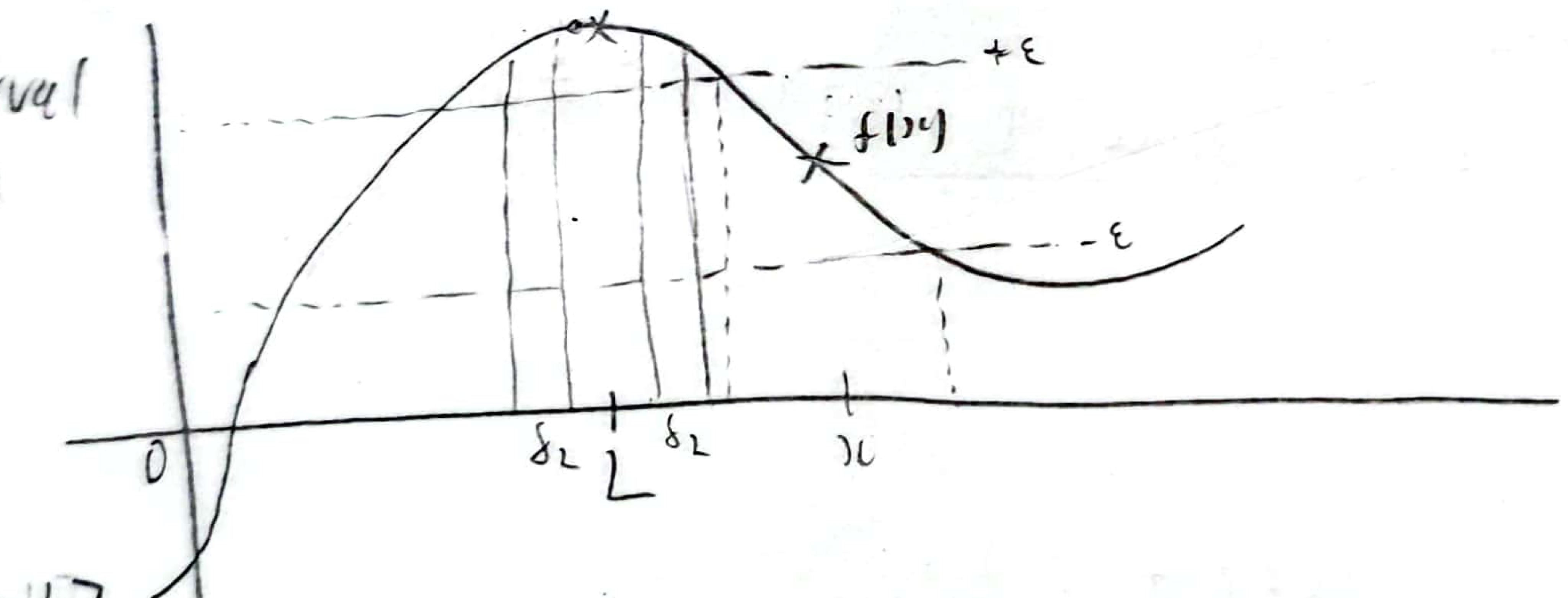
$\exists x \; \exists \varepsilon > 0 \; \forall \delta > 0$

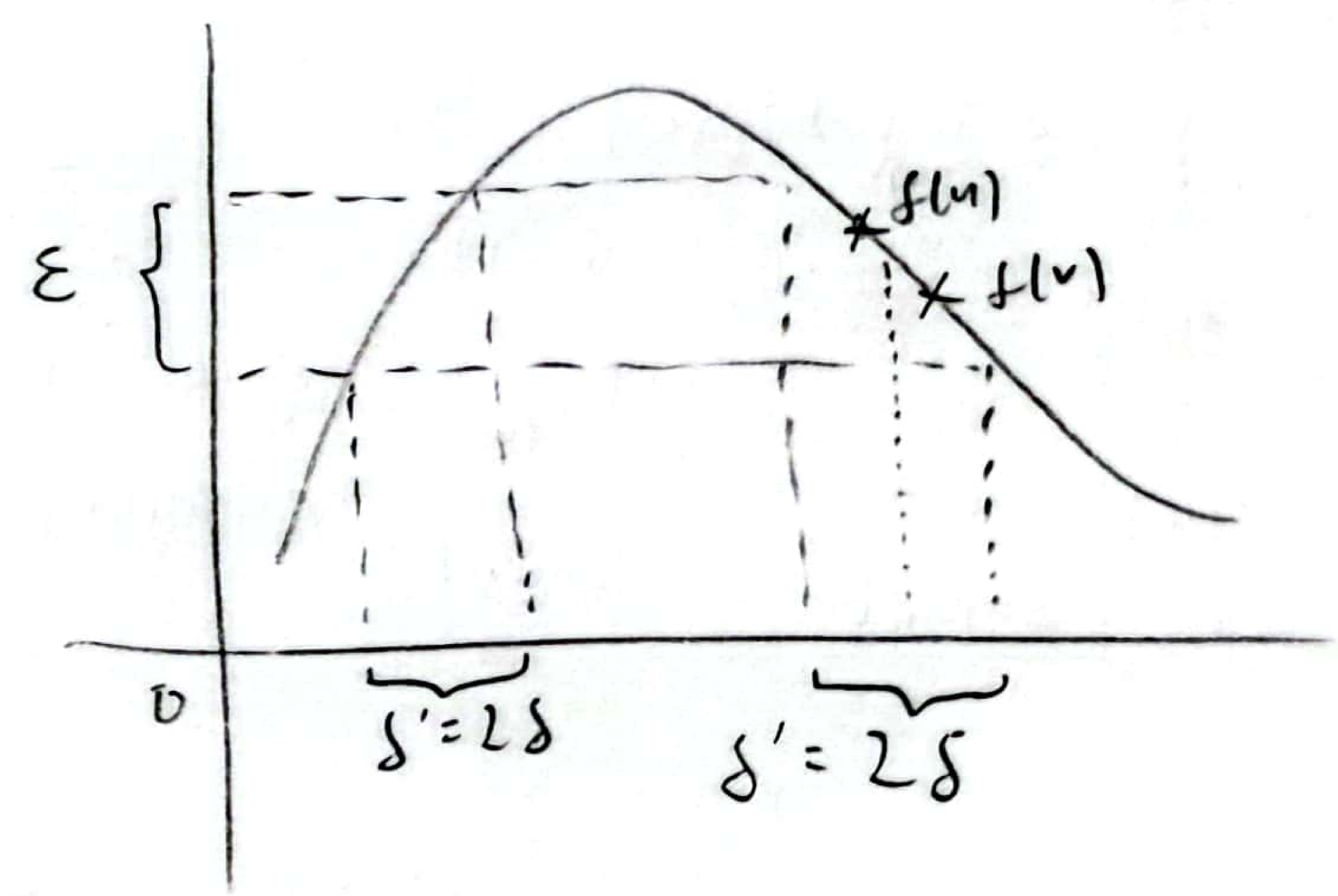
must be a maximum

$i := \inf\{\sup\{\delta_x | \cdots\} | x \in [a,b]\} \le 0$

Assume that $f : [a,b] \to \mathbb{R}$ is continuous, let $x \in [a,b]$ and $\varepsilon > 0$. By definition, there exists $\delta_x > 0$ so for every $z \in [a,b]$, if $|z-x| < \delta_x$ then $|f(z)-f(x)| < \frac{\varepsilon}{2}$. Then for any $u, v \in [a,b]$ with

☆ $[a,b]$ is a closed interval





Show $i \ne 0$

When $i = 0$, $\forall n \exists s_n \quad s_n - i < \frac{1}{n}$

$x \sim s_n < i + \frac{1}{n}$

$AC + BWT : \{x_n\}_{n=1}^{\infty} \to L \in [a,b]$

$\{s_{x_n}\}_{n=1}^{\infty} \to i = 0$

$s_{x_n} < \frac{1}{n}$

For the previously fixed $\varepsilon > 0$, $\exists \delta_L > 0 \; |z-L| < \delta_L \Rightarrow |f(z)-f(L)| < \frac{\varepsilon}{2}$

