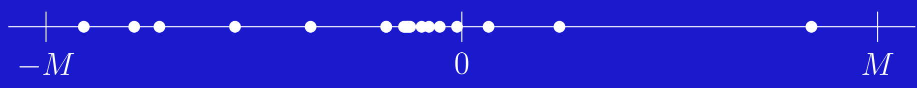

Grass

Solutions to Schröder

January 2023-Today

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Chapter 2

Sequences of Real Numbers

§2.5 Infinite Limits

Exercise 2.51. Prove **Cauchy's Limit Theorem**[a]. That is, let $\{b_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive numbers that go to infinity and let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Prove that if the sequence $\left\{\frac{a_n - a_{n-1}}{b_n - b_{n-1}}\right\}_{n=2}^{\infty}$ converges to c, then $\lim_{n\to\infty} \frac{a_n}{b_n} = c$.

Hint. Exercise 2-16 with $p_n := b_n - b_{n-1}$ and another appropriate sequence.

[a] Also known as the Stolz-Cesaro Theorem

Proof. Notice that $\lim_{n\to\infty} \sum_{k=1}^{n} b_{k+1} - b_k = \lim_{n\to\infty} b_{n+1} - b_1 = \infty$. Hence, by exercise 2-16,

$$c = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} (b_{k+1} - b_k)\left(\frac{a_{k+1} - a_k}{b_{k+1} - b_k}\right)}{\sum\limits_{k=1}^{n} (b_{k+1} - b_k)} = \lim_{n\to\infty} \frac{a_{n+1} - a_1}{b_{n+1} - b_1} = \lim_{n\to\infty} \frac{a_{n+1}}{b_{n+1} - b_1}.$$

Furthermore, we see that

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{b_{n+1} - b_1} - \frac{a_{n+1}}{b_{n+1}}\right) = \lim_{n\to\infty} b_1 \cdot \lim_{n\to\infty} \frac{a_{n+1}}{b_{n+1} - b_1} \cdot \lim_{n\to\infty} \frac{1}{b_{n+1}} = 0.$$

Consequently,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c.$$

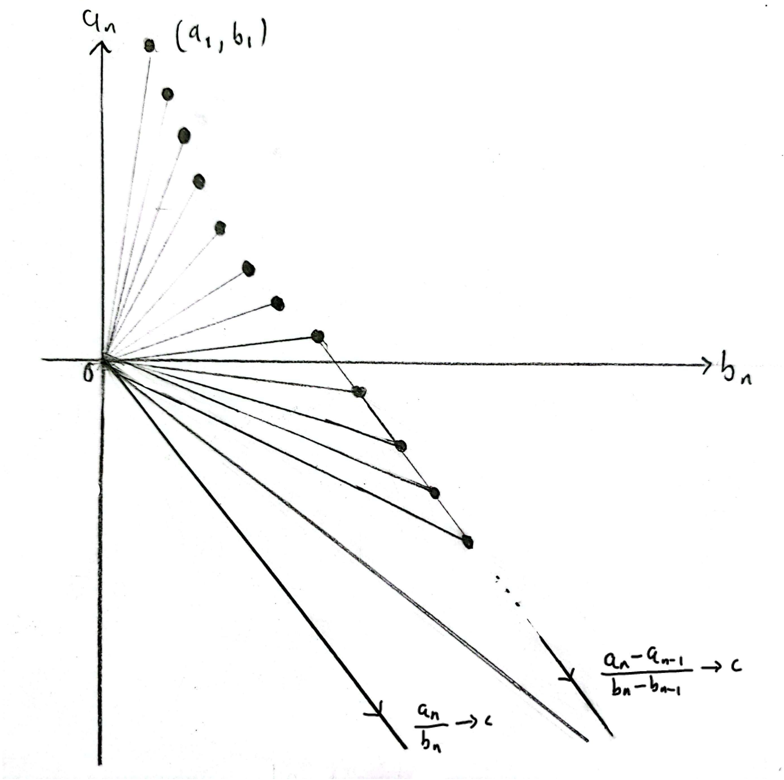

Figure 2.1: An Illustration of Cauchy's Limit Theorem

Chapter 3

Continuous Functions

§3.5 Properties of Continuous Functions

Theorem 3.34. The Intermediate Value Theorem Let $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$ (or vice versa) then there is a $c \in (a,b)$ such that $f(c) = 0$ (Also see Figure 8(a).)

V1:

Proof. Assume, for the sake of contradiction, that $f(c) \neq 0$ for all $c \in (a,b)$.

First consider the case where for all $n \in \mathbb{N}$, there exists $x_n \in [a,b]$ so $0 < f(x_n) < 1/n$. By AC and the Bolzano-Weierstrass Theorem, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ of such x_n, that converges to some limit L and whose image under f, $\{f(x_n)\}_{n=1}^{\infty}$, goes to 0. Consequently, from The Squeeze Theorem and continuity, $f(L) = \lim_{n\to\infty} f(x_n) = 0$, contradicting our initial assumption that $0 \notin f[(a,b)]$.

Now consider when there exists $\varepsilon > 0$ so for any y, if $0 < y < \varepsilon$ then $y \notin f[a,b]$. Let $\varepsilon_s > 0$ be the supremum of such ε. For each $n \in \mathbb{N}$, there exists $x_n \in [a,b]$ with $\varepsilon_s \leq f(x_n) < \varepsilon_s + 1/n$. Again, construct a sequence $\{x_n\}_{n=1}^{\infty}$ of such x_n, which converges to some limit L. Its image under f, $\{f(x_n)\}_{n=1}^{\infty}$, must also converge to ε_s. Let L_i be the infimum of such limits L. For every $m \in \mathbb{N}$, there is some limit L_m (possibly L_i itself) for which $L_m - L_i < 1/m$. Thus, also a corresponding sequence $\{x_{m,n}\}_{n=1}^{\infty}$ converging to L_m; such that for any $n \in \mathbb{N}$, there is a $k_n \in \mathbb{N}$ with $\varepsilon_s \leq f(x_{m,k_n}) < \varepsilon + 1/n$. By having $\overline{x}_m$ be some x_{m,k_m}, we can once more construct a sequence $\{\overline{x}_m\}_{m=1}^{\infty}$ converging to L_i, so $\lim_{z\to L_i} f(z) = \lim_{n\to\infty} f(\overline{x}_n) = \varepsilon_s$ by continuity.

If there is some $\delta > 0$ for which every z with $L_i - \delta < z < L_i$ has $f(z) \geq \varepsilon_s$, then let δ_s be the supremum of such δ. Then either $L_i - \delta_s = a$, hence $0 < \lim_{z\to a^+} f(z) = f(a) < 0$, or if $L_i - \delta_s < a$, then $\lim_{z\to(L_i-\delta_s)^-} f(z) < 0$ but $\lim_{z\to(L_i+\delta_s)^+} f(z) > 0$. A contradiction to continuity. Accordingly, no such $\delta > 0$ can exist. Instead, this tells us that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $[a, L_i)$ converging to L_i, having $y_n < L_i$ and $f(y_n) < 0$, for all $n \in \mathbb{N}$. Thence, $\lim_{z\to L_i} f(z) = \lim_{n\to\infty} f(y_n) \leq 0$ by continuity, a contradiction to the conclusion in the prior paragraph.

V2:

Proof. Assume, for the sake of contradiction, that $f(c) \neq 0$ for all $c \in (a, b)$.