$\exists z$

$|x_n - L| < \frac{\delta_L}{2} \qquad |z - x_n| < \frac{\delta_L}{2}$

but $|f(z) - f(x_n)| \ge \varepsilon$

$|f(z) - f(x_n)| \le |f(z) - f(L)| + |f(L) - f(x_n)| < \varepsilon$

$|u-v| \le |u - \frac{u+v}{2}| + |v - \frac{u+v}{2}|$

$\overbrace{x \quad z \quad x+s}^{s}$

$\underbrace{|z-x|} \quad \underbrace{x+s-z}$

$\frac{s + |z-x|}{2} < s \Rightarrow \left(\exists \delta_s \in \left(\frac{s+|z-x|}{2}, s\right]\right) |z-x| < \delta_s \Rightarrow |f(z)-f(x)| < \varepsilon$

$\Rightarrow$ actually max as Prop 1.21 : $s - \delta_s < s - |z-x|$

$|z-x| < \delta_s \le s$

$L - \delta_L < x - \frac{\delta_L}{2} < z < x + \frac{\delta_L}{2} < L + \delta_L$

Cont. $\Rightarrow i > 0$

$|z - x_n| \le |z - L| + |x_n - L|$

$|z - L| \le |z - x_n| + |x_n - L|$

Proposition 1.21 says there exists

Self - Proof of Lemma J.19

Ideas (Summary / tidied)

$s_x := \sup\{\delta_x \mid \cdots\}$, $i := \{s_x \mid x \in [a,b]\}$

1st Show s_x is a maximum!

Then show i is greater than 0

So, since $i \leq s_x$ — a δ_x itself — for all $x \in [a,b]$, i is a global choice of δ

Proof

Assume that $f : [a,b] \to \mathbb{R}$ is continuous and let $\varepsilon > 0$. First define s_x to be the supremum of all $\delta_x > 0$ so that for any $z \in [a,b]$, if $|z - x| < \delta_x$ then $|f(z) - f(x)| < \varepsilon$. We notice that $s_x > 0$ is a maximum (δ_x), since for any $|z - x| < s_x$, Proposition 1.21 says there exists $|z - x| < \delta_x \leq s$, thus $|f(z) - f(x)| < \varepsilon$. Now, we further define $i := \inf\{s_x \mid x \in [a,b]\}$; and suppose for a contradiction, that $i = 0$. Then for each $n \in \mathbb{N}$, Proposition 1.21 again says there exists x_n with $s_{x_n} \leq \frac{1}{n}$. Therefore, by AC and the Bolzano–Weierstrass Theorem, a sequence $\{x_n\}_{n=1}^{\infty}$ converging to some $L \in [a,b]$ exists, with $\lim_{n \to \infty} s_{x_n} = 0$. By continuity, there must some $\delta_L > 0$ such that $|z - L| < \delta_L$ implies $|f(z) - f(L)| < \frac{\varepsilon}{2}$. In addition, $|x_n - L| < \frac{\delta_L}{2}$ and $s_{x_n} < \frac{\delta_L}{2}$ for some $n \in \mathbb{N}$, by construction. Hence, there exists ($s_{x_n} <$) $|z - x_n| < \frac{\delta_L}{2}$ with $|f(z) - f(x_n)| \geq \varepsilon$. However, since $|z - L| \leq |z - x_n| + |x_n - L| < \delta_L$, we see that $|f(z) - f(x_n)| \leq |f(z) - f(L)| + |f(L) - f(x_n)| < \varepsilon$, a contradiction. Which means that $i > 0$. Consequently, for any $u, v \in [a,b]$ having $|u - v| < i \leq s_v$, $|f(u) - f(v)| < \varepsilon$ because s_v is a δ_v. Wherefore, f is uniformly continuous.

Self-Proof of Theorem 5.20

Ideas

f is uniformly continuous $\quad |R(f,P,T)-I| \quad Q \subseteq P$

Let $\varepsilon>0$,

$\|P\|<\delta \implies |P| \geq \lceil \frac{b-a}{\delta} \rceil$ Let P be a refinement of Q so $|Q| = \lceil \frac{b-a}{\delta} \rceil := n$

$U(f,P) - L(f,Q) < \varepsilon \implies R(f,P,T) - L(f,P) < \varepsilon$

Show exist $\delta>0$ so $\|P\|,\|Q\|<\delta \implies$

wlog $P \subseteq Q$

$n\|P\|E$

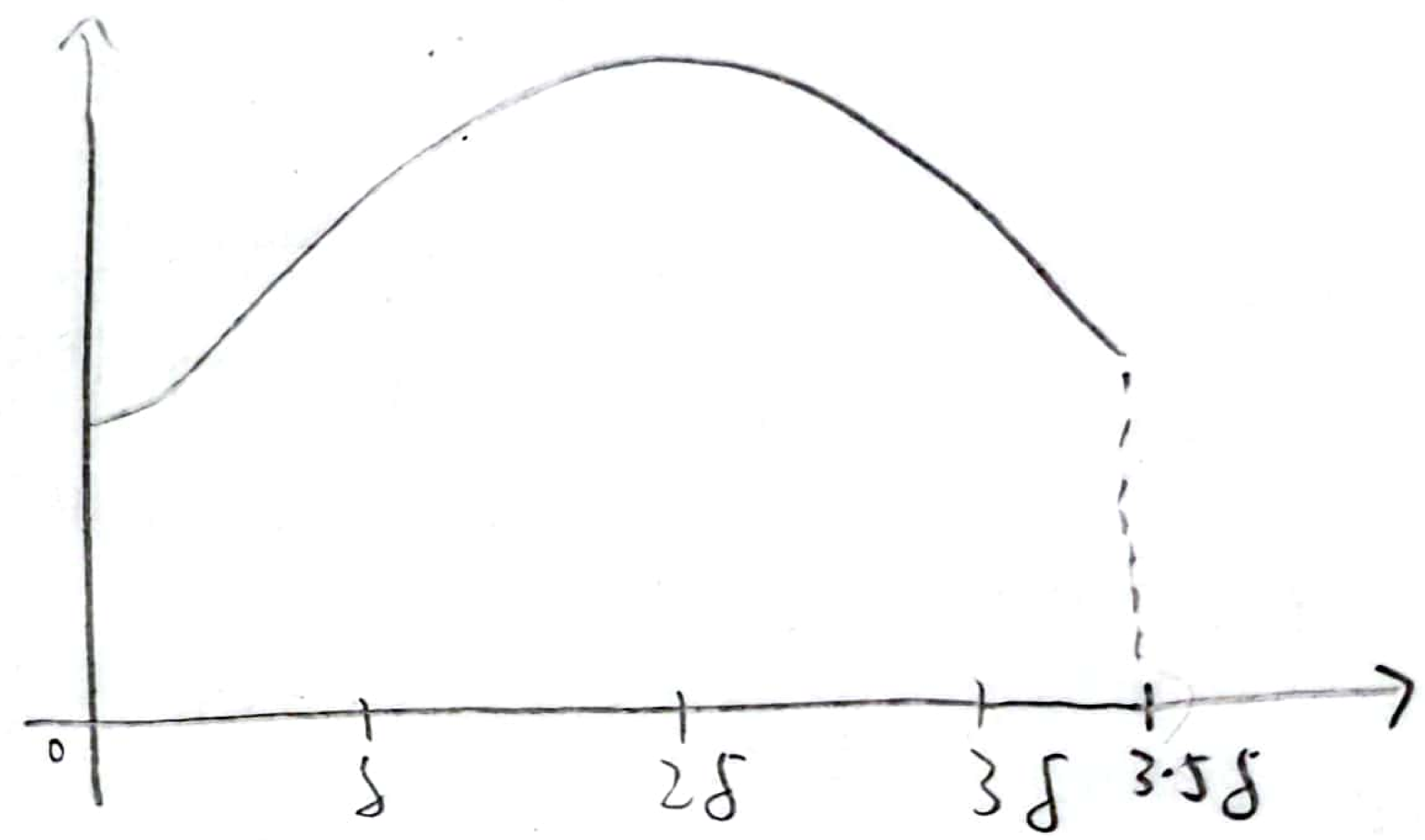
$\underbrace{\|P\|}_{\delta}E \quad |Q|=|P|+1 \quad \Downarrow\Downarrow\Downarrow \quad \delta E < \varepsilon$