First consider the case where for all $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ so $f(x_n) \in (0, 1/n)$. By AC and the Bolzano-Weierstrass Theorem, there is hence a sequence $\{x_n\}_{n=1}^{\infty}$ converging to some $L \in [a, b]$ having $f(x_n) \in (0, 1/n)$ for each n, so continuity with The Squeeze Theorem says $f(n) = \lim_{L\to\infty} f(x_n) = 0$. A contradiction.

Now consider when there exists $\varepsilon > 0$ so for any y, if $y \in (0, \varepsilon)$ then $y \notin f[a, b]$. Let $\varepsilon_s > 0$ be the supremum of all such ε, bounded above by $f(b)$; allowing us to again construct a sequence $\{x_n\}_{n=1}^{\infty}$ converging to some $L \in [a, b]$, but with $f(x_n) \in [\varepsilon_s, \varepsilon_s + 1/n)$ for each $n \in \mathbb{N}$.

Let L_i be the infimum of all possible limits $L \in [a, b]$, for which there exists any accompanying sequence $\{x_n\}_{n=1}^{\infty}$ as described above which converges to L. For every $m \in \mathbb{N}$, there therefore is a limit $L_m \in [L_i, L_i + 1/m)$ and its corresponding sequence $\{x_{m,n}\}_{n=1}^{\infty}$ with some $x_{m,k_m} \in (L_m - 1/m, L_m + 1/m) \subseteq (L_i - 1/m, L_i + 2/m)$ and $f(x_{m,k_m}) \in [\varepsilon_s, \varepsilon_s + 1/m)$. Choosing a x_{m,k_m} for each $\overline{x}_m$, we have a sequence $\{\overline{x}_m\}_{m=1}^{\infty}$ that converges to L_i, such that $\lim_{z\to L_i} f(z) = \varepsilon_s$.

Finally, let $y_i := \inf\{x \mid f(x) \geq \varepsilon_s\}$ and presume $y_i \in (a, L_i)$. Thus $f(y_i) = \lim_{z\to y_i^-} f(z) \leq 0$ and it must be that, for any $n \in \mathbb{N}$, there exists $y_n \in (y_i, y_i + 1/n)$ so $f(y_n) \geq \varepsilon$; $\lim_{z\to y_i^+} f(z) = \lim_{n\to\infty} f(y_n) \geq \varepsilon$. A contradiction, implying $y_i > L_i$. However, now $\lim_{z\to L_i^-} f(z) \leq 0$, contradicting $\lim_{z\to L_i} f(z) = \varepsilon_s$.

Wherefore, our initial assumption is impossible, suggesting that the Intermediate Value Theorem is true.

Chapter 4

Differentiable Functions

§4.3 Rolle's Theorem and the Mean Value Theorem

Exercise 4.24. Let $f\colon (a,b) \to \mathbb{R}$ be continuous and let $x \in (a,b)$. Prove that if $f'(z)$ exists for all $z \in (a,b) \setminus \{x\}$ and $\lim_{z\to x} f'(z)$ exists, then f is differentiable at x with $f'(x) = \lim_{z\to x} f'(x)$.

Proof. Let $\varepsilon > 0$, we shall unpack 3 definitions first:

1. Since $L := \lim_{z\to x} f'(x)$ exists, there be some $\delta_1 > 0$ such that whenever $y \in (a,b) \setminus \{x\}$ has $|y-x| < \delta_1$, then $\left|\lim_{z\to y} \frac{f(z)-f(y)}{z-y} - L\right| < \frac{1}{4}\varepsilon$.
2. Given any $y \in (a,b)$, we have the limit $L_y := \lim_{z\to y} \frac{f(z)-f(y)}{z-y}$. So for some $\delta_2 > 0$, if $z \in (a,b) \setminus \{y\}$ and $|z-y| < \delta_2$, then $\left|\frac{f(z)-f(x)}{z-x} - L_y\right| < \frac{1}{4}\varepsilon$.
3. Finally, by limit laws, there is a $\delta_3 > 0$, so for each $y \in (a,b) \setminus \{x\}$ and $|y-x| < \delta_3$, $\left|\frac{f(z)-f(x)}{z-x} - L\right| < \frac{1}{2}\varepsilon$.

Now, define $\delta := \min\{\delta_1, \delta_2/2, \delta_3\}$ and let $z \in (a,b) \setminus \{x\}$ so $|z-x| < \delta$, and $y \in (a,b)\setminus\{x\}$ with $|y-x| < \delta$. Then, $|z-y|-|y-x| \le |z-y-(x-y)| = |z-x| < \delta$. Accordingly, $|z-y| < |y-x| + \delta < \delta_2/2 + \delta_2/2 = \delta_2$. Therefore, the inequalities of the previous paragraph hold. Which means,

$$\left|\frac{f(z)-f(y)}{z-y} - L\right| = \left|\frac{f(z)-f(y)}{z-y} - L_y + (L_y - L)\right| < \frac{1}{2}\varepsilon.$$

As such,

$$\left|\frac{f(z)-f(x)}{z-x} - L\right| \le \left|\frac{f(z)-f(x)}{z-x} - \frac{f(z)-f(y)}{z-y}\right| + \left|\frac{f(z)-f(y)}{z-y} - L\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon.$$

In other words, we have shown that for any $\varepsilon > 0$, there is $\delta > 0$, so when $z \in (a,b) \setminus \{x\}$ such that $|z-x| < \delta$, $\left|\frac{f(z)-f(x)}{z-x} - L\right| < \varepsilon$ holds true. Consequently, we have shown that $f'(x) := \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$ exists, and is in fact just $L := \lim_{z\to x} f'(x)$.

Exercise 4.25. Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. Prove that f' (which need not be continuous) has the **intermediate value property**. That is, prove that for all $c < d$ in (a,b) and all v between $f'(c)$ and $f'(d)$ there is an $m \in (c,d)$ so that $f'(m) = v$.

Proof. *(Work in progress)*

Exercise. Is f' continuous a.e.?

Chapter 5

The Riemann Integral I

Exercise 5.20. **Power Rule** for integration. Let $r \in \mathbb{Q} \setminus \{1\}$ and let $a < b$. In case $r < 0$, let a and b either both be positive or both be negative. Prove that

$$\int_a^b x^r \, dx = \frac{1}{r+1} b^{r+1} - \frac{1}{r+1} a^{r+1}.$$

Then explain why we needed to require a and b to be both positive or both negative for $r < 0$.

Proof. This is clear from FTC and the Power Rule for differentiation. For $r < 0$, notice 0^r is undefined. Furthermore, even if we were to modify the integrand to be defined at $x = 0$, we see that x^r would still be unbounded on $[a, b]$. So, 5-10 tells us x^r is not integrable on $[a, b]$.

Exercise 5.21. **Integration by Parts.** Let $[a, b] \subset (c, d)$ and let $F, g \colon (c, d) \to \mathbb{R}$ be continuously differentiable with derivatives f and g'. Prove that

$$\int_a^b f(x) g(x) \, dx = F(b) g(b) - F(a) g(a) - \int_a^b F(x) g'(x) \, dx.$$

Proof. By the Product Rule,

$$(Fg)' = fg + Fg'.$$

Since products of continuous functions are continuous, they are Riemann integrable. Hence FTC implies

$$\int_a^b f(x) g(x) \, dx = F(b) g(b) - F(a) g(a) - \int_a^b F(x) g'(x) \, dx.$$

Exercise 5.22. Integration by Substitution. Let $[a, b] \subset (c, d)$, let $g\colon (c, d) \to \mathbb{R}$ be continuously differentiable with derivative g' and let F be continuously differentiable with derivative f such that the domain of F contains $g[\![[a, b]]\!]$. Prove that

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

Proof. By the Chain Rule,

$$(F \circ g)'(x) = f(g(x))g'(x).$$

Since products/compositions of continuous functions are continuous by limit laws/Theorem 3.30, they are Riemann integrable. By FTC,

$$\int_a^b f(g(x))g'(x)\,dx = F(g(b)) - F(g(a)).$$

Theorem 5.29. Let $f\colon [a, b] \to \mathbb{R}$ be a bounded function. Then f is Darboux integrable on $[a, b]$ iff f is Riemann integrable on $[a, b]$. Moreover, the Darboux and Riemann integrals are equal in this case, that is,

$$\int_a^b f(x)\,dx = \mathcal{L} = \mathcal{U}.$$

Proof. Let $\varepsilon > 0$.
First consider f being Darboux integrable. From Lemma 5.16, there must be a partition R of $[a, b]$ so $\mathcal{L}_f - L(f, R) < \frac{1}{2}\varepsilon$ and $U(f, R) - \mathcal{U}_f < \frac{1}{2}\varepsilon$. Hence, f is Riemann integrable since

$$U(f, R) - L(f, R) \leq U(f, R) - \mathcal{L}_f + \mathcal{L}_f - L(f, R) < \varepsilon.$$

Conversely, when f is Riemann integrable, there is a partition S with $\mathcal{U}_f - L(f, S) < \frac{1}{2}\varepsilon$ and $U(f, S) - \mathcal{L}_f < \frac{1}{2}\varepsilon$. Consequently, f is Darboux integrable as

$$\mathcal{U}_f - \mathcal{L}_f \leq \mathcal{U}_f - L(f, R) - \mathcal{L}_f + U(f, R) < \varepsilon.$$

Furthermore, Lemma 5.14 guarantees the equality

$$\int_a^b f(x)\,dx = \mathcal{L} = \mathcal{U}.$$

Theorem. If $g\colon [a, b] \to \mathbb{R}$ is Riemann integrable, so is $|g|$.

Proof. Let $\varepsilon > 0$. Define $n_i := \min\{|m_i|, |M_i|\}$ and $N_i = |m_i| + |M_i| - n_i$. Wlog, $n_i = |m_i|$. Notice that

$$N_i - n_i = |M_i| - n_i + |m_i| - n_i \leq |M_i - m_i|.$$

By Riemann's Condition, there is a partition P for which $U(g, P) - L(g, P) < \varepsilon$. As such,

$$U(|g|, P) - L(|g|, P) = \sum (N_i - n_i)\Delta x_i \leq \sum (M_i - m_i)\Delta x_i < \varepsilon.$$

Lemma. Let $g\colon [a, b] \to \mathbb{R}$ be Riemann integrable. Then, for any sequence of partitions $\{P_k\}_{k=1}^{\infty}$ with norm $\|P_k\|$ converging to 0,

$$\lim_{k\to\infty} L(g, P_k) = \int_a^b g(x)\, dx.$$

Proof. Let $\varepsilon > 0$. For each k, there is t_i such that

$$g(t_i) - m_i < \frac{\varepsilon}{2k\Delta x_i}.$$

Accordingly, for the evaluation set T_k consisting of all such t_i,

$$R(g, P_k, T_k) - L(g, P_k) < \frac{1}{2}\varepsilon.$$

By Lemma 5.6, there is some K, such that when $k \geq K$, we have

$$\left| R(g, P_k, T_k) - \int_a^b g(x)\, dx \right| < \frac{1}{2}\varepsilon.$$

Hence, using the triangular inequality,

$$\left| L(g, P_k) - \int_a^b g(x)\, dx \right| < \varepsilon.$$

Exercise 5.26. Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$.

(a) For each $n \in \mathbb{N}$, let $P_n = \{a = x_0^{(n)} < \cdots < x_{k_n}^{(n)} = b\}$ be a partition of the interval $[a, b]$ with $\|P_n\| < \frac{1}{n}$ and let

$$s_n := \left(\sum_{i=1}^{k_n - 1} m_i^{(n)} \mathbf{1}_{\left[x_{i-1}^{(n)}, x_i^{(n)}\right)}\right) + m_{k_n}^{(n)} \mathbf{1}_{\left[x_{k_n-1}^{(n)}, x_{k_n}^{(n)}\right]},$$

where $m_i^{(n)} = \inf\left\{ f(x) \,\middle|\, x \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right]\right\}$.

i. Prove that for all $n \in \mathbb{N}$ and all $x \in [a, b]$ we have $s_n(x) \leq f(x)$.

ii. Prove that if f is continuous at $x \in [a, b]$, then $\{s_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$.

iii. Prove that

$$\lim_{n\to\infty} \int_a^b |f - s_n| \, dx = 0.$$

(b) Prove that there is a sequence $\{c_n\}_{n=1}^{\infty}$ of continuous functions on $[a, b]$ such that for all $n \in \mathbb{N}$ we have $|c_n| \leq |f|$ and so that

$$\lim_{n\to\infty} \int_a^b |f - c_n| \, dx = 0.$$

Proof.

(a)

i. Fix $n \in \mathbb{N}$ and $x \in [a, b]$. There exists i with $x \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right)$. So,

$$s_n(x) = m_i^{(n)} \leq f(x).$$

ii. Let f be continuous at $x \in [a, b]$, and $\varepsilon > 0$. Then, there is $\delta > 0$, such that if $|z - x| < \delta$, then $|f(z) - f(x)| < \varepsilon/2$. Moreover, $\delta > 1/K$ for some $K \in \mathbb{N}$. When $k \geq K$, there also exists $|y - x| < 1/k$ with $f(y) - s_k(x) < \varepsilon/2$. Consequently,

$$f(x) - s_k(x) \leq |f(x) - f(y)| + |f(y) - s_k(x)| < \varepsilon.$$

iii. Let $\varepsilon > 0$. By our lemma, there exists $N \in \mathbb{N}$ for which when $n \geq N$,

$$\int_a^b |f - s_n| \, dx = \int_a^b f(x) \, dx - \int_a^b s_n(x) \, dx = \int_a^b f(x) \, dx - L(f, P_n) < \varepsilon.$$

(b) *(Work in progress)*

Exercise 5.27. Prove **Riemann's Condition** for Riemann-Stieltjes integrals. THat is, let $f\colon [a, b] \to \mathbb{R}$ be bounded. Let $g\colon [a, b] \to \mathbb{R}$ be nondecreasing and prove that f is Riemann-Stieltjes integrable on $[a, b]$ with respect to g iff for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U_g(f, P) - L_g(f, P) < \varepsilon$.

Chapter 6

Series of Real Numbers I

§6.1 Series as a Vehicle To Define Infinite Sums

Definition. Let $\{x_n\}_{n=1}^{\infty}$ be sequence converging to 0, and m_n to be the least number such that $|x_{m_n} - L| < 1/n$. The *1-rate of convergence* and *2-rate of convergence* are, respectively,

$$r_1 := \lim_{n\to\infty} \frac{m_{n+1}}{m_n} \qquad \text{and} \qquad r_2 := \lim_{n\to\infty} \frac{m_{n+2} - m_{n+1}}{m_{n+1} - m_n}.$$

Question. Does r_2 always exist?