Let $\varepsilon>0 \quad \|P\|\varepsilon$

$\exists\ \delta := \min\{\delta_1, 1\}$ so

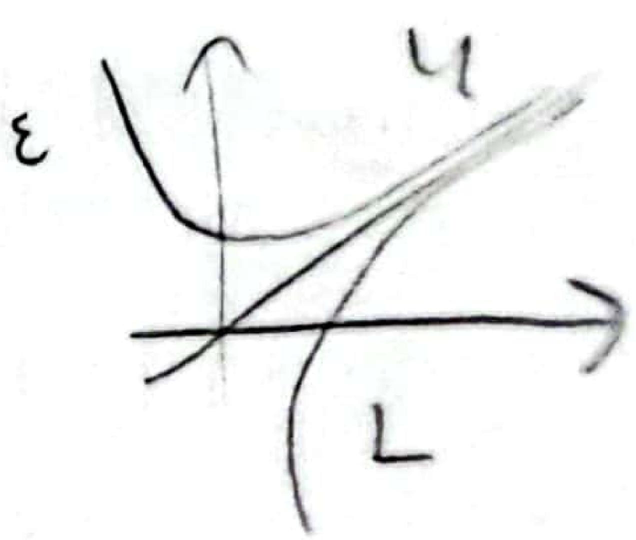
$|u-v|<\delta \implies |f(u)-f(v)| < \varepsilon$

... formalisation



$$\underset{P_n}{|U(f,P)} - \underset{P_{n+1}}{L(f,Q)|} < \delta\frac{\varepsilon}{b-a} \leq \varepsilon \quad \text{for } \|P\|,\|Q\|<\delta \quad (\text{similarly for } |L(f,Q) - U(f,P)| < \varepsilon)$$

$\& \ |L(f,Q)-L(f,P)| < \frac{1}{2}\delta\varepsilon < \varepsilon$



A simple inductive proof gives $|U(f,P)-L(f,Q)|<\varepsilon$ for $\|P\|,\|Q\|<$ some δ for any $|P|$ and $|Q|$

$R_n := \{a + \frac{b-a}{n}\cdot i \mid 0 \leq i \leq n\}$

$\forall\varepsilon\exists\delta \quad \|R_n\|,\|R_m\|<\delta \implies |L(f,R_m)-L(f,R_n)| < \varepsilon \xRightarrow{\text{Cauchy sequence}} \lim_{n\to\infty} L(f,R_n) = L$ for some L

$\hookrightarrow n,m$ sufficiently big

$0 \leq R(f,Q,T) - L(f,P) \leq U(f,Q) - L(f,P)$

$|U(f,P)-L| \leq |U(f,P)-L(f,R_n)| + |L(f,R_n)-L| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$

$\leq$

$|R(f,P,T)-L|$

as $(U(f,P) \leq L(f,R_n)\ \forall n\in\mathbb{N}) \implies U(f,P) < L\ )$

Self-Proof of Theorem 5.20

Proof

Let $f:[a,b]\to\mathbb{R}$ be continuous and $\varepsilon>0$. Then, as f is uniformly continuous according to Lemma 5.19, there exists $\delta>0$ for which if $|u-v|<\delta$ then $|f(u)-f(v)|<\frac{\varepsilon}{2(b-a)}$. So, for any partitions P and Q with $\|P\|, \|Q\|<\delta$, $|U(f,P)-L(f,Q)| \leq |\sup(f[a,b])\cdot(b-a)-\inf(f[a,b])\cdot(b-a)| < b-a \cdot \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{2}$. Similarly, $|L(f,P)-L(f,Q)|<\frac{\varepsilon}{2}$. Now define the sequence of partitions $\{R_n\}_{n=1}^{\infty}$ by $R_n := \{a+\frac{b-a}{n}\cdot i \mid 0\leq i\leq n\}$. Letting $k\in\mathbb{N}$ such that $\frac{b-a}{k}<\delta$, we see that for every $n,m\geq k$, we have $\|R_n\| = \frac{b-a}{n}<\delta$ and $\|R_m\|=\frac{b-a}{m}<\delta$. Hence, $|L(f,R_n)-L(f,R_m)|<\frac{\varepsilon}{2}$. which means $\{L(f,R_n)\}_{n=1}^{\infty}$ is a Cauchy sequence, i.e. it converges to some $L\in\mathbb{R}$. Consequently, given $\|P\|<\delta$, and any evaluation set T, $|R(f,P,T)-L| \leq |U(f,P)-L| \leq |U(f,P)-L(f,R_n)| + |L(f,R_n)-L| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. Therefore, $\int_a^b f(x)\,dx = L$. $\square$

Self-Proof of Proposition 5.21

Idea: $R(f,P,T) \le R(g,P,T)$

By cont.: If $I_f > I_g$, $\frac{I_f - I_g}{2} > 0 \Rightarrow \exists \delta > 0\ \forall P \forall T : \|P\| < \delta \Rightarrow R(g,P,T) < I_g + \frac{I_f - I_g}{2} = \frac{I_f + I_g}{2} = I_f - \frac{I_f - I_g}{2} < R(f,P,T)$

Cont.

Direct?

$I_f < R(f,P,T) + \varepsilon \le R(g,P,T) + \varepsilon$

We can use the fact that if $a \le b + \varepsilon$ for all $\varepsilon > 0$, then $a \le b$.

Proof

Assume, for the sake of contradiction, that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$ instead. Then, since $\frac{\int_a^b f(x)\,dx - \int_a^b g(x)}{2} > 0$ there exists $\delta > 0$, so if a partition P has norm $\|P\| < \delta$, then for any associated evaluation set T, $R(g,P,T) < \int_a^b g(x)\,dx + \frac{\int_a^b f(x)\,dx - \int_a^b g(x)\,dx}{2} = \frac{\int_a^b f(x) + \int_a^b g(x)}{2} = \int_a^b f(x)\,dx - \frac{\int_a^b f(x) - \int_a^b g(x)}{2} < R(f,P,T)$.

But simultaneously, $f(t_i) \le g(t_i)$ for each $t_i \in T$ means $R(f,P,T) \le R(g,P,T)$. A contradiction. □

Self-Proof of Theorem 5.22

Proof

By Theorem 3.44, there is a absolute maximum value $f(x_M)$ of f. By defining $g : [a,b] \to \mathbb{R}$ by $g(x) := f(x_M)$, Proposition 5.21 tells us $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. In fact, it is easy to see that $\int_a^b g(x)\,dx = (b-a)f(x_M)$ because $R(g,P,T) = \sum_{i=1}^n g(t_i)\Delta t_i = f(x_M)\sum_{i=1}^n \Delta t_i = (b-a)f(x_M)$ for any partition $P := \{a = x_0 < \cdots < x_n = b\}$ and associated evaluation set $T := \{t_1, \dots, t_n\}$. Similarly, there is a absolute minimum value $f(x_m)$ of f, such that, in conjunction with the previous result, $(b-a)f(x_m) \le \int_a^b f(x)\,dx \le (b-a)f(x_M)$. Thus, $f(x_m) \le \frac{\int_a^b f(x)\,dx}{b-a} \le f(x_M)$. From the Intermediate Value Theorem, it is clear that there is some $c \in (a,b)$ so $f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$. In other words, $\int_a^b f(x)\,dx = f(c)(b-a)$. □