Question. Is there any range of values of r_1 and/or r_2 for which $\sum_{n=1}^{\infty} x_n$ is guareenteed to converge? Can we establish a biconditional?

Exercise 6.2. Compute the value of each of the series below

(d)

$$\sum_{j=4}^{\infty} \frac{2^{j-2}}{5^{j+4}}$$

Proof. (d) By theorem 6.2, we compute

$$\sum_{j=4}^{\infty} \frac{2^{j-2}}{5^{j+4}} = \frac{1}{2500} \cdot \frac{2}{5} \left[\frac{1}{1 - \frac{2}{5}} - \frac{1 - \left(\frac{2}{5}\right)^3}{1 - \frac{2}{5}} \right] = \frac{4}{234375}.$$

Exercise 6.3. More on the artihmetic of series (Theorem 6.4).

(c) Give an example of two series $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ such that $\sum_{j=1}^{\infty} a_j + b_j$ converges but neither $\sum_{j=1}^{\infty} a_j$ nor $\sum_{j=1}^{\infty} b_j$ converges.

(d) Is it possible to find two series $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ such that $\sum_{j=1}^{\infty} a_j + b_j$

converges, and exactly one of $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ converges.

(e) Is there a series $\sum_{j=1}^{\infty} a_j$ and a $c \in \mathbb{R}$ such that $\sum_{j=1}^{\infty} ca_j$ converges, but $\sum_{j=1}^{\infty} a_j$ diverges?

Proof. (c) Let $a_j = 1$ and $b_j = -1$ for all j. Then, $\sum_{j=1}^{\infty} a_j = \infty$ and $\sum_{j=1}^{\infty} b_j = -\infty$, while $\sum_{j=1}^{\infty} a_j + b_j = 0$.

(d) It is impossible. Let $\sum_{j=1}^{\infty} a_j = S$ be a convergent series, and $\sum_{j=1}^{\infty} b_j$ a divergent series.

Also let $L \in \mathbb{R}$. Then, there exists $\varepsilon > 0$ and some $n \in \mathbb{N}$, for which $\left|\sum_{j=1}^{\infty} a_j - S\right| < \varepsilon$ and $\left|\sum_{j=1}^{\infty} b_j - (L - S)\right| \geq 2\varepsilon$. Then, by the reverse triangle inequality,

$$\left|\sum_{j=1}^{\infty} a_j + b_j - L\right| \geq \left|\sum_{j=1}^{\infty} b_j - (L-S)\right| - \left|\sum_{j=1}^{\infty} a_j - S\right| \geq 2\varepsilon - \varepsilon = \varepsilon.$$

(e) For any divergent series $\sum_{j=1}^{\infty} a_j$, such as the harmonic series, $\sum_{j=1}^{\infty} 0 \cdot a_j = 0$. But if $c \neq 0$, then $\sum_{j=1}^{\infty} ca_j$ clearly must diverge.

Lemma. Let $\{d_j\}_{j=1}^{n}$ be a sequnece of digits from 0 to 9. Then,

$$0.\overline{d_1 d_2 \dots d_n} = \frac{d_1 d_2 \dots d_n}{\underbrace{99\dots9}_{n \text{ times}}}.$$

Proof. It is clear that

$$0.\overline{d_1 d_2 \dots d_n} = d_1 d_2 \dots d_n \sum_{j=1}^{\infty} 10^{-n} = \frac{d_1 d_2 \cdots d_n \cdot 10^{-n}}{1 - 10^{-n}} = \frac{d_1 d_2 \dots d_n}{\underbrace{99\dots9}_{n \text{ times}}}.$$

Exercise 6.4. Convert each of the following infinite repeating decimals below into a fraction. A bra over a set of digits means that these digits repeat indefintely.

(a) 0.25, (b) $0.\overline{25}$, (c) $0.\overline{9462}$, (d) $0.\overline{1473}$, (e) $12.00\overline{495}$.

Proof. By the above lemma, we see that

$$\text{(a) } 0.25 = \tfrac{1}{4}, \quad \text{(b) } 0.\overline{25} = \tfrac{25}{99}, \quad \text{(c) } 0.\overline{9462} = \tfrac{9462}{9999} = \tfrac{3153}{3333},$$
$$\text{(d) } 0.\overline{1473} = \tfrac{1473}{9999}, \quad \text{(e) } 12.004\overline{95} = 12 + \tfrac{4}{1000} + \tfrac{95}{99} \cdot 10^{-3} = \tfrac{1188491}{99000}.$$

Exercise 6.8. 2^k **test**. Let $\sum_{j=1}^{\infty} a_j$ be a nonincreasing sequence with nonnegative terms. Prove that $\sum_{j=1}^{\infty} a_j$ converges iff $\sum_{j=1}^{\infty} 2^k a_{2^k}$ converges.
Hint. Use Example 6.8 as guidance.

Proof. Assume $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges. By nonnegativity, $\sum_{j=1}^{\infty} 2^k a_{2^k} = \infty$ is clear. Hence,

$$\sum_{j=1}^{2^n} a_j = a_1 + \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} a_j \geq a_1 + \sum_{k=0}^{n-1} 2^k a_{2^{k+1}} \geq \frac{1}{2} a_1 + \frac{1}{2} \sum_{k=0}^{n} 2^k a_{2^k}.$$

So, $\sum_{j=1}^{\infty} a_j = \infty$.
Now suppose $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges to $S \geq 0$. Similarly,

$$\sum_{j=1}^{2^n} a_j = a_1 + \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} a_j \leq a_1 + \sum_{k=0}^{n-1} 2^k a_{2^k} \leq a_1 + S.$$

Since $\sum_{j=1}^{n} a_j$ is nondecreasing and bounded, it definitely converges.

Exercise 6.9. Prove that $\sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j}$ converges by showing that the partial sums form a Cauchy sequence.

Proof. Let $\varepsilon > 0$ and pick $N \in \mathbb{N}$, such that $\varepsilon > N^{-1}$. Then, for every $m \geq n \geq N$ we have

$$\left| \sum_{j=n+1}^{m} (-1)^{j+1} \frac{1}{j} \right| = \left| \sum_{j=1}^{m-n} (-1)^{n+1+j} \frac{1}{n+j} \right| \leq \sum_{\substack{j=1 \\ j \text{ odd}}}^{m-n} \frac{1}{n+j} - \sum_{\substack{j=1 \\ j \text{ even}}}^{m-n} \frac{1}{n+j+1} \leq \frac{1}{n+1}$$

which is less than ε.

Exercise 6.10. Translating between sequences and series. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Prove that $\{a_n\}_{n=1}^{\infty}$ converges iff the series $\sum_{j=1}^{\infty} a_{j+1} - a_j$ converges and that in this case we obtain the limit as $\lim_{n \to \infty} a_n = a_1 + \sum_{j=1}^{\infty} a_{j+1} - a_j$.

Proof. This is apparent by noticing that $\sum_{j=1}^{n} a_{j+1} - a_j = a_{n+1} - a_1$, for any $n \in \mathbb{N}$ and sequence $\{a_n\}_{n=1}^{\infty}$.

Exercise 6.11. Use Lemma 6.3 to prove the Monotone Sequence Theorem.

Proof. Wlog, let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence bounded above by u. So, $a_{j+1} - a_j \geq 0$ for each j, and $\{\sum_{j=1}^{n} a_{j+1} - a_j\}_{n=1}^{\infty}$ is bounded above by $u + |a_1|$. Hence, Lemma 6.3 tells us $\sum_{j=1}^{\infty} a_{j+1} - a_j$ converges. The preceding exercise implies $\{a_n\}_{n=1}^{\infty}$ converges.