$[c \in (\min\{x_m, x_M\}, \max\{x_m, x_M\}) \subseteq (a,b)]$

5-14

Let $\delta > 0$. Then, since $2\delta^2+\delta > \delta$ means $\delta > \frac{\delta}{2\delta+1} > 0$, $\delta - \frac{\delta}{2\delta+1} < 0$. In fact, $\left|\frac{2\delta+1}{\delta} - \frac{1}{\delta}\right| = \left|2+\frac{1}{\delta}-\frac{1}{\delta}\right| = 2$. So, $\frac{1}{x}$ is not uniformly continuous. However, by limit laws and the continuity of $g:(0,1] \to \mathbb{R}$ defined by $g(x)=x$, f is continuous. This does not contradict Lemma 5.19 since it requests specifically for functions with closed interval domains, which $(0,1]$ isn't. $\square$

5-15 (a) Ideas

Lipschitz continuous $\Rightarrow$ basically bounded gradient

$$\left|\frac{f(u)-f(v)}{u-v}\right| > \left|\frac{\varepsilon}{\delta}\right| \quad \left|\frac{\varepsilon}{\frac{\varepsilon}{L}}\right| = L$$

trat: If $\neg$ U.C., $\exists \varepsilon\ \forall \delta\ \exists u \exists v$: $|u-v| < \delta$ but $|f(u)-f(v)| \geq \varepsilon$

Let $L \in \mathbb{R}^+$, $\delta := \frac{\varepsilon}{L}$

~~all for x-y ... times~~

direct: $\exists L > 0\ \forall x\ \forall y\ |f(x)-f(y)| \leq L|x-y| \ (< \varepsilon)$ $\quad L|x-y| < \varepsilon$

Let $\varepsilon > 0$. $\delta := \frac{\varepsilon}{L}$ $\quad |x-y| < \frac{\varepsilon}{L}$

$$|x-y| < \frac{\varepsilon}{L} \Rightarrow |f(x)-f(y)| \leq L|x-y| < L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

Proof 1: Contrapositive

Assume f is not uniformly continuous. Then there exists $\varepsilon > 0$ so, for all $\delta > 0$, there is $u, v \in [a,b]$ with $|u-v| < \delta$ but $|f(u)-f(v)| \geq \varepsilon$. Let $L > 0$, $\delta := \frac{\varepsilon}{L}$: Thus, $\left|\frac{f(u)-f(v)}{u-v}\right| > \frac{\varepsilon}{\delta} = \frac{\varepsilon}{\left(\frac{\varepsilon}{L}\right)} = L$ for some $u, v \in [a,b]$ with $|u-v| < \delta$. Hence, f is not Lipschitz continuous.

Proof 2. Direct

Now assume f is Lipschitz continuous, so there exists $L > 0$ such that for every $x, y \in [a,b]$ we have $|f(x)-f(y)| \leq L|x-y|$. As usual, let $\varepsilon > 0$. We define $\delta := \frac{\varepsilon}{L}$ and behold the results: when $|x-y| < \frac{\varepsilon}{L}$, $|f(x)-f(y)| \leq L|x-y| < L \cdot \frac{\varepsilon}{L} = \varepsilon$.

5-15(b)

$\exists l, u \ \forall x \in [a,b] \quad f'(x) \geq l \ \& \ f'(x) \leq u$

Idea

$\forall x \exists G \forall \varepsilon \exists \delta \forall z : |z-x| < \delta \Rightarrow \left| \frac{f(z)-f(x)}{z-x} - G \right| < \varepsilon$

Find $L > 0 \ \forall x \forall y \quad |f(x) - f(y)| \leq L|x-y|$

$\left| \frac{f(x)-f(y)}{x-y} \right| \leq L$ if $x \neq y$

Let $L = \max\{|l|, |u|\} \Rightarrow$ use MVT !!

Proof

Let l and u be lower and upper bounds of f' respectively, and thus, define $L := \max\{|l|, |u|\} + 1$. For any $x < y$ in $[a,b]$ there exists $c \in (a,b)$ with $\left| \frac{f(x)-f(y)}{x-y} \right| = |f'(c)| < L$ by the Mean Value Theorem. Even if $x = y$, $|f(x) - f(y)| = 0 \leq 0 = L|x-y|$.

(c) Idea

$|\sqrt{\varepsilon^2} - \sqrt{0}| = \varepsilon$

$|u - v| < \varepsilon^2$

$u < v + \varepsilon^2$

Since $\sqrt{\ }$ is increasing, & wlog $u < v$

$|\sqrt{u} - \sqrt{v}| < \sqrt{u+\varepsilon^2} - \sqrt{u}$
$\leq \sqrt{u + 2u\varepsilon + \varepsilon^2} - u$
$= u + \varepsilon - u$
$= \varepsilon$

$u \leq 1$
$u^2 \leq u$
$u \leq \sqrt{u}$

Let $L > 0$.

$m := \min\{\frac{1}{L^2}, 1\}$ $\quad m \leq \frac{1}{L^2}$, $L^2 \leq \frac{1}{m}$ $\quad L \leq \frac{1}{\sqrt{m}}$

$\left| \frac{\sqrt{m} - \sqrt{0}}{m - 0} \right| = \frac{1}{\sqrt{m}} \geq L$

$|\sqrt{m} - \sqrt{0}| \geq L|m - 0|$

Proof

Let $\varepsilon > 0$, then we have that for all $u < v$ with $|u - v| < \varepsilon^2$; since f is (strictly) increasing, $|\sqrt{u} - \sqrt{v}| < \sqrt{u+\varepsilon^2} - \sqrt{u} \leq \sqrt{u + 2u\varepsilon + \varepsilon^2} - u = u +$ $= \varepsilon$. So, f is indeed uniformly continuous. However, letting $L > 0$, we have that $m := \min\{\frac{1}{(L+1)^2}, 1\}$ is such that $\left| \frac{\sqrt{m} - \sqrt{0}}{m - 0} \right| = \frac{1}{\sqrt{m}} \geq L +$ which means f is not Lipschitz continuous.

5-16 Ideas for direct proof $\qquad \|P\| < \delta$

Let $x \in [a,b]$, P be a partition of $[a,b]$ and u be the largest i so $x \in [x_{i-1}, x_i]$,

$$\varepsilon > \left| R(f,P,T) - 0 \right| = \left| \left[\sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i)(x_i - x_{i-1}) \right] + f(x)(x_u - x_{u-1}) - 0 \right| \geq \left[\left[\sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i) \right] + f(x) \right] \cdot \overbrace{\min\{x_i - x_{i-1} \mid 1 \leq i \leq n\}}^{m}$$

$$\varepsilon > \left[\left[\sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i) \right] + f(x) \right]$$

$$\varepsilon \geq \varepsilon - \sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i) > f(x)$$

/qed. :)

Direct Proof

Let $x \in [a,b]$, $\varepsilon > 0$, the partition $P_n := \{a = x_0 < \cdots < x_n = b\}$ where $x_i := a + \frac{b-a}{n} \cdot i$, and the associated evaluation set $T_n := \{t_i \mid 1 \leq i \leq n\}$ where $t_u := x$ — for u being the largest $1 \leq i \leq n$ so $x \in [x_{i-1}, x_i]$ — and $t_i := x_i$ otherwise. (since $\lim \|P_n\| = 0$ is clear) Lemma 5.6 guarantees the existance of some $N \in \mathbb{N}$ for which $n \geq N$ implies $\frac{b-a}{n} \cdot \varepsilon > \left| R(f, P_n, T_n) - 0 \right| = \left[\sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i)(x_i - x_{i-1}) \right] + f(x)(x_u - x_{u-1}) = \left[\left[\sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i) \right] + f(x) \right] \cdot \frac{b-a}{n}$. Thus, $\varepsilon > \left[\sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i) \right] + f(x)$. Therefore, by nonnegativity, $\varepsilon \geq \varepsilon - \sum_{\substack{i=1 \\ i \neq u}}^{n} f(x_i) > f(x)$. Since $\varepsilon > f(x) \geq 0$ for all $\varepsilon > 0$, $f(x) = 0$ is certain. $\square$

Proof by Contraposition

Suppose that $f : [a,b] \to [0, \infty)$ is a continuous nonnegative function so there exists $x \in [a,b]$ with $f(x) \neq 0$. By continuity, integrability follows. Let $\varepsilon > 0$. There hence exists $\delta > 0$ for which if a partition P has norm $\|P\| < \delta$, then given any associated evaluation set, $R(f,P,T)$ -

(Ah well I feel like moving on so maybe I'll come back to finish this contrapositive proof.)