Exercise 6.12. Let $\sum_{j=1}^{\infty} a_j$ be a convergent series, let $\{j_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of natural numbers with $j_0 = 1$ and for all $k \in \mathbb{N}$ let $A_k := \sum_{j=j_{k-1}}^{j_k - 1} a_j$. Prove that $\sum_{k=1}^{\infty} A_k$ converges and $\sum_{k=1}^{\infty} A_k = \sum_{j=1}^{\infty} a_j$.

Proof. We see that $\sum_{k=1}^{n} A_k = \sum_{j=1}^{j_n - 1} a_j$ for each $n \in \mathbb{N}$. Hence, the result follows immediately. (Recall that for any convergent sequence $\{s_n\}_{n=1}^{\infty}$, all subsequences of $\{s_n\}_{n=1}^{\infty}$ have the same limit as $\{s_n\}_{n=1}^{\infty}$.)

§6.2 Absolute Convergence and Unconditional Convergence

Theorem 6.11 (Alternating Series Test). Let $\{b_j\}_{j=1}^{\infty}$ be a nonincreasing nonnegative sequence such that $\lim_{j\to\infty} b_j = 0$. Then, $\sum_{j=1}^{\infty} (-1)^{j+1} b_j$ converges.

Proof. This is clear from imitating what was done for exercise 6.9.

Proposition 6.13. If the series $\sum_{j=1}^{\infty} a_j$ converges absolutely, then it converges. Moreover, the **triangular inequality** $|\sum_{j=1}^{\infty} a_j| \leq \sum_{j=1}^{\infty} |a_j|$ holds.

Proof. Let $\varepsilon > 0$ and pick $N \in \mathbb{N}$, such that for $n \geq m \geq N$,

$$\sum_{j=m}^{n} |a_j| < \varepsilon.$$

So, by the triangular inequality (exercise 1-42),

$$\left| \sum_{j=m}^{n} a_j \right| \leq \sum_{j=m}^{n} |a_j| < \varepsilon.$$

Furthermore, from the reverse triangular inequality,

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} a_j \right| = \left| \sum_{j=1}^{\infty} a_j \right|.$$

It is hence clear that

$$\left| \sum_{j=1}^{\infty} a_j \right| \leq \sum_{j=1}^{\infty} |a_j|.$$

Theorem 6.15 (Comparison Test). Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be series with $0 \leq a_j \leq b_j$ for all $j \in \mathbb{N}$. If $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges, too.

Proof. When $\sum_{j=1}^{\infty} b_j$ converges, the partial sums $\sum_{j=1}^{n} a_j$ are bounded above by $\sum_{j=1}^{\infty} b_j$. By Lemma 6.3, $\sum_{j=1}^{\infty} a_j$ must converge.

Theorem 6.16. Let series $\sum_{j=1}^{\infty} a_j$ converges absolutely iff it converges unconditionally.

Proof. Assume $\sum_{j=1}^{\infty} a_j$ converges absolutely. Let σ be a permutation of $\mathbb{N}$ and $\varepsilon > 0$. So, pick $N \in \mathbb{N}$ such that if $n \geq m \geq N$,

$$\sum_{j=m}^{n} |a_j| < \varepsilon.$$

Fix $M \in \mathbb{N}$, such that for any $i \geq M$ we have $\sigma(i) \geq N$. Further define $u_n := \max\{\sigma^{-1}(j) \mid j \leq n\} \geq n$. For every $n \geq m \geq M$, we see that

$$\sum_{i=m}^{n} |a_{\sigma(i)}| \leq \sum_{j=N}^{u_n} |a_j| < \varepsilon.$$

As such, $\sum_{i=1}^{n} a_{\sigma(i)}$ converges. Therefore, $\sum_{j=1}^{\infty} a_j$ converges unconditionally.
Now consider when $\sum_{j=1}^{\infty} a_j$ does not converge absolutely. Let $\{a_{p_k}\}_{k=1}^{\infty}$ and $\{a_{q_k}\}_{k=1}^{\infty}$ be the subsequences of all nonnegative and all negative a_j, respectively. Wlog,

$$\sum_{k=1}^{\infty} a_{p_k} = \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} a_{q_k} = -\infty.$$

Thus, for each n, there is the least natural number ℓ_n such that

$$\sum_{k=1}^{\ell_n} a_{p_k} - \sum_{k=1}^{n} a_{q_k} \geq n.$$

We hence have the permutation σ defined by

$$\sigma = (\underbrace{p_1, p_2, \ldots, p_{\ell_1}}_{1 \text{ to } \ell_1}, q_1, \underbrace{p_{\ell_1+1}, p_{\ell_1+2}, \ldots, p_{\ell_2}}_{\ell_1+1 \text{ to } \ell_2}, q_2, \ldots, \underbrace{p_{\ell_n+1}, p_{\ell_n+2}, \ldots, p_{\ell_{n+1}}}_{\ell_n+1 \text{ to } \ell_{n+1}}, q_{n+1}, \ldots).$$

Hence,

$$\sum_{i=1}^{\infty} a_{\sigma(i)} = \infty.$$

does not converge unconditionally

Remark. If exactly one of $\sum_{k=1}^{\infty} a_{p_k}$ or $\sum_{k=1}^{\infty} a_{q_k}$ diverges, then $\sum_{j=1}^{\infty} a_j$ diverges; it does not converge unconditionally.
When that both series converge, $\sum_{j=1}^{\infty} |a_j| \leq \sum_{k=1}^{\infty} a_{p_k} - a_{q_k}$ also converges, a contradiction.

Remark. To be pedantic, we would write $\sigma(1) = p_1$ and

$$\sigma(i+1) = \begin{cases} \sigma(i)+1 & \text{if } \sigma(i) = p_m \text{ for some } \ell_n \leq m < \ell_{n+1}, \\ p_{\ell_n+1} & \text{if } \sigma(i) = q_n \text{ for some } n, \\ q_{n+1} & \text{if } \sigma(i) = p_{\ell_n} \text{ for some } n. \end{cases}$$

Proposition 6.22. Let $\{a_{(i,j)}\}_{i,j=1}^{\infty}$ be a family of nonnegative numbers. Then, the double series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)}$ converges iff for all bijections $\sigma\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ the sum $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{\sigma(i,j)}$ converges. Furthermore, in this case the values are equal.

Proof. Assume the double series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)} = \sum_{i=1}^{\infty}|\sum_{j=1}^{\infty} a_{(i,j)}|$ converges. The preceding theorem guarantees unconditional convergence. So $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{\sigma(i,j)}$ converges to the same value. (The necessary permutation on $\mathbb{N}$ is obvious.)

The converse is trivial. Simply let σ be the identity function on $\mathbb{N}\times\mathbb{N}$.

Exercise 6.19. Give an example of an absolutely convergent series $\sum_{j=1}^{\infty} a_j$ so that $\sum_{j=1}^{\infty} a_j \neq \sum_{j=1}^{\infty} |a_j|$.

Proof. Consider the geometric sequence $\{(-1/2)^j\}_{j=1}^{\infty}$. Then, the sum

$$\sum_{j=1}^{\infty}\left(-\frac{1}{2}\right)^j = -\frac{1}{2}$$

whilst

$$\sum_{j=1}^{\infty}\left|-\frac{1}{2}\right|^j = \frac{1}{2}.$$

Exercise 6.21. Determine which of the following series converges. If it converges, determine if it converges absolutely.

(a) $\sum\limits_{j=1}^{\infty} \frac{1}{j!}$, (b) $\sum\limits_{j=1}^{\infty} \frac{(-1)^j}{j+\sqrt{j}}$, (c) $\sum\limits_{j=1}^{\infty} \frac{(-1)^j}{4^j}$

Proof.

(a) Notice that $0 \leq 1/j! \leq (1/2)^{j-1}$ for all j. Hence, by the comparison test, $\sum_{j=1}^{\infty} \frac{1}{j!}$ converges (absolutely).

(b) For each j, we have

$$0 \leq \frac{1}{2j} \leq \frac{1}{j+\sqrt{j}}.$$

By the comparison test, since the harmonic series diverges, so must $\sum_{j=1}^{\infty} \frac{(-1)^j}{j+\sqrt{j}}$. Hence, absolute convergence is impossible. But regular convergence is possible. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$, such that $\varepsilon > N^{-1}$. So, if $n \geq m \geq N$,

$$\begin{aligned}\left|\sum_{j=m}^{n} \frac{(-1)^j}{j+\sqrt{j}}\right| &= \left|\sum_{j=0}^{m-nn-m} \frac{(-1)^{m+j}}{m+j+\sqrt{m+j}}\right| \\ &\leq \sum_{\substack{j=0 \\ j \text{ even}}}^{n-m} \frac{1}{m+j+\sqrt{m+j}} - \sum_{\substack{j=0 \\ j \text{ odd}}}^{n-m} \frac{1}{m+j+1+\sqrt{m+j+1}} \\ &\leq \frac{1}{m+\sqrt{m}} < \varepsilon.\end{aligned}$$

(c) We see that this is just a geometric series with constant ratio $-1/4$. Thus, it converges (absolutely).

Exercise 6.22 (Limit Comparison Test for series). Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be series with positive terms. Prove that if $\lim_{j\to\infty} a_j/b_j = c > 0$, then either both series converge or both series diverge.

Proof. Since $c > 0$, we can assume (for the sake of argument) wlog that $\sum_{j=1}^{\infty} b_j$ converges. Moreover, notice $\{a_j/b_j\}_{j=1}^{\infty}$ is bounded from above by some $u > 0$. As such, $0 < a_j \leq u b_j$ for all j. Therefore, $\sum_{j=1}^{\infty} a_j$ converges.

Note. The importance of $c > 0$ should be noted. If $c = 0$, then $\lim_{j\to\infty} b_j/a_j = \infty$. As such, the "wlog" part of the proof fails to hold. It is possible that $\sum_{j=1}^{\infty} a_j$ converges, while $\sum_{j=1}^{\infty} b_j$ diverges.

Example. Consider $a_j = 2^{-j}$ and $b_j = 2^j$. Then, $\lim_{j\to\infty} a_j/b_j = \lim_{j\to\infty} 2^{-2j} = 0$. But it is clear that $\sum_{j=1}^{\infty} a_j = 0$, whilst $\sum_{j=1}^{\infty} b_j = \infty$.

The definition of the permutation σ in our proof of theorem 6.16 was a little inelegant. Let's try defining a little tool to help us have better notation (hopefully).

Definition. Given a sequence of permutations $\sigma_i := (a_{i1}, a_{i2}, \ldots, a_{ik_i})$ of k_i numbers, we define

$$(\sigma_1, \sigma_2, \ldots) := (\underbrace{a_{11}, a_{12}, \ldots, a_{1k_1}}_{\sigma_1}, \underbrace{a_{21}, a_{22}, \ldots, a_{2k_2}}_{\sigma_2}, \ldots).$$

Exercise 6.23. Let $\sum_{j=1}^{\infty} a_j$ be a conditionally convergent series and let $z \in \mathbb{R}$. Prove that there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\sum_{j=1}^{\infty} a_{\sigma(j)} = z$.

Proof. As in theorem 6.16, we let $\{a_{p_k}\}_{k=1}^{\infty}$ and $\{a_{q_k}\}_{k=1}^{\infty}$ be the subsequences of all nonnegative and all negative a_j, respectively. Conditional convergence implies that and $\sum_{j=1}^{\infty} a_{p_k} = \infty$, and $\sum_{j=1}^{\infty} a_{q_k} = -\infty$. Hence, we define ν_n recursively by $\nu_0 := 0$, and having ν_n be the least natural number with

$$\sum_{k=1}^{\nu_n} a_{p_k} + \sum_{k=1}^{\mu_{n-1}} a_{q_k} > z,$$

where μ_{n-1} is the least natural number with

$$\sum_{k=1}^{\nu_{n-1}} a_{p_k} + \sum_{k=1}^{\mu_{n-1}} a_{q_k} < z.$$

Now define $P_n := (p_{\nu_n+1}, p_{\nu_n+2}, \dots, p_{\nu_{n+1}})$ and $Q_n := (q_{\mu_n+1}, q_{\mu_n+2}, \dots, q_{\mu_{n+1}})$. We obtain the permutation

$$\sigma := (P_0, Q_0, P_1, Q_1, \dots, P_n, Q_n, \dots).$$

Let $\varepsilon > 0$ and note that $\lim_{k\to\infty} a_{p_k} = \lim_{k\to\infty} a_{q_k} = 0$. Thus, there is some $K \in \mathbb{N}$ such that if $k \geq K$, we have

$$0 \leq a_{p_k} < \varepsilon \qquad \text{and} \qquad 0 < -a_{q_k} < \varepsilon.$$

Consequently,

$$\left| \sum_{j=1}^{k} a_{\sigma(j)} - z \right| < \varepsilon.$$

Exercise 6.24. Let $\{a_{(i,j)}\}_{i,j=1}^{\infty}$ be a family of real numbers so that the double series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{(i,j)}|$ converges. Prove that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{(i,j)}$ converges to a number L and for all bijections $\sigma\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ the sum $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{\sigma(i,j)}$ converges to the same number L.

Proof. Recall that $0 \leq |\sum_{j=1}^{\infty} a_{(i,j)}| \leq \sum_{j=1}^{\infty} |a_{(i,j)}|$ always holds. So, the series

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} a_{(i,j)} \right| \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{(i,j)}|$$

converges absolutely, and hence, unconditionally to some limit L.

Chapter 7

Some Set Theory

§7.1 The Algebra of Sets

Exercise 7.5. Let Xbe a set. An **algebra** is a set of sets $\mathscr{A} \subseteq \mathscr{P}(X)$ such that $\varnothing \in \mathscr{A}$, if $A \in \mathscr{A}$, then $X \setminus A \in \mathscr{A}$, and if $A_j \in \mathscr{A}$ for all $j = 1, \ldots, n$, then $\bigcup_{j=1}^{n} A_j \in \mathscr{A}$.

(a) Prove that if $A_j \in \mathscr{A}$ for all $j = 1, \ldots, n$, then $\bigcap_{j=1}^{n} A_j \in \mathscr{A}$.

(b) Let X be a set. Prove that the power set of X is an algebra.

(c) Let X be a set. Prove that $\mathscr{A} := \{A \subseteq X \mid A \text{ or } X \setminus A \text{ is finite}\}$ is an algebra.

(d) Prove that an algebra need not contain countable unions of its elements.

Proof.