5-17 Ideas

$m_i := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

$\forall n \exists x_n$: $f(x_n) - m_i < \frac{1}{n}$

AC + BWT: $\{x_n\} \to L \in [a,b]$

$\{f(x_n)\} \to m_i$

Proof

Let $\{P_k\}_{k=1}^{\infty}$ be a sequence of partitions with $\lim_{n\to\infty} \|P_n\| = 0$.
Since m_i is infimal, for any $n \in \mathbb{N}$, there exists $z_n \in [x_{i-1}, x_i]$ with $f(z_n) - m_i < \frac{1}{n}$. By AC and the Bolzano Weierstass Theorem, we have a sequence $\{x_n\}_{n=1}^{\infty}$ so $\lim_{n\to\infty} x_n = L_i$ and $\lim_{n\to\infty} f(x_n) = m_i$. Clearly, because $x_n \in [a,b]$, we have that $L_i \in [a,b]$.
Therefore, we define the associated sequence of evaluation sets $\{T_k\}_{k=1}^{\infty}$ by $T_k := \{L_i \mid 1 \le i \le k\}$. Thence, by Lemma 5.6, $\int_a^b f(x)\,dx = \lim_{k\to\infty} R(f, P_k, T_k) = \lim_{k\to\infty} L(f, P_k)$ as continuity ensures $f(L_i) = m_i$.

$\square$

5-18 Assume that $f: [a,b] \to [0,\infty)$ is a nonnegative Riemann integrable function with $\int_a^b f(x)\,dx > 0$ such that, for all $\varepsilon > 0$ and $c < d$, there exists $z \in [c,d]$ having $f(x) \le \varepsilon$. Now let $x \in [a,b]$ and fix some $\varepsilon > 0$. For each $n \in \mathbb{N}$, there is some $z_n \in [x - \frac{1}{n}, x + \frac{1}{n}]$ with $f(z_n) \le \frac{\varepsilon}{2}$. By AC, there is hence a sequence $\{z_n\}_{n=1}^{\infty}$ converging to x, so $f(z_n) \le \frac{\varepsilon}{2}$ for every $n \in \mathbb{N}$. Accordingly, $f(x) = \lim_{n\to\infty} f(z_n) \le \frac{\varepsilon}{2} < \varepsilon$.
Since such a sequence $\{z_n\}_{n=1}^{\infty}$ exists for any and all $\varepsilon > 0$, it holds that $f(x) < \varepsilon$ for any ε too. By nonnegativity it is clear that $f(x) = 0$.
In other words, f is the zero function; $\int_a^b f(x)\,dx = \int_a^b 0\,dx = 0$.

$\square$

Now, consider the case where for all $n \in \mathbb{N}$, there exists $x_n \in [a,b]$ so $0 < f(x_n) < \frac{1}{n}$. Once more, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of such x_n, that converges to some limit $L \in [a,b]$ and whose image under f, $\{f(x_n)\}_{n=1}^{\infty}$, goes to 0 — by virtue of AC, the Bolzano Weierstrass Theorem, and the Squeeze Theorem. Consequently,

$$f(L) = \lim_{z \to L} f(z) = \lim_{n \to \infty} f(x_n) = 0,$$ contradicting our initial assumption that $0 \notin f[[a,b]]$.

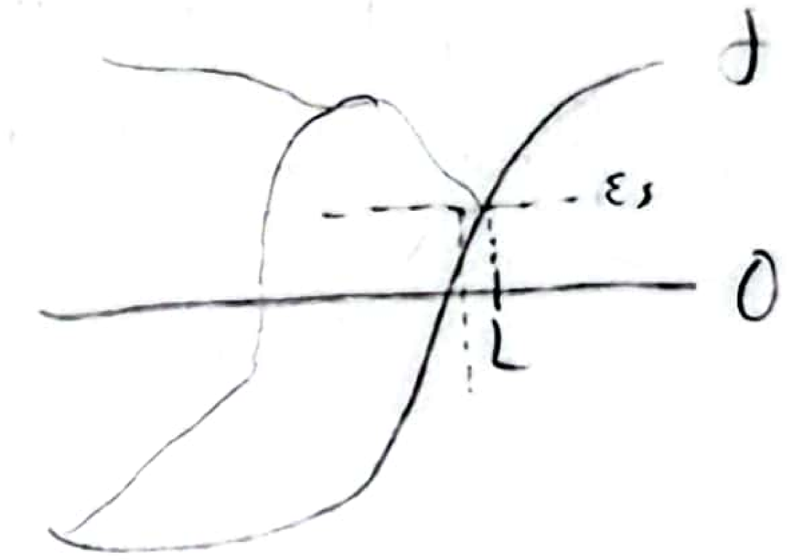
$\square$

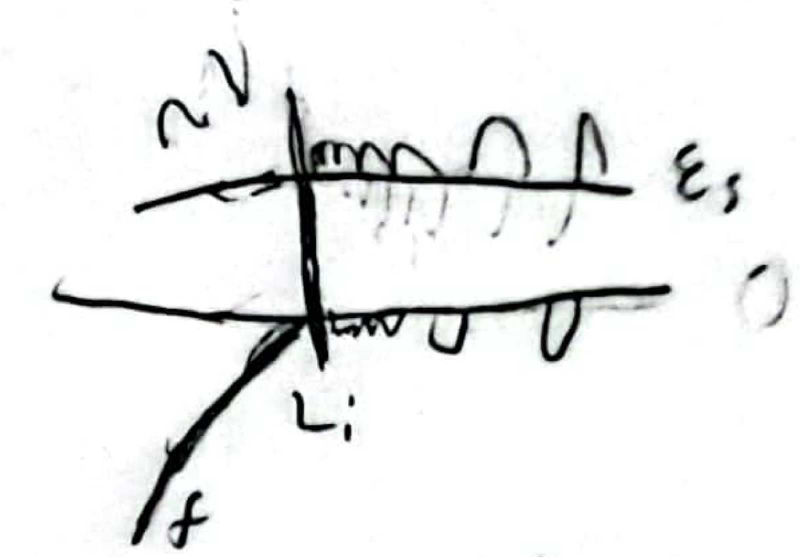
Wherefore, it must be that there is a $c \in (a,b)$ such that $f(c) = 0$.

Either $\exists \varepsilon > 0$ by $|y-0|<\varepsilon \Rightarrow y \notin f[a,b]$

$\varepsilon_i := \sup\{\text{all possible } \varepsilon>0\}$ $\forall y$ $|y|<\varepsilon_i \Rightarrow y \notin f[a,b]$ & $\exists y$ $|y| \geq \varepsilon_i$ & $y \in f[a,b]$

$y - \varepsilon_i$





$\lim_{z \to L_i^-} f(L_i)$

Or $\forall \varepsilon > 0$ $\exists y$ $|y| < \varepsilon$ and $y = f(x_y)$ for some $x_y \in [a,b]$.

By AC we can choose unique y_n for each $\varepsilon = \frac{1}{n}$ and unique x_n so that $x_n \to$ some L (by BWT)

$\{y_n\}_{n=1}^{\infty} = \{f(x_n)\}_{n=1}^{\infty}$

$f(L) = \lim_{n\to\infty} f(x_n) = 0$ by continuity.