(a) Notice that $X - \bigcap_{j=1}^{n} A_j = \bigcup_{j=1}^{n} (X - A_j) \in \mathscr{A}$. So, $\bigcap_{j=1}^{n} A_j \in \mathscr{A}$ follows.

(b) We see that $\varnothing \subseteq X$, and $X - A \subseteq X$ for all $A \subseteq X$. Moreover, a union of subsets of X is still a subset of X. Hence, the power set of X is indeed an algebra.

(c) Since $\varnothing \subseteq X$ is obviously finite, $\varnothing \in \mathscr{A}$. For $A \in \mathscr{A}$, clearly A or $X - A$ is finite. Thus, $X - A \in \mathscr{A}$. Now let $A_1, A_2, \ldots, A_n \in \mathscr{A}$. If A_j is infinite for some j, then $X - \bigcup_{j=1}^{n} A_j \subseteq X - A_j$ is finite. When each A_j is finite, $\bigcup_{j=1}^{n} A_j$ is finite. Either ways, $\bigcup_{j=1}^{n} A_j \in \mathscr{A}$ holds.

(d) Consider the set $\daleth$ of all finite subsets of $\mathbb{N}$. It is easily verified to be an algebra. However, the countable union $\bigcup_{j=1}^{\infty} \{j\} = \mathbb{N} \notin \daleth$.

§7.2 Countable Sets

Lemma. Let $\{a_{(i,j)}\}_{j=1}^{\infty}$ be a family of numbers, such that the double series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{(i,j)}$ converges.

Exercise 7.17. Let $\{a_{(i,j)}\}_{j=1}^{\infty}$ be a family of nonnegative numbers. Then the double series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)}$ converges iff for all bijections $\sigma\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ the sum $\sum_{i=1}^{\infty} a_{\sigma(i)}$ converges. Furthermore, in this case the values are equal.

Proof. Assume $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)}$ and let $\sigma\colon \mathbb{N}\to\mathbb{N}\times\mathbb{N}$ be a bijection. Fix n and m. So, define $\daleth := \max\{(\pi_1\circ\pi)(\iota) \mid \iota \le n\}$ and

$$\gimel_i := \max\{(\pi_2\circ\pi)(j) \mid (\pi_1\circ\pi)(\iota) = i \ \&\ \iota \le n\}.$$

Hence,

$$0 \le \sum_{i=1}^{n} a_{\sigma(i)} \le \sum_{i=1}^{\daleth}\sum_{j=1}^{\gimel_i} a_{(i,j)} \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)}.$$

The series $\sum_{i=1}^{\infty} a_{\sigma(i)}$ must converge.
Conversely, suppose $\sum_{i=1}^{\infty} a_{\sigma(i)}$ converges for all bijections $\sigma\colon\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ and fix $n\in\mathbb{N}$. Now letting $\mathcal{M}_i := \max\{\sigma^{-1}(i,j) \mid j \le m\}$, we have that

$$0 \le \sum_{j=1}^{m} a_{(i,j)} \le \sum_{i=1}^{\mathcal{M}_i} a_{\sigma(i)} \le \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

Therefore, $\sum_{j=1}^{\infty} a_{(i,j)}$ converges for all i.
Now let $\varepsilon > 0$. Pick $M\in\mathbb{N}$ such that, for every $1\le m\le n$,

$$\sum_{j=1}^{\infty} a_{(i,j)} - \sum_{j=1}^{m} a_{(i,j)} < \frac{\varepsilon}{n}.$$

Thus, for $\mathscr{M} := \max\{\sigma^{-1}(i,j) \mid i\le n \ \&\ j \le m\}$, we see that

$$0 \le \sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{(i,j)} < \varepsilon + \sum_{i=1}^{n}\sum_{j=1}^{m} a_{(i,j)} \le \varepsilon + \sum_{i=1}^{\mathscr{M}} a_{\sigma(i)} \le \varepsilon + \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

As such, the series $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)}$ converges.
In fact, since

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)} \le \sum_{i=1}^{\infty} a_{\sigma(i)},$$

it holds that

$$\sum_{i=1}^{\infty} a_{\sigma(i)} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{(i,j)}.$$

§7.3 Uncountable Sets

Exercise 7.23. Let $F\colon [a,b] \to \mathbb{R}$ be a **nondecreasing** bounded function. Prove that F can have at most countably many discontinuities.

Proof. Let $x \in [a,b]$, $L := \sup f[a,x)$ and $\varepsilon > 0$. Then, there is $y \in [a,x)$ for which $L - f(y) < \varepsilon$. Consequently, since F is nondecreasing, for any $|z - x| < x - y$ we have $L - f(z) \leq L - f(y) < \varepsilon$. So, only jump discontinuities are present.
(Work in progress)

Exercise 7.24. Prove that for every countable subset $A \subseteq \mathbb{R}$ there is a **nondecreasing** function $f\colon \mathbb{R} \to [0,1]$ that is continuous on $\mathbb{R} \setminus A$ and discontinuous at every $a \in A$.

Proof. *I assume the author means to prove continuity/discontinuity with respect to* $\mathbb{R} - A$.
Wlog, consider A being countably infinite. Let there f be a bijection from $\mathbb{N}$ to A. Define the function $h\colon \mathbb{N} \to \mathbb{R}$ by $h(0) := 1/2$ and the following.

i. If $f(k+1) < f(i)$ for all $i \leq k$, then let $h(n+1) := h(\ell)/2$, where $h(\ell) := \min\{h(i) \mid i \leq k\}$.
ii. When $f(k+1) > f(i)$ for every $i \leq k$, let $h(n+1) := \frac{h(M)+1}{2}$, where $h(M) := \max\{h(i) \mid i \leq k\}$.
iii. Otherwise, let $h(k+1) = \frac{h(n)+h(m)}{2}$ for the largest $f(n) < f(k+1)$ and least $f(m) < f(k+1)$ such that $n, m \leq k$.

(Work in progress)

Exercise 7.25. Cantor sets.

(a) Prove that for any sequence $Q = \{q_n\}_{n=1}^{\infty}$ of numbers $q_n \in (0, 1/2)$ there is a bijective function from C^Q to the set of all sequences of zeroes and ones.
(b) Prove that C^Q is uncountable.
(c) Prove that the set of endpoints of the intervals $I_{j,n}^Q$ that make up the C_n^Q is countable.
(d) Prove that every $x \in C^Q$ is the limit of a sequence of endpoints of the intervals $I_{j,n}^Q$.

Proof. (a) Every
(Work in progress)

Chapter 8

The Riemann Integral II

Proposition 8.3. Countable subsets of $\mathbb{R}$ have outer Lebesgue measure 0.

Proof. Wlog, let $C \subseteq \mathbb{R}$ be a countably infinite set. Then, there is a bijection $f\colon \mathbb{N} \to C$. So, for each $\varepsilon > 0$ we define $h_\varepsilon\colon \mathbb{N} \to \mathscr{P}(\mathbb{R})$ by

$$h_\varepsilon(j) := (f(j) - \varepsilon^j,\ f(j) + \varepsilon^j).$$

We see that for $\varepsilon \in (0, 1)$:

$$\sum_{j=1}^{\infty} |h_\varepsilon(j)| = \sum_{j=1}^{\infty} 2\varepsilon^j = \frac{2\varepsilon}{1-\varepsilon} = -2 + \frac{2}{1-\varepsilon}.$$

Furthermore,

$$\lim_{\varepsilon \to 0} -2 + \frac{2}{1-\varepsilon} = 0.$$

Therefore, $\lambda(C) = 0$.

Proposition 8.5. Let $a, b \in \mathbb{R}$ and $a < b$. Then, $\lambda([a, b]) = b - a$.

Proof. For each $\varepsilon > 0$, let $I_1 := (a-\varepsilon, b+\varepsilon)$, and for $j \geq 2$, $I_j^\varepsilon := [0, \varepsilon^j]$. Therefore, given $\varepsilon \in (0, 1)$,

$$\sum_{j=1}^{\infty} |I_j| \geq b - a + 2\varepsilon + \frac{\varepsilon}{1-\varepsilon}.$$

As $\varepsilon \to 0$, this converges to $b - a$. i.e. $\lambda([b - a]) \leq b - a$.
Now let $I_j = (a_j, b_j)$ be a sequence of intervals that covers $[a, b]$. So by the Heine-Borel Theorem, $\sum_{j=1}^{n} |I_j| < b - a$ for some n. Wlog, $a_j \leq a_{j+1} \leq b_j \leq b_{j+1}$ for each j. Therefore,

$$\sum_{j=1}^{n} |I_j| \geq b_1 - a_1 + \sum_{j=2}^{n} b_j - b_{j-1} \geq b_n - a_1 \geq b - a.$$

i.e. $\lambda([b-a]) \geq b-a$.

Theorem 8.6 (The **properties of the outer Lebesgue measure** λ)**.** With ∞ defined to be greater than all real numbers and the sum of a divergent series of nonnegative numbers being ∞ we have the following.

1. $\lambda(\varnothing) = 0$.
2. If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.
3. Outer Lebesgue measure is **countably subadditive**. That is, for all sequences $\{A_n\}_{n=1}^{\infty}$ of subsets $A_n \subseteq \mathbb{R}$ the inequality

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda(A_n)$$

holds.

Proof.

1. Let $\varepsilon > 0$ and $I_j = (0, \varepsilon 2^{-j})$. Then, $\sum_{j=1}^{\infty} I_j \leq \varepsilon$. Thus, $\lambda(\varnothing) = 0$.
2. Let $\varepsilon > 0$, and $\{I_j\}_{j=1}^{\infty}$ be a sequence of open intervals that covers B, such that $\sum_{j=1}^{\infty} |I_j| < \lambda(B) + \varepsilon$. Hence, it covers A, which implies

$$\lambda(A) \leq \sum_{j=1}^{\infty} |I_j| < \lambda(B) + \varepsilon.$$

As such, $\lambda(A) \leq \lambda(B)$.

3. Let $\varepsilon > 0$ and $\sigma\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. For each n, choose a sequence of open intervals $\{I_{(n,j)}\}_{j=1}^{\infty}$ that cover A_n, such that $\sum_{j=1}^{\infty} |I_{(n,j)}| < \lambda(A_n) + \varepsilon 2^{-n}$. Notice that $\{I_{\sigma(i)}\}_{i=1}^{\infty}$ covers $\bigcup_{n=1}^{\infty} A_n$. By exercise 7.17,

$$\lambda\left(\sum_{n=1}^{\infty} A_n\right) \leq \sum_{i=1}^{\infty} |I_{\sigma(i)}| = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |I_{(n,j)}| < \varepsilon + \sum_{n=1}^{\infty} \lambda(A_n).$$

Consequently,

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \lambda(A_n).$$

Exercise 8.3. Let $Q = \{q_n\}_{n=1}^{\infty}$ be a sequence of numbers $q_n \in (0, 1/2)$ and let C^Q be the associated **Cantor set** as in Definition 7.22. We will use the notation of Definition 7.22 throughout this exercise.

(a) Prove that $\lambda(C_n^Q) = \prod_{j=1}^{n} 2q_j$.

(b) Prove that $\{\prod_{j=1}^{n} 2q_j\}_{n=1}^{\infty}$ converges.

(c) Prove that $\lambda(C^Q) = \lim_{n\to\infty} \prod_{j=1}^{n} 2q_j$.

(d) Prove that for any $q \in (0, 1/2)$ the constant sequence $Q = \{q\}_{n=1}^{\infty}$ yields a

Cantor set C^Q of measure zero.

Note. By exercise 7-25(b) in section 7.3 Cantor sets are uncountable. This means there are uncountable sets of measure zero.

(e) Use $Q = \left\{\frac{2^{n+1}-1}{2^{n+2}}\right\}_{n=1}^{\infty}$ to prove that there are Cantor sets that are not of measure zero.

(f) Prove that there are Cantor sets whose Lebesgue measure is arbitrarily close to 1.

Proof.

(a) If $n = 0$, then $\lambda([0,1]) = \lambda(C_n^Q) = \prod_{j=1}^{0} 2q_j = 1$. So, assume $n = k$ and consider $n = k+1$. It is clear that, $\lambda(I_{i,m}^Q) = \lambda(I_{j,m}^Q)$ for all m and $i, j \leq m$. Therefore, $\lambda(I_{i,k}^Q) = \prod_{j=1}^{k} q_j$. Thus, $\lambda(I_{i,k+1}^Q) = \prod_{j=1}^{k+1} q_j$, and hence, $\lambda(C_{n+1}^Q) = \prod_{j=1}^{k+1} 2q_j$.

(b) Since $2q_j < 1$ for each j, this sequence is decreasing. Furthermore, it is bounded below by 0, and hence, must converge.

(c) For any n, since $C^Q \subseteq C_n^Q$, we have that $\lambda(C^Q) \leq \prod_{j=1}^{n} 2q_j$. That is, $\lambda(C^Q) \leq \lim_{n\to\infty} \prod_{j=1}^{n} 2q_j$.
This implies $\lambda(C^Q) \in \mathbb{R}$. So let $\{\mathcal{I}_j\}_{j=1}^{\infty}$ be a sequence of open intervals that covers C^Q, such that $\sum_{j=1}^{\infty} |\mathcal{I}_j|$ converges. Fix some $x \in C^Q$; then $x \in \mathcal{I}_k$ for some k. Since $\lim_{n\to\infty} \prod_{j=1}^{n} q_j = 0$, pick i such that $\prod_{j=1}^{i} q_j < |\mathcal{I}_k|$.

$$\sum_{j=1}^{\infty} \mathcal{I}_j \geq \prod_{j=1}^{m} 2q_j \geq \lim_{n\to\infty} \prod_{j=1}^{n} 2q_j.$$

(Work in progress)
Probably related to the funnei uncountability of the Cantor set.

(d) It follows from the above, that since $2q < 1$,

$$\lambda(C^Q) = \lim_{n\to\infty} (2q)^n = 0.$$

(e)

Exercise 8.4. Use the Heine-Borel Theorem and the axioms for $\mathbb{R}$ except for Axiom 1.19 to prove the Bolzano-Weierstrass Theorem.

Exercise 8.5. Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous and $\lambda(\{x \in [a,b] \mid f(x) \neq 0\}) = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

Exercise 8.6. Prove that if $f, g\colon [a,b] \to \mathbb{R}$ are continuous almost everywhere, then $f + g$ is continuous almost everywhere.

Chapter 9

Resources

§9.1 Chapter 4: Differentiable Functions

1. Information on discontinuous derivatives
   (a) https://math.stackexchange.com/questions/292275/discontinuous-derivative
   (b) https://math.stackexchange.com/questions/112067/how-discontinuous-can-a-derivative-be