Self Proof of The Intermediate Value Theorem

Assume, for the sake of contradiction, that $f(c) \neq 0$ for all $c \in (a,b)$. Consider when there exists $\varepsilon > 0$ so for any y, if $0<y<\varepsilon$ then $y \notin f[a,b]$. Let $\varepsilon_s > 0$ be the supremum of such ε. For each $n \in \mathbb{N}$, there exists $x_n \in [a,b]$ with $\varepsilon_s \leq f(x_n) < \varepsilon_s + \frac{1}{n}$. By AC and the Bolzano–Weierstrass Theorem, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ of such x_n, which converges to some limit L. Its image $\{f(x_n)\}_{n=1}^{\infty}$ under f must also converge to ε_s by the Squeeze Theorem. So we can assume, without loss of generality, that $x_n \neq L$. Let L_i be the infimum of all such limits L. ~~If L_i is least, we can take any~~ corresponding ~~sequence $\{x_n\}_{n=1}^{\infty}$ as described earlier~~ (possibly L_i itself) ~~where we can suppose without loss of generality that $f(x_n)>0$, since $\lim_{n\to\infty} f(x_n) = \varepsilon_s > 0$. Otherwise,~~ For every $m \in \mathbb{N}$, there is some limit L_m for which $L_m - L_i < \frac{1}{m}$ by exercise 1-16. Thus also a corresponding sequence $\{(x_m)_n\}_{n=1}^{\infty}$ converging to L_m; such that for any $n \in \mathbb{N}$, there is $k_n \in \mathbb{N}$ with $\varepsilon_s \leq f((x_m)_{k_n}) < \varepsilon_s + \frac{1}{n}$ ~~for some $k_n \in \mathbb{N}$~~. Again, we can construct a sequence $\{\bar{x}_n\}_{n=1}^{\infty}$ so each $\bar{x}_n$ is some $(x_n)_{k_n}$. Therefore, in any case we have a sequence $\{\bar{x}_n\}_{n=1}^{\infty}$ converging to L_i and has $\lim_{n\to\infty} f(\bar{x}_n) = \varepsilon_s$. Now, $\lim_{z \to L_i} f(x) = \varepsilon_s$ by continuity. If there is some $\delta > 0$ for which every z with $L_i - \delta < z < L_i$ has $z \geq \varepsilon_s$, then let δ_s be the supremum of such δ. Then either $L_i - \delta = a$, hence $0 < \lim_{z\to a^-} f(z) = f(a) < 0$, or if $L_i - \delta > a$, then $\lim_{z \to a^-} f(z) < 0$ but $\lim_{z \to a^+} f(z) > 0$. A contradiction. Accordingly, no such $\delta > 0$ exists. Instead, this tells us there exists a sequence $\{y_n\}_{n=1}^{\infty}$ converging to L_i, having $y_n < L_i$ and $f(y_n) < 0$ for all $n \in \mathbb{N}$. Thence, $\lim_{z \to L_i} f(z) = \lim_{n\to\infty} f(y_n) \leq 0$ by continuity, a contradiction.

5-19 (e)

i. ~~Clearly, $\mathbb{1}_{[c,b]}$~~ $: [a,b] \to \mathbb{R}$ ~~is nondecreasing.~~ Let $\varepsilon > 0$, so by continuity, there is $\delta > 0$ such that $|z-c| < \delta$ implies $|f(z) - f(c)| < \varepsilon$. Now for any partition $P := \{a = x_0 < \cdots < x_n = b\}$ with norm $\|P\| < \delta$, and any associated evaluation set $T := \{t_1, \cdots, t_n\}$, let l be the least i with $c \in [x_{i-1}, x_i]$. It is clear that $c \in (x_{l-1}, x_l]$. Thus, $|S_g(f, P, T) - f(c)| = |\sum_{i=1}^{n} f(t_i)\Delta g_i - f(c)| = |f(t_l) - f(c)| < \varepsilon$ because $t_l, c \in [x_{l-1}, x_l]$ means $|t_l - c| \le |x_l - x_{l-1}| \le \|P\| < \delta$, and $\Delta g_i = \begin{cases} 0-0=0 & \text{if } i<l, \\ 1-0=1 & \text{if } i=l, \\ 1-1=0 & \text{if } i>l. \end{cases}$ Hence, $\int_a^b f \, d\mathbb{1}_{[c,b]} = f(c)$.

ii. If $f : [a,b] \to \mathbb{R}$ is not continuous at c, then for some $\varepsilon > 0$ and all $\delta > 0$, there exists $|z-c| < \delta$ with $|f(z) - f(c)| \ge \varepsilon$. Let $\delta > 0$; define the partition $P := \{a + \frac{b-a}{n} \cdot i \mid 0 \le i \le n\}$ for some $\frac{1}{n} < \delta$. There is a least l for which $c \in [x_{l-1}, x_l]$. Again, to be specific, $c \in (x_{l-1}, x_l]$. So $c - x_{l-1} > 0$. As such, there is $|t_l - c| < c - x_{l-1}$ with $|f(t_l) - f(c)| \ge \varepsilon$. As before, by defining $T := \{t_i \mid 1 \le i \le n\}$ where $t_i := a + \frac{b-a}{n} i$ for $i \ne l$, we note that $|S_g(f, P, T)| - f(c)| = |f(t_l) - f(c)|$. However, this is now instead at least as large as ε. Consequently, f is not Riemann-Stieltjes ~~integrable~~ integrable with respect to g. $\square$

(f) Ideas

$$\left| \sum_{i=1}^{n} f(t_i)\Delta g_i - \int_a^b f(x)\, g'(x)\, dx \right| \le \left| \sum_{i=1}^{n} \overbrace{f(t_i)}^{\sup|f[a,b]|} \underbrace{(\Delta g_i - g'(t_i)\Delta x_i)}_{g(t_i) - g(t_{i-1}) - g'(t_i)(x_i - x_{i-1})} \right| + \left| \sum_{i=1}^{n} f(t_i) g'(t_i) \Delta x_i - \int_a^b f(x) g'(x)\, dx \right|$$

$$\frac{\varepsilon}{2 \cdot \sup|f[a,b]| \cdot b-a}$$

$$\left| \frac{g(t_i) - g(t_{i-1})}{x_i - x_{i-1}} - g'(t_i) \right| < \varepsilon \qquad |t_{i-1} - t_i| < \delta_i$$

$$l := \inf\{\delta_x^{\max} > 0 \mid |z - x| < \delta_i \Rightarrow \left|\frac{g(z) - g(x)}{z - x} - g'(x)\right| < \varepsilon\}$$

$$\left| g(t_i) - g(t_{i-1}) - g'(t_i)(x_i - x_{i-1}) \right| < \varepsilon$$

if $l = 0$, $\exists \{z_n\}_{n=1}^{\infty} \to L$ + $\{\delta_{z_n}\}_{n=1}^{\infty} \to l = 0$

$$\left| \sum_{i=1}^{n} f(t_i)(\Delta g_i - g'(t_i)\Delta x_i) \right| \le \sum_{i=1}^{n} |f(t_i)| \, |\Delta g_i - g'(t_i)\Delta x_i| \le (b-a) \sup\{|f(x)| \mid x \in [a,b]\} \sum_{i=1}^{n} \left[\frac{g(t_i) - g(t_{i-1})}{x_i - x_{i-1}} - g'(t_i) \right]$$

$P \subseteq Q$: $|P| := n$, $|Q| := n+1$ $\qquad |P| < \delta \qquad b-a < |P|(b-a) < \frac{1}{m}$

Obviously any eval set T associated to P can be extended to an eval set T' associated to Q $\qquad m < \frac{b-a}{|P|}$

$t_i > t_{i-1} \ge t_i - \|P\|$

$g(t_i) > g(t_i - \|P\|)$ $\qquad g(t_i) - g(t_i - \overbrace{\|P\|}^{\frac{1}{m}}) \ge g(t_i) - g(t_{i-1})$ $\qquad \lim_{m \to \infty} \sum_{i=1}^{n} \left[\frac{g(t_i) - g(t_i - \frac{1}{m})}{\frac{1}{m}} - g'(t_i) \right] = 0$

$x_i - (x_i - \frac{1}{m}) = \frac{1}{m} \ge x_i - x_{i-1}$

Theorem 6.18

Assume $\sum_{j=1}^{\infty} a_j$ converges absolutely, and let σ be a permutation of $\mathbb{N}$. For ε 0, pick ...

$$\sum_{j=1}^{\hat{}} |a_j| < \varepsilon + \sum_{j=1}^{\infty} |a_j| .$$

Letting $u_n := \max\{\sigma^{-1}(j) \mid 1 \le j \le n\}$, we see that

$$\sum_{i=1}^{n} a_{\sigma(i)} \le \sum_{j=1}^{u_n} a_j \le \sum_{j=1}^{u_n} |a_j| < \varepsilon + \sum_{j=1}^{\infty} |a$$

So, ...

Now consider when $\sum_{j=1}^{\infty} a_j$ does not converge absolutely. Then for all $L \in \mathbb{R}$ there exists $\varepsilon > 0$ such that ...

$$\left| \sum_{j=1}^{n} |a_j| - L \right| \ge \varepsilon$$

$\sum p_n \to \infty$ $\sum q_n \to -\infty$

$[l_0, l_1]$ q_1 $[l_1, l_2]$

$[l_{n-1}, l_n]$ q_n $[l_n, l_{n+1}]$

$\sigma(1) = p_1$

$\sigma(2) = p_2$

$\vdots$

$\sigma($

Proposition 2.22

Exercise 6-22

(since $c>0$) Wlog, $\sum_{j=1}^{\infty} b_j = S$

$$\left| \frac{a_j}{b_j} - c \right| < \varepsilon$$

$$| a_j - b_j c | < b_j \varepsilon < \varepsilon \quad \text{(for large } j\text{)}$$

Eventually: $a_j \approx c b_j$

$$\lim_{j\to\infty} (a_j - c b_j) = 0$$

$$\lim_{j\to\infty} (b_j - c^{-1} a_j) = 0$$

What if $c=0$?

$$\sum_{j=m}^{n} a_j \leq c \left| \sum_{j=m}^{n} b_j \right| ? \quad \text{if } c \geq 1$$

exists $N \in \mathbb{N}$ with $N > c \iff \frac{c}{N} < 1$.

Ftsoc, $\forall x > 0 \ \forall K \in \mathbb{N} \ \exists k \geq K$: $a_j > x b_j$

$$\frac{a_j}{b_j} > x$$

E.g. $\frac{a_j}{b_j} > c+1$

$$\frac{a_j}{b_j} = \frac{5 b_j}{b_j} = 5$$

$\cancel{\forall} c > 0$?

$$0 < \frac{a_j}{b_j} < u$$

$$0 < a_j < u b_j \quad \square$$

$$0 < \frac{\frac{1}{2^n}}{2^n} < 1$$

$$\frac{1}{2^n} < 2^n$$

Random Scribbles

$p_1, p_2, \ldots, p_{l_1}, q_1, q_2, \ldots, q_{l_1},$

$p_{l_1+1}, p_{l_1+2}, \ldots, p_{l_{l_1}}, q_{l_{l_1}+1}, q_{l_{l_1}+2}, \ldots$

$U(f,P) - L(f,P) \leq R($

$a < b < c < d$

$0 < b-a < c-a < d-a$ $\quad$ $a-b < 0 < c-b < d-b$ $< d-$

$a < b < c$ $\quad$ $0 < c-b < c-a$

$c-a > c-b$?

Proposition 6.13

$\forall \varepsilon \exists N \forall n \geq N$

$$\left| \sum_{j=1}^{n} |a_j| - S \right| < \varepsilon$$

$$\left| \sum_{j=m}^{n} a_j \right|$$

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} a_j \right| = \left| \underbrace{\lim_{n\to\infty} \sum_{j=1}^{n} a_j}_{L} \right|$$

$$\left| \left| \sum_{j=1}^{n} a_j \right| - |L| \right| < \left| \left(\sum_{j=1}^{n} a_j \right) - L \right|$$

$$a_n \geq b_n$$

$$a_n - b_n \geq 0$$

$$\lim_{n\to\infty} (a_n - b_n) \geq 0$$

$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$$

Theorem 6.15 ✓

$(-1)^{n+2}\frac{1}{n+1}+$

6-9 $\left|\sum_{j=n+1}^{m}(-1)^{j+1}\frac{1}{j}\right| = \left|\sum_{j=1}^{m-n}(-1)^{n+1+j}\frac{1}{n+j}\right|$

$= \left|(-1)^{n+1}\sum_{j=1}^{m-n}(-1)^{j}\frac{1}{n+j}\right|$

$< \sum_{j=1}^{m-n}(-1)^{j}\frac{1}{n+j+\frac{1+(-1)^{j+1}}{2}}$

$= \frac{1}{n+1}$ or $\frac{1}{n+1}-\frac{1}{m-n+1}$

$< \frac{1}{n+1}-\frac{1}{m-n+1}$

$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots$

$\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+3}-\frac{1}{n+4}+\frac{1}{n+4}-\cdots-\frac{1}{m-n}$ or $-\frac{1}{m-n+1}$

1 if j odd

0 if j even

$\frac{1+(-1)^{j+1}}{2}$

6-11 Let $\{a_j\}_{j=1}^{\infty}$ be a nondecreasing sequence of ~~nonnegative numbers~~, which is bounded above by u.

$a_{j+1}-a_j \geq 0$ (non decreasing)

b_{m+1} or $b_{m+1}-b_{n+1}$

Theorem 6.11

$\left|\sum_{j=m+1}^{n}(-1)^{j+1}b_j\right| \leq \left|\sum_{j=1}^{n-m}(-1)^{m+1+j}b_{m+j}\right| \leq \left|\sum_{j=1}^{n-m}(-1)^{j+1}b_{m+j}\right| \leq \sum_{\substack{j=1\\ j\text{ odd}}}^{n-m}b_{m+j} - \sum_{\substack{j=1\\ j\text{ even}}}^{n-m}b_{m+j+1} \leq b_{m+1} < \varepsilon$ since $\lim_{j\to\infty}b_j=0$

Nonincreasing

Nonnegativity

$$\overbrace{a + q(b-a)}^{qb - qa} < \overbrace{b - q(b-a)}^{-qb + qa}$$

$$\Leftrightarrow \quad a - 2qa < b - 2qb \qquad (1-2q) \text{ where } q < \tfrac{1}{2}$$

$$\Leftrightarrow \quad a < b$$

4-25

Let $x \in (a,b)$.

$\exists \delta > 0 \ \forall z \in (a,b) - \{x\}$: f' is continuous at z ?

$\forall \delta > 0 \ \exists z \in (a,b) - \{x\}$: f' is not continuous at z.

wlog, $f'(d) > f'(c)$.

$m := \inf \{ x \in (c,d) \mid f'(x) \in [v, f'(d)] \}$

7-23

$g : \mathbb{N} \to$ set of all discontinuities $\mathcal{D}$

$H : \mathbb{N} \to \mathbb{R}$

$H(n) =$ (

$s : \mathbb{N}_{\mathbb{N}} \to {}^{\mathbb{N}}\mathcal{D}$

$s(h) \in {}^{\mathbb{N}}\mathcal{D} - s[h]$ where ${}^{\mathbb{N}}\mathbb{N}$ is ordered lexicographically

$\sum_{i=1}^{\infty} \delta^i = \frac{\delta}{1-\delta} = -1 + \frac{1}{1-\delta}$

$\Sigma : \mathbb{N} \to \mathbb{R}$

$\Sigma(n) = \sum_{i_n=1}^{\infty} \sum_{i_{n-1}=1}^{\infty} \cdots \sum_{i_1=1}^{\infty} s(i_1, i_2, \ldots, i_n, 1, 1, \ldots)$

$\Sigma(n) \to S \le \underbrace{(u-l)}_{\delta}$

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$, such that when $n \ge N$,

$S - \Sigma(n) < \varepsilon$

$n \ge m \ge N$

$\Sigma(n) - \Sigma(m) < \varepsilon$

$\sum \cdots \sum s(i_1, i_2, \ldots, i_{m+1}, \ldots) < \varepsilon$

7-24

$h : \mathbb{N} \to [0,1]$

$h(1) = \frac{1}{2} \qquad h(n+1) = \frac{h(n)+1}{2}$

$h \circ f$

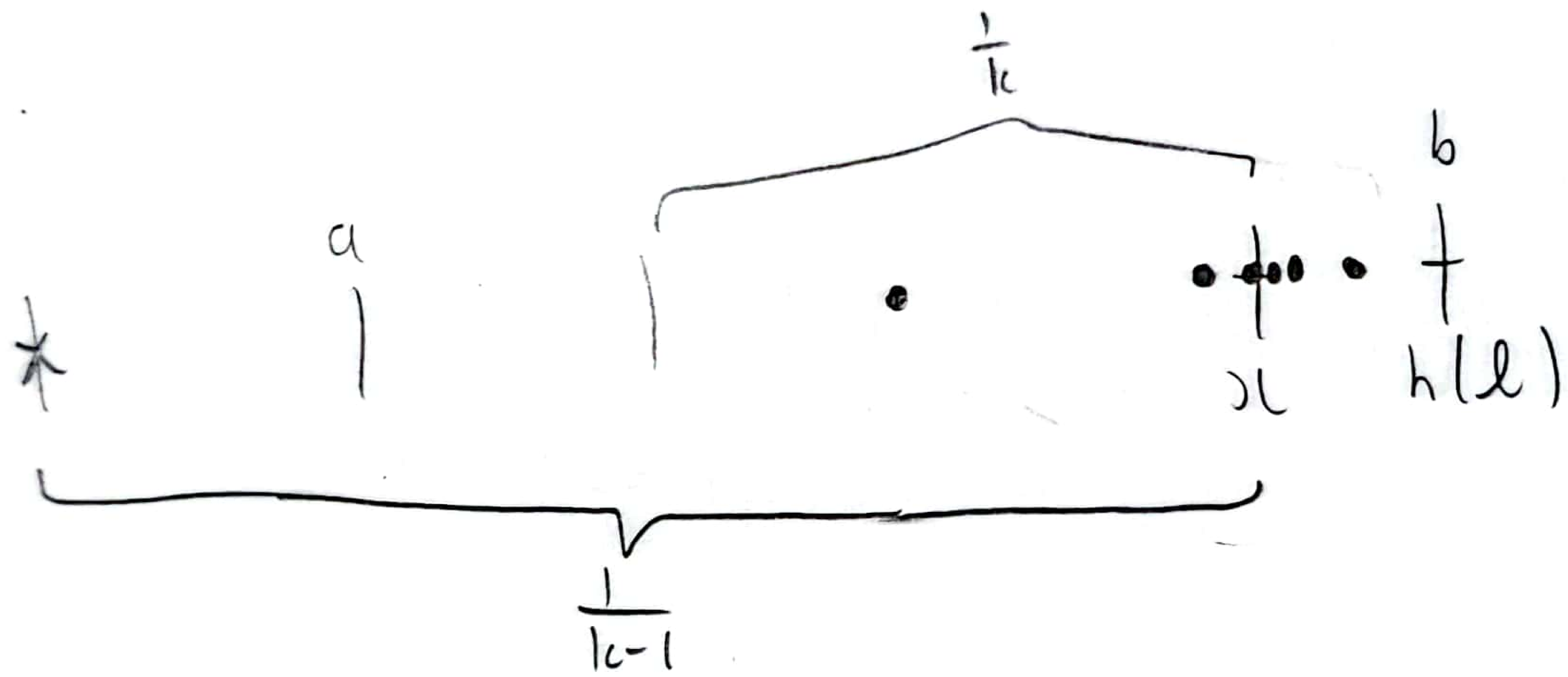
$\left(\frac{1}{2}\right)^{n-1}$

Define $\psi := h \circ \varphi$

$$\frac{b-\psi(n)}{b-\psi(1)} = \left(\frac{1}{2}\right)^{n-1}$$

$n=1: \; 1=1$ ✓

$$n=k+1: \; \frac{b - \frac{\psi(k)+b}{2}}{b-\psi(1)} = \frac{\frac{b-\psi(k)}{2}}{b-\psi(1)} = \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} = \left(\frac{1}{2}\right)^{k}$$



$E > x - f(k) - \frac{2}{\alpha}$

$E - (x - f(k)) > -\frac{2}{\alpha}$

$x + E + x - (x - f(k)) > 2x - \frac{2}{\alpha}$

$$\frac{+E+f(k)}{2} = \frac{x + E + x - (x - f(k))}{2} > x - \frac{1}{\alpha}$$

ε

x

$\frac{1}{\alpha}$

$\frac{1}{\alpha-1}$

$$\frac{x+\varepsilon+x-\frac{1}{\alpha-1}}{2} \le x -$$

$\varepsilon - \frac{1}{\alpha-1} \le 0$

$\varepsilon \le \frac{1}{\alpha-1}$

$$\frac{x+\varepsilon+x-\frac{1}{\alpha-1}}{2} > x - \frac{1}{\alpha}$$

$\varepsilon - \frac{1}{\alpha-1} > -\frac{2}{\alpha}$

$\varepsilon > \frac{1}{\alpha-1} - \frac{2}{\alpha}$

$\varepsilon \in \left(\frac{1}{\alpha-1} - \frac{2}{\alpha}, \frac{1}{\alpha-1}\right]$

$\varepsilon \in \left(x - b - \frac{2}{\alpha}, x - b\right]$

8.3 (c)

$$\prod_{j=1}^{i} q_j \geq \prod_{j=1}^{m} 2q_j = 2^m \prod_{j=1}^{m} q_j$$

$$\prod_{j=1}^{m-i} q_j \leq 2^{-m}$$

2^{j+1}

$$\prod_{j=1}^{n} \frac{2^{j+1}-1}{2^{j+1}} = \frac{\cancel{(2-1)}(4-1)(8-1)\cdots(2^{n+1}-1)}{}$$

(e)

Let $\varepsilon > 0$. There exists $M \in \mathbb{N}$, such that for all $m \geq M$,

$$1 - \left(1 - \frac{1}{2^{m+1}}\right) < \varepsilon$$

$$1 - \varepsilon < 1 - \frac{1}{2^{m+1}}$$

$$\frac{2^2-1}{2^2} = \frac{3}{4}$$

Q is bounded below by $\frac{3}{4}$

$$1 - \frac{1}{2^{m+1}} > 1 - \frac{1}{m+3} = \frac{m+2}{m+3}$$

$(1-z_1)(1-z_2)\cdots(1-z_n)$

$$1 + \sum_{i=1}^{n} \sum_{j=1}^{n-i} \binom{n-i}{j}$$

(1, 1) (1, 2)

(2, 1) (2, 2)

$\vdots$ $\vdots$

$j=1$ $j=n$

3 4 5 6

4 5 6 7

$$1 - \frac{1}{2^{m+1}} > 1 - \frac{1}{m^2+3} = \frac{m^2+2}{m^2+3}$$

$$\frac{\prod_{j=1}^{n+1}(2^{j+1}-1) - \prod_{j=1}^{n}(2^{j+1}-1)}{\prod_{j=1}^{n+1} 2^{j+1} - \prod_{j=1}^{n} 2^{j+1}} = \frac{(2^{n+2}-2)\prod_{j=1}^{n}(2^{j+1}-1)}{(2^{n+2}-1)\prod_{j=1}^{n} 2^{j+1}}$$

7-24

$$\frac{f(a)+f(k)}{2} \in \left(x-\frac{1}{\alpha},\ x\right)$$

$$f(a) > 2x - f(k) - \frac{?}{?}$$

$\Rightarrow 2x-\frac{2}{\alpha}-f(k) < f(a) < 2x-f(k)$

$2x-\frac{2}{\alpha}-\left(x-\frac{1}{\alpha}\right) <$

$= x-\frac{1}{\alpha}$

$2x-\left(x-\frac{1}{\alpha}\right)$

$= x+\frac{1}{\alpha}$

$2x-\frac{2}{\alpha}-\left(x-\frac{1}{\alpha-1}\right)$

$= x-\frac{2}{\alpha}-\frac{1}{\alpha-1}$